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Inference Based On Marshall-Olkin Extended Rayleigh Lomax Distribution*

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Abstract

In this paper, a new distribution called the Marshall-Olkin Rayleigh Lomax distribution is introduced and studied. Marshall and Olkin [1] proposed an interesting method of adding a parameter to well-known distributions, so we extended the Rayleigh Lomax distribution by the Marshall-Olkin method. For this new distribution, the probability density function, cumulative distribution function, hazard rate, survival function, moments and order statistics are derived. Then, maximum likelihood estimators and the confidence intervals of the model parameters by observed Fisher information matrix are obtained. Further, Bayes estimates under squared error and linear exponential loss functions using Markov Chain Monte Carlo methods are obtained. Finally, the comparative study of different method of estimations is carried out using Monte Carlo simulation method.

1 Introduction

An important topic for many researchers is to expand the family of distributions because it adds flexibility to the original distribution. There are many ways to generate new distributions from original distributions. One of these methods is proposed by Marshall and Olkin [1], in which the original distributions could be found as a special case of new distribution.

The Rayleigh distribution was introduced by Rayleigh [2] and Lomax distribution was given by Lomax [3]. These two important distributions in statistical science are widely used in the field of medicine, engineering, business and actuarial sciences. Fatima et al. [4] introduced a new probability model called Rayleigh Lomax distribution as a mixture of above two models and studied its distributional properties. This distribution is more flexible as compared to its base distributions. Sankaran and Jayakumar [5] studied the physical interpretation of MO family by considering odd models. Jose [6] provided the applications of MO family in reliability theory. Bdair [7] studied some methods of estimation for MO exponential distribution. Many researchers have used extended family distribution to generate new forms of the distributions. Some statistical properties of these new distributions were illustrated. For example, Alice and Jose [8] introduced MO Pareto distributions and its reliability applications. MO Extended Lomax distribution has been introduced by Ghitany et al. [9]. MO extended Weibull distribution was studied by Cordeiro and Lemonte [10]. Gui [11] studied MO power log-normal distribution. Singh et al. [12] presented Bayesian estimation of MO extended exponential parameters. For more details one can see, [13, 14, 15, 16, 17, 18]. The aim of this paper is to provide a new extension of Rayleigh Lomax distribution called Marshall–Olkin Rayleigh Lomax (MORL) distribution and study some of its properties and methods of estimation for the unknown parameters. The paper is organized as follows:

MORL distribution and some properties are introduced in section 2. The maximum likelihood estimations (MLEs) of the unknown parameters for MORL distribution are derived and confidence intervals by Fisher information matrix are obtained in section 3. In section 4, MCMC is used to obtain the Bayes estimates of

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the parameters. In section 5, simulation study to compare the efficiency of different methods is provided. The discussion of this work is concluded in section 6.

2 New Model and its Properties

The probability density function (PDF) and cumulative distribution function (CDF) of the Rayleigh Lomax distribution respectively are given by

$$f(x) = \frac{\beta\lambda}{\theta}\left(\frac{\theta}{\theta+x}\right)^{-2\lambda+1} e^{-\frac{\beta}{2}\left(\frac{\theta}{\theta+x}\right)^{-2\lambda}}, \; x \geq -\theta; \;\; \theta,\, \lambda,\, \beta > 0 \tag{1}$$

and

$$F(x) = 1 - e^{-\frac{\beta}{2}\left(\frac{\theta}{\theta+x}\right)^{-2\lambda}}, \; x \geq -\theta; \;\; \theta,\, \lambda,\, \beta > 0. \tag{2}$$

Without loss of generality throughout the discussion we shall consider $\lambda = 0.5$. Therefore (1) and (2) becomes

$$f(x) = \frac{\beta}{2\,\theta} e^{-\frac{\beta}{2\theta}(\theta+x)}, \; x \geq -\theta; \;\; \theta,\, \beta > 0 \tag{3}$$

and

$$\bar{F}(x) = e^{-\frac{\beta}{2\theta}(\theta+x)}, \; x \geq -\theta; \;\; \theta,\, \beta > 0. \tag{4}$$

Marshall and Olkin [1] proposed a method to expand families of distributions based on the survival function of a distribution by adding a new parameter. If $\bar{F}(x)$denote the survival function of a continuous random variable X. Then, the survival function of Marshall Olkin (MO) distribution is given by

$$\bar{G}(x,\alpha) = \frac{\alpha\bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}, \;\; -\infty < x < \infty, \; \alpha > 0, \; \bar{\alpha} = 1-\alpha. \tag{5}$$

Then the survival function of the three parameters MORL distribution is defined as

$$\bar{F}(x,\alpha,\beta,\theta) = \frac{\alpha e^{-\frac{\beta}{2\theta}(\theta+x)}}{1-\bar{\alpha}e^{-\frac{\beta}{2\theta}(\theta+x)}}, \;\; x > -\theta. \tag{6}$$

