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An Identity Derived By A 90° Turn Of A Classical Fluid-Mechanics Problem*

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Abstract

We take the classical problem of laminar flow in a rectangular duct and turn the duct 90°. Driven by the same pressure gradient, the volume flow rate should remain the same. This leads to an identity that relates a polynomial to infinite series of tanh functions. We discuss the mathematical properties of this identity and verify it by two different methods.

1 Introduction

A well-behaved function can be expanded as an infinite series of another function, if the other function forms a complete orthogonal base set [1]. The most commonly known base functions are the sine and cosine functions [2]. It is more difficult to express a function as an infinite series of hyperbolic functions because they do not form a complete set. Hence, identities involving infinite series of hyperbolic functions are interesting and require special derivation methods [3–6]. Here, we show a simple method for deriving such an identity.

2 A Classical Fluid-Mechanics Problem

Fully-developed incompressible laminar flow of a Newtonian fluid in a horizontal rectangular duct obeys [7]

$$\mu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=\frac{\partial p}{\partial x}, \tag{1}$$

where u is the axial velocity, p is the fluid pressure, μ is the fluid viscosity, and (x, y, z) is a Cartesian coordinate system positioned at the center of the duct, with x pointing downstream. At the duct walls located at $y = \pm a$ and $z = \pm b$, the fluid cannot slip so that $u = 0$. Since the flow is assumed fully-developed, $u = u(y, z)$, and the driving pressure gradient $\partial p/\partial x$ must be a constant [7]. An analytic solution of u has been found by the method of separation of variables [7]. Thus, the axial volume flow rate then can be calculated as [7]

$$Q=\int_{-b}^{b}\int_{-a}^{a}u(y,z)\,dy\,dz=\frac{4ba^3}{3\mu}\left(-\frac{\partial p}{\partial x}\right)\left[1-\frac{192a}{\pi^5 b}\sum_{n=1,3,5,\ldots}^{\infty}\frac{\tanh(n\pi b/2a)}{n^5}\right]. \tag{2}$$

This result has been used in our research of dual-wet micro heat pipes [8] and drop flow in rectangular microchannels [9].

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3 A Mathematical Identity

If the duct is rotated 90° (i.e., swap a and b), but with the same pressure gradient $\partial p/\partial x$ and viscosity μ, then the volume flow rate Q should remain unchanged. Thereby, the following identity arises:

$$r - r^3 = \frac{192}{\pi^5} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^5} \left[\tanh\left(\frac{n\pi r}{2}\right) - r^4 \tanh\left(\frac{n\pi}{2r}\right)\right], \tag{3}$$

where $r = a/b$ is the aspect ratio. Equation (3) shows that a polynomial can be written as infinite series of hyperbolic tangent functions. An interesting property of this identity is that if we replace r by $1/r$, the same identity is recovered (as expected). Therefore, equation (3) is invariant in the $r \to 1/r$ transformation. We note that the right-hand side of (3) contains a function of r that is differentiable only to the third order at $r = 0$. (Thus, the identity can be differentiated once and twice to give two new identities that relate two lower-order polynomials to infinite series of hyperbolic functions.) Both series on the right-hand side converge absolutely as demonstrated by the direct comparison test. Equation (3) is also valid when r is complex. Particularly, when $r = i\Psi$, where Ψ is real, (3) becomes

$$\Psi + \Psi^3 = \frac{192}{\pi^5} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^5} \left[\tan\left(\frac{n\pi\Psi}{2}\right) + \Psi^4 \tan\left(\frac{n\pi}{2\Psi}\right)\right]. \tag{4}$$

It should be noted that equation (4) has singularities at $\Psi = (2m-1)/n$ and $\Psi = n/(2m-1)$, where $m = 0, \pm 1, \pm 2, \dots$.

4 Verifications and Conclusions

Equations (3) and (4) have been numerically verified for r and Ψ in the $\pm[10^{-6}, 10^6]$ interval (away from the singularities). For small values of r and Ψ ($|r|, |\Psi| \le 2$), four terms in the summations in equation (3) are sufficient to achieve an accuracy of four significant figures, whereas in (4) three terms will yield the same level of accuracy. For $2 < |r|, |\Psi| < 15$, six terms give an accuracy of four significant figures for (3) and (4). For large values of r and Ψ ($|r|, |\Psi| \ge 15$), eight terms are needed to give an accuracy of five significant figures for (3) and (4).

We search the literature for this identity and find a recent manuscript that contains several identities involving infinite series of hyperbolic functions [10]. The identities were derived by the contour integration and residue theorem. Equation (**2.17**) in [10] reads

$$\alpha\beta^k \sum_{n=1}^{\infty} \frac{\tanh\left(\frac{2n-1}{2}\alpha\right)}{(2n-1)^{2k-1}} + (-1)^k \alpha^k \beta \sum_{n=1}^{\infty} \frac{\tanh\left(\frac{2n-1}{2}\beta\right)}{(2n-1)^{2k-1}}$$
$$= \sum_{\substack{k_1+k_2=k \\ k_1,k_2 \ge 1}}^{\infty} \frac{(2^{2k_1}-1)(2^{2k_2}-1)\,|B_{2k_1}|\,B_{2k_2}}{(2k_1)!\,(2k_2)!} \alpha^{k_2} \beta^{k_1} \pi^{2k}, \tag{5}$$

where k is a positive integer greater than 1, α and β are real numbers such that $\alpha\beta = \pi^2$, and B_{2k_1} and B_{2k_2} are the Bernoulli numbers. If we set $k = 3$, $\alpha = \pi r$, and $\beta = \pi/r$ in (5), then our identity in (3) is recovered. Thus, our identity is a special case of a more general identity involving infinite series of hyperbolic functions. Since our identity is derived by a method different from that of the general identity, the agreement provides a mutual verification of both results.

This research is initiated by an incidental observation while reading a textbook fluid-mechanics problem. It shows that classical problems can still yield interesting results if one is willing to think harder.

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