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Some Existence, Uniqueness And Stability Results Of Nonlocal Random Impulsive Integro-Differential Equations*

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Abstract

This paper is concerned with random impulsive integro-differential equations with nonlocal conditions. At first, some sufficient conditions which can guarantee existence and uniqueness of mild solution are derived using Banach fixed point theorem. Secondly, combining with Banach fixed point theorem with some inequality techniques, we give stability of the solution. Finally some examples are given to establish the effectiveness of our results.

1 Introduction

The impulsive system has been considered to be one of the most important models in mathematical ecology, and many perfect existence as well as stability results of its modified models have been obtained. For example, in order to maintain the long-term sustainable development of fishery industry, the government puts a lot of little fish into the sea in spring and allows the fishermen to catch the adult fish in autumn and winter, which can be described by impulsive differential equations. Also we must choose the impulse perturbation coefficients based on the actual situation, which may oscillate in some ranges or change irregularly.

There will be instantaneous and great changes of population density in the form of perturbations if we take into account the disturbance of environmental factors at certain time moments, which cannot be neglected. So naturally we can introduce impulsive effects into differential equations (see Bainov and Simeonov [4], Lakshmikantham, Bainov and Simeonov [11] and Saker [17]).

Many authors [9, 22] have studied the existence of solutions of impulsive differential equations of the form

$$x^{'}(t) = f(t, x(t), S(t), T(t)), \quad 0 < t < T_0,\ t \neq t_i, \tag{1}$$

$$x_{t_0} = x_0, \tag{2}$$

$$\Delta x(t_i) = I_i(t_i), \quad i = 1, 2, \ldots, p. \tag{3}$$

Guo and Liu [9] also established the existence theorems of maximal and minimal solutions for (1)–(3) with strong conditions provided f is uniformly continuous. Guo and Liu [12], Liu [10] also considered the case when f does not contain integral operator S in (1) and obtained the same conclusion by using monotone iterative technique. Again recently, Liu [10] considered the special case where (1)–(3) has no impulses and Liu [10] obtained a unique solution by using monotone iterative technique with coupled upper and lower quasi-solutions when $f = f(t, x(t), S(t), T(t))$. Liu [14] also obtained a similar conclusion. In [9, 12, 10] the assumptions that f satisfies some compactness-type conditions is required. But it is difficult and inconvenient to verify in abstract spaces. By using the successive approximations for the evolution equation with an

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unbounded operator A, Rogovchenko [16] studied the existence and uniqueness of the classical solutions. That is equations of the form

$$x^{'}(t) = Ax(t) + f(t, x(t)), \quad t > 0, \ t \neq t_i$$

with impulsive condition in (2) − (3), where A is sectorial operator with some conditions given on the fractional operators $A^{\alpha}, \alpha \geq 0$. Liu [13] studied the existence of mild solutions of the impulsive evolution equation

$$x^{'}(t) = Ax(t) + f(t, x(t)), \quad 0 < t < T_0, \ t \neq t_i$$

where A is the infinitesimal generator of C_0 semigroup with the impulsive condition in (2) − (3) by using semigroup theory.

Most of the published papers on impulsive differential systems deals with the problems related to fixed time impulses. However, actual jumps do not always happen at fixed points but usually at random points. Recently the properties of solutions to some differential equations with random impulses have been studied [21, 2, 20, 3, 8, 1].

The existence of solution for non local differential equations have been extensively researched in recent years taking into account the theoretical and practical significance. Byszewski initiated the nonlocal initial conditions for evolution equations [5, 6]. There are many applications for nonlocal condition in physics and it is more natural than the classical initial condition $x(0) = x_0$. Recently, Sayooj Aby Jose and Venkatesh Usha [19] extended the results of [5, 6] to random impulsive differential equations with non local initial conditions and proved the existence of the solutions by a fixed point theorem.

There are several papers which include the study of impulsive integrodifferential equations involving random impulses [18, 9, 7]. Random impulsive integro-differential equation with non local initial conditions is studied in this paper, hoping that the results obtained will contribute to the area. And it is a well known fact that, if we consider integro-differential equations, in some applications, we will be able to obtain better descriptions of the phenomena under study. Thus, the main objective of this work is to present non local random impulsive integro-differential equations.

This paper is summarized as follows: Section 2 includes some preliminaries. Some hypotheses are included in Section 3. The existence and uniqueness of solution of random impulsive integro-differential equation with nonlocal condition is investigated in section 4. And we have used Lipschitz condition for deriving the main results, followed by stability results in section 5. In the last section two examples are discussed.

2 Preliminaries

Consider a real separable Hilbert space X and a non empty set Ω. Assume that τ_k is a random variable defined from Ω to D_k, where $D_k = (0, d_k)$ for all $k \in \mathbb{N}$ (collection of natural numbers) and $0 < d_k < +\infty$. Also for $i, j = 1, 2, \ldots$ assume that if $i \neq j$ then τ_i and τ_j are independent with each other. Let τ be a real constant. Denote $\Re_\tau = [\tau, T]$. Next we consider the nonlocal random impulsive integro differential equations of the form

$$\begin{cases} x^{'}(t) = Ax(t) + f(t, x(t)) + \int_0^T f_1(\eta, x(t+\eta))d\eta, \quad t \neq \xi_k, \ t \geq \tau, \\ x(\xi_k) = b_k(\tau_k)x(\xi_k^-), \quad k = 1, 2, 3, \ldots, \\ x_{t_0} + g(x) = x_0, \end{cases} \tag{4}$$

where A is the infinitesimal generator of a strongly continuous semi group of bounded linear operator $S(t)$ in X, $f, f_1 : \Re_\tau \times X \to X$, $b_k : D_k \to \Re$ for each $k \in \mathbb{N}$, $g : X \to X$ is a given function; $\xi_0 = t_0 \in [\tau, T]$ and $\xi_k = \xi_{k-1} + \tau_k$ for each $k \in \mathbb{N}$, here $t_0 \in \Re_\tau$ is arbitrary real number. Obviously

$$t_0 = \xi_0 < \xi_1 < \xi_2 < \xi_3 \cdots < \xi_k < \ldots ; x(\xi_k -) = \lim_{t \uparrow \xi_k} x(t)$$

according to their path with the norm $\|x\| = \sup_{\tau \leq \eta \leq t} |x(\eta)|$ for each t satisfying $t \in [\tau, T]$.