The PDF and hazard rate function (HRF) of MORL distribution, respectively are

$$f(x,\alpha,\beta,\theta) = \frac{\alpha\beta e^{-\frac{\beta}{2\theta}(\theta+x)}}{2\theta\left(1-\bar{\alpha}e^{-\frac{\beta}{2\theta}(\theta+x)}\right)^2}, \;\; \alpha,\, \beta,\, \theta > 0,\, x > -\theta \tag{7}$$

and

$$h(x,\alpha,\beta,\theta) = \frac{\beta}{2\theta\left(1-\bar{\alpha}e^{-\frac{\beta}{2\theta}(\theta+x)}\right)}, \;\; \alpha,\, \beta,\, \theta > 0,\, x > -\theta. \tag{8}$$

Figures 1–6 describe different forms of the PDF, CDF, HRF of the MORL distribution for different values of the parameters β and θ at $\alpha = 3$.

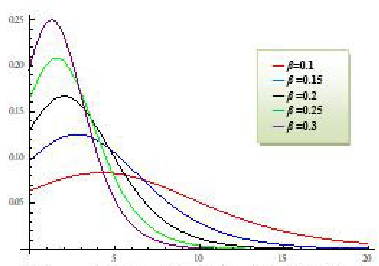

Figure 1: Plots of the PDF of the MORL for $\alpha = 3,\ \theta = 0.2$

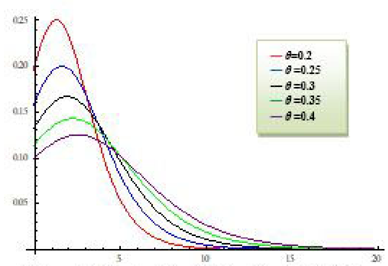

Figure 2: Plots of the PDF of the MORL for $\alpha = 3,\ \beta = 0.3$

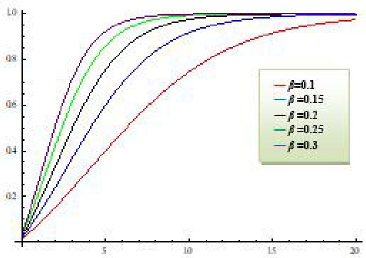

Figure 3: Plots of the CDF of the MORL for $\alpha = 3,\ \theta = 0.2$

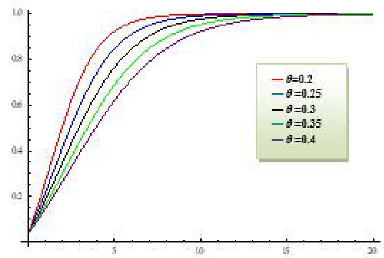

Figure 4: Plots of the CDF of the MORL for $\alpha = 3,\ \beta = 0.3$

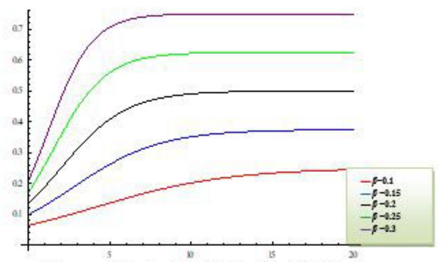

Figure 5: Plots of the HRF of the MORL for $\alpha = 3,\ \theta = 0.2$

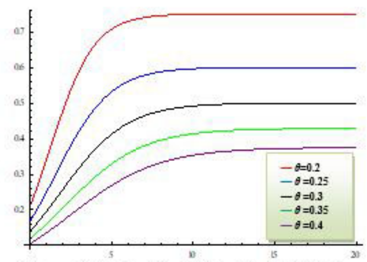

Figure 6: Plots of the HRF of the MORL for $\alpha = 3,\ \beta = 0.3$

2.1 Moments

In any statistical analysis, moments are important and have significant applications. The j^{th} moment of the MORL distribution is given as

$$E(X^j) = \int_{-\theta}^{\infty} x^j \frac{\alpha\beta e^{-\frac{\beta}{2\theta}(\theta+x)}}{2\theta(1-\bar{\alpha}e^{-\frac{\beta}{2\theta}(\theta+x)})^2}\,dx. \tag{9}$$

Let $\frac{\beta}{2\theta}(\theta+x) = t$ in (9). Then

$$\begin{aligned}
E(X^j) &= \alpha\theta^j \int_0^{\infty} \left(\frac{2t}{\beta}-1\right)^j \frac{e^{-t}}{(1-\bar{\alpha}e^{-t})^2}\,dt \\
&= \alpha\theta^j \sum_{i=0}^{j}(-1)^i \left(\frac{2}{\beta}\right)^{j-i} \int_0^{\infty} t^{j-i}\frac{e^{-t}}{(1-\bar{\alpha}e^{-t})^2}\,dt \\
&= \alpha\theta^j \sum_{i=0}^{j}(-1)^i \left(\frac{2}{\beta}\right)^{j-i} \sum_{l=1}^{\infty} l(\bar{\alpha})^{l-1} \int_0^{\infty} e^{-lt}t^{j-i}dt \\
&= \alpha\theta^j \sum_{i=0}^{j}\sum_{l=1}^{\infty}(-1)^i \left(\frac{2}{\beta}\right)^{j-i} l(\bar{\alpha})^{l-1}\frac{\Gamma(j-i+1)}{(l^{j-i+1})}. \qquad (10)
\end{aligned}$$

2.2 Order Statistics

In a random sample of size n from a continuous distribution, the probability density function of r^{th} order statistic $X_{r:n}$ is

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x). \tag{11}$$

The expression (11) can also be expressed as

$$f_{r:n}(x) = C_{r:n}\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}[1-F(x)]^{n+j-r}f(x), \tag{12}$$

where $C_{r:n} = \frac{n!}{(r-1)!(n-r)!}$. Thus using (12), the PDF of r^{th} order statistic $X_{r:n}$ from MORL in view of (6) and (7) can be written as

$$\begin{aligned} f_{r:n}(x) &= \frac{\beta}{2\theta}C_{r:n}\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}\alpha^{n+j-r+1}e^{-\frac{\beta}{2\theta}(n+j-r+1)(\theta+x)} \\ &\quad\times(1-\bar{\alpha}e^{-\frac{\beta}{2\theta}(\theta+x)})^{-(n+j-r+2)}. \end{aligned}$$

Since $(1-x)^{-n} = \sum_{m=0}^{\infty}\binom{m+n-1}{m}x^m$, the above PDF of $X_{r:n}$ can be expressed as

$$\begin{aligned} f_{r:n}(x) &= \frac{\beta}{2\theta}C_{r:n}\sum_{j=0}^{r-1}\sum_{m=0}^{\infty}(-1)^j\binom{r-1}{j}\binom{m+n+j-r+1}{m} \\ &\quad\times\alpha^{n+j-r+1}\bar{\alpha}^m e^{-\frac{\beta(\theta+x)}{2\theta}(m+n+j-r+1)}. \end{aligned} \tag{13}$$

The CDF of $X_{r:n}$ is

$$F_{r:n}(x) = \sum_{i=r}^{n}\binom{n}{i}[F(x)]^i[1-F(x)]^{n-i} = \sum_{i=r}^{n}\sum_{j=0}^{i}(-1)^j\binom{n}{i}\binom{i}{j}[1-F(x)]^{n-i+j}.$$

Now using (6), the CDF of r^{th} order statistics from MORL distribution is given as

$$F_{r:n}(x) = \sum_{i=r}^{n}\sum_{j=0}^{i}\sum_{k=0}^{\infty}(-1)^i\binom{n}{i}\binom{i}{j}\binom{k+n-i+j}{k}\alpha^{n-i+j}(\bar{\alpha})^k e^{-\frac{\beta}{2\theta}(\theta+x)(n-i+j+k)}. \tag{14}$$

3 Classical Estimation

In this section, the MLEs and confidence intervals based on approximate Fisher information matrix of the unknown parameters α, β and θ are evaluated.

3.1 Maximum Likelihood Estimation

Let $X_1, X_2, ..., X_n$ be a random sample from MORL distribution. Then the likelihood function is given by

$$L(\psi;\underset{\sim}{x}) = \alpha^n\beta^n(2\theta)^{-n}e^{-\frac{\beta}{2\theta}(n\theta+\sum_{i=1}^{n}x_i)}\prod_{i=1}^{n}\left[1-\bar{\alpha}e^{-\frac{\beta}{2\theta}(\theta+x_i)}\right]^{-2}, \tag{15}$$

where $\psi = (\alpha, \beta, \theta)$. The log-likelihood function $\ell(\theta)$ is given by

$$\ell(\psi;\underset{\sim}{x}) = n\ln\alpha + n\ln\beta - n\ln(2\theta) - \frac{\beta}{2\theta}(n\theta+\sum_{i=1}^{n}x_i) - 2\sum_{i=1}^{n}\ln\left[1-\bar{\alpha}e^{-\frac{\beta}{2\theta}(\theta+x_i)}\right].$$

Thus, we have

$$\frac{\partial\ell(\psi;\underset{\sim}{x})}{\partial\alpha} = \frac{n}{\alpha} - 2\sum_{i=1}^{n}[C_i^{-1}(\psi)\,e^{-\frac{\beta}{2\theta}(\theta+x_i)}], \tag{16}$$

$$\frac{\partial \ell(\psi;\underset{\sim}{x})}{\partial \beta}=\frac{n}{\beta}-\frac{n(\theta+\bar{x})}{2\theta}-2\sum_{i=1}^{n}\frac{\bar{\alpha}}{2\theta}[C_i^{-1}(\psi)(\theta+x_i)e^{-\frac{\beta}{2\theta}(\theta+x_i)}], \quad (17)$$

and

$$\frac{\partial \ell(\psi;\underset{\sim}{x})}{\partial \theta}=-\frac{n}{\theta}+\frac{\beta n\bar{x}}{2\theta^2}+\sum_{i=1}^{n}\frac{\bar{\alpha}\beta}{\theta^2}[C_i^{-1}(\psi)x_i e^{-\frac{\beta}{2\theta}(\theta+x_i)}], \quad (18)$$

where $C_i(\psi)=1-\bar{\alpha}e^{-\frac{\beta}{2\theta}(\theta+x_i)}$; $\psi=(\alpha,\,\beta,\,\theta)$, $i=1,2,...,n$ and $\bar{x}=\frac{1}{n}\sum_{i=1}^{n}x_i$.

Making use of $\frac{\partial \ell(\psi;\underset{\sim}{x})}{\partial \alpha}=0$, $\frac{\partial \ell(\psi;\underset{\sim}{x})}{\partial \beta}=0$ and $\frac{\partial \ell(\psi;\underset{\sim}{x})}{\partial \theta}=0$ in (16), (17) and (18), we get a system of three nonlinear equations in three unknowns parameters α, β and θ. To find a numerical solution of the system, one can use the Newton-Raphson method.

3.2 Asymptotic Confidence Intervals

Now, we can use the asymptotic distributions of the MLEs for the parameters $\psi=(\alpha,\beta,\theta)$, to find the approximate confidence intervals for the parameters. The asymptotic distribution of the MLEs of ψ is given by

$$\left[(\hat{\alpha}-\alpha),\,(\hat{\beta}-\beta),\,(\hat{\theta}-\theta)\right]\rightarrow N[0,I^{-1}(\alpha,\,\beta,\theta)],$$

where $I^{-1}(\alpha,\,\beta,\theta)$ is the variance-covariance matrix of the parameters$\psi=(\alpha,\,\beta,\theta)$, and it can be approximated by the inverse of observed Fisher-information matrix. Observed Fisher-information matrix is given by

$$I(\hat{\alpha},\,\hat{\beta},\hat{\theta})=\begin{pmatrix}\frac{\partial^2\ell}{\partial\alpha^2} & \frac{\partial^2\ell}{\partial\alpha\partial\beta} & \frac{\partial^2\ell}{\partial\alpha\partial\theta}\\ \frac{\partial^2\ell}{\partial\alpha\partial\beta} & \frac{\partial^2\ell}{\partial\beta^2} & \frac{\partial^2\ell}{\partial\beta\partial\theta}\\ \frac{\partial^2\ell}{\partial\alpha\partial\theta} & \frac{\partial^2\ell}{\partial\beta\partial\theta} & \frac{\partial^2\ell}{\partial\theta^2}\end{pmatrix}_{(\alpha,\,\beta,\,\theta)=(\hat{\alpha},\,\hat{\beta},\,\hat{\theta})}=\begin{pmatrix}I_{11} & I_{12} & I_{13}\\ I_{21} & I_{22} & I_{23}\\ I_{31} & I_{32} & I_{33}\end{pmatrix}$$

The elements of matrix $I(\hat{\alpha},\,\hat{\beta},\hat{\theta})$ can be expressed using following equations

$$\frac{\partial^2 \ell(\psi;\underset{\sim}{x})}{\partial \alpha^2}=-\frac{n}{\alpha^2}+2\sum_{i=1}^{n}[C_i^{-2}(\psi)\,e^{-\frac{\beta}{\theta}(\theta+x_i)}],$$

$$\frac{\partial^2 \ell(\psi;\underset{\sim}{x})}{\partial \alpha\partial\beta}=\sum_{i=1}^{n}\frac{(\theta+x_i)}{\theta}\,C_i^{-2}(\psi)\,e^{-\frac{\beta}{2\theta}(\theta+x_i)},$$

$$\frac{\partial^2 \ell(\psi;\underset{\sim}{x})}{\partial \alpha\partial\theta}=-\sum_{i=1}^{n}\frac{\beta x_i}{\theta^2}\,C_i^{-2}(\psi)\,e^{-\frac{\beta}{2\theta}(\theta+x_i)},$$

$$\frac{\partial^2 \ell(\psi;\underset{\sim}{x})}{\partial \beta^2}=-\frac{n}{\beta^2}+\sum_{i=1}^{n}\frac{\bar{\alpha}(\theta+x_i)^2}{2\theta^2}\,C_i^{-2}(\psi)\,e^{-\frac{\beta}{2\theta}(\theta+x_i)},$$

$$\frac{\partial^2 \ell(\psi;\underset{\sim}{x})}{\partial \beta\partial\theta}=\frac{n\bar{x}}{2\theta^2}-\frac{\bar{\alpha}}{2\theta^3}\sum_{i=1}^{n}e^{-\frac{\beta}{2\theta}(\theta+x_i)}[C_i^{-1}(\psi)x_i(\beta x_i+\beta\theta-2\theta)+\bar{\alpha}\beta x_i C_i^{-2}(\psi)e^{-\frac{\beta}{2\theta}(\theta+x_i)}(\theta+x_i)],$$

$$\frac{\partial^2 \ell(\psi;\underset{\sim}{x})}{\partial \theta^2}=\frac{n}{\theta^2}-\frac{\beta}{\theta^3}\sum_{i=1}^{n}x_i+\frac{\beta\bar{\alpha}}{2\theta^4}\sum_{i=1}^{n}x_i e^{-\frac{\beta}{2\theta}(\theta+x_i)}[-4\theta C_i^{-1}(\psi)+\beta x_i C_i^{-1}(\psi)+\beta\bar{\alpha}x_i C_i^{-2}(\psi)e^{-\frac{\beta}{2\theta}(\theta+x_i)}].$$

Then, the approximate $100(1-\gamma)\%$ two-sided confidence intervals for α, β and θ are, respectively, given by

$$\left[\hat{\alpha}_{ML}\pm Z_{\gamma/2}\sqrt{I_{11}^{-1}(\hat{\alpha}_{ML})},\,(\hat{\beta}_{ML}\pm Z_{\gamma/2}\sqrt{I_{22}^{-1}(\hat{\beta}_{ML})},\,(\hat{\theta}_{ML}\pm Z_{\gamma/2}\sqrt{I_{33}^{-1}(\hat{\theta}_{ML})}\right],$$

where $Z_{\gamma/2}$ is the upper $100(\gamma/2)^{th}$ percentile of the standard normal distribution.

4 Bayesian Estimation

In this section, we shall consider square error (SE) and linear exponential (LINEX) loss function to find the Bayes estimates (BEs) of the parametersα, β and θ. For obtaining the BEs, informative priors are used. Suppose the parameters α, β and θ are independent, then

$$\pi_1(\alpha) \propto \alpha^{\lambda_1-1}e^{-\alpha/\eta_1};\ \lambda_1,\eta_1>0,$$

$$\pi_2(\beta) \propto \beta^{\lambda_2-1}e^{-\beta/\eta_2};\ \lambda_2,\eta_2>0,$$

$$\pi_3(\theta) \propto \theta^{\lambda_3-1}e^{-\theta/\eta_3};\ \lambda_3,\eta_3>0.$$

Many authors used the gamma prior because it covers all prior information of the experimenter. For more details, see [19], [20]. Then the joint prior is given by

$$\pi(\alpha,\beta,\theta) \propto \alpha^{\lambda_1-1}\beta^{\lambda_2-1}\theta^{\lambda_3-1}e^{-\alpha/\eta_1-\beta/\eta_2-\theta/\eta_3};\ \lambda_1,\lambda_2,\lambda_3,\eta_1,\eta_2,\eta_3>0. \tag{19}$$

From (15) and (19), the joint posterior density function is

$$\pi^*(\alpha,\beta,\theta|\underset{\sim}{x}) \propto L(\alpha,\beta,\theta;\underset{\sim}{x})\pi(\alpha,\beta,\theta)$$

$$\propto \alpha^{n+\lambda_1-1}\beta^{n+\lambda_2-1}\theta^{\lambda_3-n-1}e^{-\alpha/\eta_1-\beta/\eta_2-\theta/\eta_3}\ e^{\frac{-\beta}{2\theta}(n\theta+\sum_{i=1}^n x_i)}\prod_{i=1}^n\left[1-\bar{\alpha}e^{-\frac{\beta}{2\theta}(\theta+x)}\right]^{-2}. \tag{20}$$

Based on SE loss and LINEX loss functions, the BE of the function $\Phi(\psi)=\Phi(\alpha,\beta,\theta)$ of the model parameters α, β and θ can be written, respectively, as

$$\begin{aligned}
\hat{\Phi}_{SE}(\psi) &= E[\Phi(\psi)|\underset{\sim}{x}] = \int_\psi \Phi(\psi)\pi^*(\psi|\underset{\sim}{x})d\psi \\
&= \int_0^\infty\int_0^\infty\int_0^\infty \Phi(\alpha,\beta,\theta)\pi^*(\alpha,\beta,\theta|\underset{\sim}{x})d\alpha d\beta d\theta \\
&= \int_0^\infty\int_0^\infty\int_0^\infty \Phi(\alpha,\beta,\theta)\alpha^{n+\lambda_1-1}\beta^{n+\lambda_2-1}\theta^{\lambda_3-n-1}e^{-\alpha/\eta_1-\beta/\eta_2-\theta/\eta_3} \\
&\quad\times e^{-\frac{\beta}{2\theta}(n\theta+\sum_{i=1}^n x_i)}\prod_{i=1}^n[1-\bar{\alpha}e^{-\frac{\beta}{2\theta}(\theta+x)}]^{-2}d\alpha d\beta d\theta,
\end{aligned} \tag{21}$$

$$\begin{aligned}
\hat{\Phi}_{LINEX}(\psi) &= -\frac{1}{c}\ln\left[E(\exp(-c\ \Phi(\psi)|\underset{\sim}{x})\right] \\
&= -\frac{1}{c}\ln\left[\int_0^\infty\int_0^\infty\int_0^\infty e^{-c\Phi(\alpha,\beta,\theta)}\pi^*(\alpha,\beta,\theta|\underset{\sim}{x})d\alpha d\beta d\theta\right] \\
&= -\frac{1}{c}\ln[\int_0^\infty\int_0^\infty\int_0^\infty e^{-c\Phi(\alpha,\beta,\theta)}\alpha^{n+\lambda_1-1}\beta^{n+\lambda_2-1}\theta^{\lambda_3-n-1}e^{-\alpha/\eta_1-\beta/\eta_2-\theta/\eta_3}e^{-\frac{\beta}{2\theta}(n\theta+\sum_{i=1}^n x_i)} \\
&\quad\times\prod_{i=1}^n[1-\bar{\alpha}e^{-\frac{\beta}{2\theta}(\theta+x)}]^{-2}\ d\alpha d\beta d\theta],
\end{aligned} \tag{22}$$

where $E(\cdot)$ is the expected value and c is an arbitrary constant. This integration cannot be in closed form. Therefore, we adopt MCMC method to approximate this integral.

4.1 MCMC Method

In Bayesian analysis, methods such as MCMC have revolutionized Bayesian statistical computation. For full details on the advantages of the method, one can see [21, 22, 23]. In this section, we have studied Gibbs sampling, which is MCMC algorithm for obtaining a sequence of random samples from posterior function to compute the BEs and credible intervals under the SE and LINEX loss functions. From (20), the conditional posterior density functions of the parameters α, β and θ can be written, respectively, as follows

$$\pi^*_\alpha(\alpha|\beta,\theta;\underset{\sim}{x}) \propto \alpha^{n+\lambda_1-1}e^{-\alpha/\eta_1}\prod_{i=1}^{n}\left[1-\bar{\alpha}e^{-\frac{\beta}{2\theta}(\theta+x)}\right]^{-2}, \tag{23}$$

$$\pi^*_\beta(\beta|\alpha,\theta;\underset{\sim}{x}) \propto \beta^{n+\lambda_2-1}e^{-\beta/\eta_2}e^{-\frac{\beta}{2\theta}(n\theta+\sum_{i=1}^{n}x_i)}\prod_{i=1}^{n}\left[1-\bar{\alpha}e^{-\frac{\beta}{2\theta}(\theta+x)}\right]^{-2}, \tag{24}$$

and

$$\pi^*_\theta(\theta|\alpha,\beta;\underset{\sim}{x}) \propto \theta^{\lambda_3-n-1}e^{-\theta/\eta_3}e^{-\frac{\beta}{2\theta}(n\theta+\sum_{i=1}^{n}x_i)}\prod_{i=1}^{n}\left[1-\bar{\alpha}e^{-\frac{\beta}{2\theta}(\theta+x)}\right]^{-2}. \tag{25}$$

The equations (23), (24) and (25) cannot be reduced analytically to known distributions from the conditional posterior distributions of α, β and θ. So, to solve this problem, we use the Metropolis-Hastings (MH) algorithm, which is used to generate random samples by using normal proposal distribution. For more information, see [24]. To compute the Bayes estimators, we used the following algorithm.

Algorithm 1

1. Start with an ($\alpha^{(0)}=\hat{\alpha}_{MLE}$, $\beta^{(0)}=\hat{\beta}_{MLE}$ and $\theta^{(0)}=\hat{\theta}_{MLE}$).
2. Set $t=1$.
3. Use MH algorithm to generate $\alpha^{(t)}$ from $\pi^*_\alpha(\alpha^{(t-1)}|\beta^{(t-1)},\theta^{(t-1)};\underset{\sim}{x})$ with the proposal distribution $N\left(\alpha^{(t-1)},Var(\hat{\alpha}_{ML})\right)$, $\beta^{(t)}$ from $\pi^*_\beta(\beta^{(t-1)}|\alpha^{(t)},\theta^{(t-1)};\underset{\sim}{x})$ with the proposal distribution $N\left(\beta^{(t-1)},Var(\hat{\beta}_{ML})\right)$ and then generate $\theta^{(t)}$ from $\pi^*_\theta(\theta^{(t-1)}|\beta^{(t)},\theta^{(t)};\underset{\sim}{x})$ with the proposal distribution $N\left(\theta^{(t-1)},Var(\hat{\theta}_{ML})\right)$.
4. Set $t=t+1$.
5. Repeat steps 2-4 N times.
6. Obtain $\alpha^{(t)}$, $\beta^{(t)}$ and $\theta^{(t)}$, $t=M+1,\ldots..,N$, and now, the approximate means of $\Phi(\alpha,\beta,\theta)$ and $\exp\left[-c\,\Phi(\alpha,\beta,\theta)\right]$ are given, respectively, by

$$E\left[\Phi(\alpha,\beta,\theta)|\underset{\sim}{x}\right]=\frac{1}{N-M}\sum_{t=M+1}^{N}\Phi(\alpha^{(t)},\beta^{(t)},\ \theta^{(t)}),$$

$$E\left[\exp(-c\Phi(\alpha,\beta,\theta)|\underset{\sim}{x})\right]=\frac{1}{N-M}\sum_{t=M+1}^{N}\exp\left[-c\Phi(\alpha^{(t)},\beta^{(t)},\theta^{(t)})|\underset{\sim}{x}\right],$$

where M is the burn-in period.

Therefore, the Bayes MCMC point estimate of $\Phi(\alpha,\beta,\theta)$ based on *SE* and *LINEX* loss function are given, respectively, by

$$\hat{\Phi}_{SE}(\alpha,\beta,\theta)=E\left[\Phi(\alpha,\beta,\theta)|\underset{\sim}{x}\right],$$

$$\hat{\Phi}_{LINEX}(\alpha,\beta,\theta)=-\frac{1}{c}\ln\left[E(\exp(-c\Phi(\alpha,\beta,\theta)|\underset{\sim}{x})\right].$$

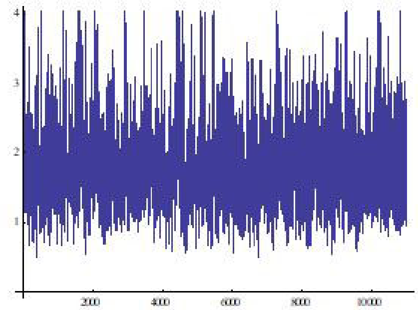

Figure 7: Simulation number of $\alpha = 2$ obtained by MCMC method

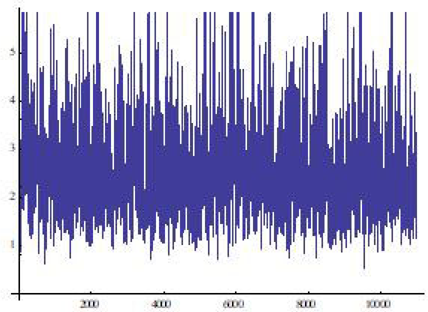

Figure 8: Simulation number of $\alpha = 2.5$ obtained by MCMC method

7. Repeat 1-6 H times and arrange each estimate in order as $\hat{\alpha}_{SE}^{[1]} \ldots \hat{\alpha}_{SE}^{[H]}$, $\hat{\beta}_{SE}^{[1]} \ldots \hat{\beta}_{SE}^{[H]}$ and $\hat{\theta}_{SE}^{[1]} \ldots \hat{\theta}_{SE}^{[H]}$. Then, the $100(1-\gamma)\%$ credible intervals for α, β and θ are $(\hat{\alpha}_{SE}^{[\frac{\gamma}{2}H]}, \hat{\alpha}_{SE}^{[(1-\frac{\gamma}{2})H]})$, $(\hat{\beta}_{SE}^{[\frac{\gamma}{2}H]}, \hat{\beta}_{SE}^{[(1-\frac{\gamma}{2})H]})$ and $(\hat{\theta}_{SE}^{[\frac{\gamma}{2}H]}, \hat{\theta}_{SE}^{[(1-\frac{\gamma}{2})H]})$ respectively.

5 Simulation Study

In this section, simulation studies for different sample sizes are conducted to evaluate the performances of the MLEs and Bayes estimates, the estimation procedure is performed according to the following algorithm.
Algorithm 2

1. Set values for n and c.
2. For given values of the prior parameters $(\lambda_1, \lambda_2, \lambda_3, \eta_1, \eta_2, \eta_3)$ generate α, β and θ from $\pi_1(\alpha)$, $\pi_2(\beta)$ and $\pi_3(\theta)$, respectively.
3. For given values of the parameters (α, β, θ), use the model given by equation (5) to generate random sample $x_1, x_2, \ldots, x_n$.
4. Use the random sample to compute the MLEs of the parameters α, β and θ by solving the nonlinear equations (16), (17) and (18).
5. Compute the Bayes estimates of (α, β, θ) based on SE and LINEX loss function using MCMC approximation.
6. Use Metropolis algorithm to generate a sequence of 11000 random samples iteratively with $N = 11000$ and $M = 1000$.
7. Compute the approximate confidence bounds with confidence level 95% and compute 95% credible CIs for the parameters α, β and θ.
8. Repeat the steps 3–7, 1000 times.
9. Compute the average values of the MSEs and BEs of the parameters α, β and θ.

From Figures 7–12, one can see that most of the simulation numbers for parameters α, β and θ are centered around the suggested values of these parameters. Thus, we can conclude that the mean of the parameters is an estimate for this parameter.

Table 1: MLE and BE of the parameters α, β and θ for the given prior values $(\eta_1 = 0.5, \eta_2 = 0.5, \eta_3 = 0.4, \lambda_1 = 0.7, \lambda_2 = 0.7, \lambda_3 = 0.7)$ with $c = 2$

α β θ	n	$\hat{\alpha}$			$\hat{\beta}$			$\hat{\theta}$		
		ML	BE		ML	BE		ML	BE	
			SE	LINEX		SE	LINEX		SE	LINEX
2.0 0.3 0.2	10	1.6627	1.7342	1.7452	0.2425	0.2521	0.2611	0.1463	0.1594	0.1562
	30	1.8523	1.8821	1.8993	0.2697	0.2733	0.2721	0.1627	0.1732	0.1811
	50	1.8921	1.9244	1.9623	0.2711	0.3212	0.3189	0.1801	0.1911	0.1932
2.5 0.5 0.5	10	2.2442	2.334	2.38121	0.3345	0.4231	0.4177	0.2976	0.3521	0.3421
	30	2.2213	2.4111	2.3993	0.3724	0.4575	0.4356	0.3421	0.4032	0.3987
	50	0.2298	2.5120	2.5199	0.3987	0.5113	0.4876	0.3965	0.4664	0.4529

Table 2 : Lengths of 95% CIs for estimates of the parameters α, β and θ for the given prior values $(\eta_1 = 0.5,\ \eta_2 = 0.5,\ \eta_3 = 0.4,\ \lambda_1 = 0.7,\ \lambda_2 = 0.7,\ \lambda_3 = 0.7)$ with $c = 2$

α β θ	n	$\hat{\alpha}$		$\hat{\beta}$		$\hat{\theta}$	
		ML	BE	ML	BE	ML	BE
2.0 0.3 0.2	10	1.1342	1.0826	0.5593	0.5224	0.4620	0.3753
	30	0.9623	0.8201	0.5132	0.4891	0.3591	0.3129
	50	0.8721	0.8532	0.4504	0.3439	0.2961	0.2101
2.5 0.5 0.5	10	0.8864	0.7721	0.5065	0.4654	0.3321	0.3110
	30	0.7123	0.6129	0.4325	0.3218	0.2562	0.1974
	50	0.5463	0.3224	0.2381	0.1974	0.1894	0.1103

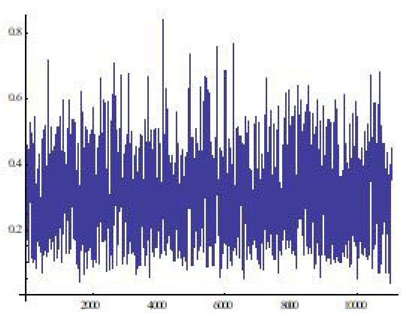

Figure 9: Simulation number of $\beta = 0.3$ obtained by MCMC method

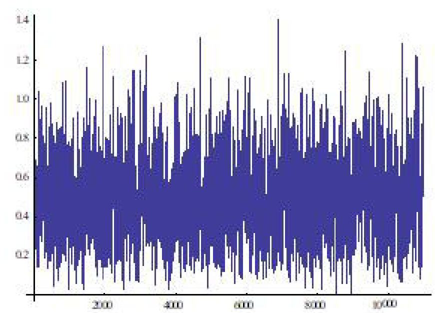

Figure 10: Simulation number of $\beta = 0.5$ obtained by MCMC method

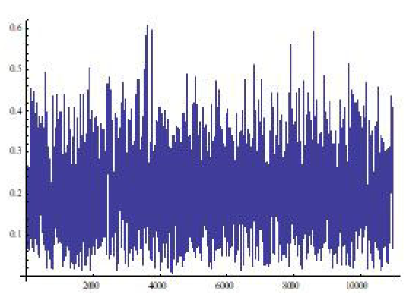

Figure 11 : Simulation number of $\theta = 0.2$ obtained by MCMC method

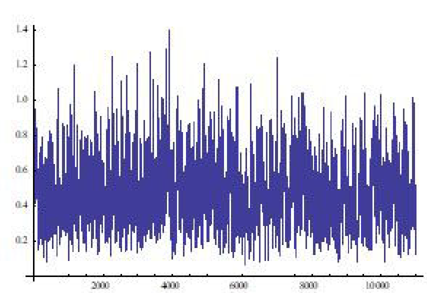

Figure 12: Simulation number of $\theta = 0.5$ obtained by MCMC method

6 Concluding Remarks

In this paper, a new distribution is proposed, which we call as MORL distribution. For this distribution, the probability density function, cumulative distribution function, hazard rate, survival function, moments and order statistics are derived. Then, maximum likelihood estimators and Bayes estimates using MCMC of the model parameters are discussed. Finally, a simulation study is conducted to compare the performance of the estimators. We found from Tables 1 and 2 that:

- The parameters' estimations based on Bayesian method are much better than those based on the ML method.
- The ML and Bayes methods give more precise estimate by increasing the sample size.
- The length of the CIs by ML and Bayes methods are decreasing by increasing the sample size n.

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