Let $\{B_t, t \geq 0\}$ be the simple counting process generated by $\{\xi_n\}$, that implies $\{B_t \geq t\}$, also denote $\mathbb{F}_t$ as the notation for the σ-algebra generated by $\{B_t, t \geq 0\}$. The $(\Omega, P, \{\mathbb{F}_t\})$ is a probability space. And the Hilbert space of all $\{\mathbb{F}_t\}$-measurable square integrable random variables with values in X is denoted as $L_2 = L_2(\Omega, \{\mathbb{F}_t\}, X)$.

Let $\mathbb{B}$ denote Banach space $\mathbb{B}([\tau, T], L_2)$, the family of all $\{\mathbb{F}_t\}$-measurable random variable ψ with the norm

$$\|\psi\|^2 = \sup_{t\in[\tau,T]} E\|\psi\|^2.$$

Definition 1 ([15]) *Let A be the infinitesimal generator of a C_0 semigroup $S(t)$. Let $u_0 \in X$ and $f \in L^1(0, T; X)$ with nonlocal condition $g(u)$. Then the function $u \in C([0, T]; X)$ is given by*

$$u(t) = S(t)(u_0 - g(u)) + \int_0^t S(t-s)f(s)ds, \quad 0 \le t \le T$$

is the mild solution of the initial value problem

$$\begin{cases} u'(t) = Au(t) + f(t), & 0 < t < T, \\ u(0) + g(x) = u_0. \end{cases} \tag{5}$$

Definition 2 ([15]) *A semigroup $\{S(t), 0 \le t < \infty\}$ of bounded linear operators on X is uniformly bounded if there exists a constant $\mathcal{K} \ge 1$ such that*

$$\|S(t)\| \le \mathcal{K}, \quad \text{for } t \ge 0.$$

Definition 3 *For a given $T \in (\tau, +\infty)$, a stochastic process $\{x(t) \in \mathbb{B}, \tau \le t \le T\}$ is called a mild solution to equation (4) in $(\Omega, p, \{\mathbb{F}_t\})$, if*

(i) $x(t) \in X$ is $\mathbb{F}_t$-adapted;

(ii)

$$\begin{aligned} x(t) = \sum_{k=0}^{+\infty} \Bigg[& \prod_{i=1}^{k} b_i(\tau_i) S(t-t_0)(x_0 - g(x)) + \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} S(t-s) f(s, x(s)) ds \\ & + \int_{\xi_k}^{t} S(t-s) f(s, x(s)) ds + \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} S(t-s) \int_0^T f_1(\eta, x(s+\eta)) d\eta ds \\ & + \int_{\xi_k}^{t} S(t-s) \int_0^T f_1(\eta, x(s+\eta)) d\eta ds \Bigg] I_{[\xi_k, \xi_{k+1})}(t), \qquad t \in [\tau, T] \end{aligned}$$

where

$$\prod_{j=m}^{n} (.) = 1 \text{ as } m > n, \prod_{j=i}^{k} b_j(\tau_i) = b_k(\tau_k) b_{k-1}(\tau_{k-1}) \dots b_i(\tau_i)$$

and $I_A(.)$ is the index function, i.e.,

$$I_A(t) = \begin{cases} 1, & \text{if } t \in A, \\ 0, & \text{if } t \notin A \end{cases}$$

3 Assumptions

In this section, we deals with some hypotheses which are used in our results.

(H_1) The function f satisfies the Lipschitz condition. That is, for $x, y \in X$ and $\tau \le t \le T$ there exist constants $\mathbb{L}_0, \mathbb{M}_0 \ge 0$ such that

$$E\|f(t, x) - f(t, y)\|^2 \le \mathbb{L}_0 E\|x - y\|^2 \text{ and } E\|f(t, 0)\|^2 \le \mathbb{M}_0.$$

(H_2) The function f_1 satisfies the following condition. That is, for $x, y \in X$ and $\tau \leq t \leq T$ there exist constants $\mathbb{L}_1, \mathbb{M}_1 \geq 0$ such that

$$E\|\int_0^T \big[f_1(\eta, x(t+\eta)) - f_1(\eta, y(t+\eta))\big]d\eta\|^2 \leq \mathbb{L}_1 E\|x(t+\eta) - y(t+\eta)\|^2,$$

$$E\|f_1(\eta, 0)\| \leq \mathbb{M}_1.$$

(H_3) The condition $\max_{i,k}\left\{\prod_{j=1}^k \|b_j(\tau_j)\|\right\}$ is uniformly bounded if, there is a constant $\vartheta > 0$ such that

$$\max_{i,k}\left\{\prod_{j=1}^k \|b_j(\tau_j)\|\right\} \leq \vartheta, \quad \text{for all } \tau_j \in D_j, j = 1, 2, 3, \ldots.$$

(H_4) $g : X \to X$ satisfies the Lipschitz condition. That is, for $x, y \in X$ and $\tau \leq t \leq T$, there exists a constant $\mathbb{L}_* \geq 0$ such that

$$E\|g(x) - g(y)\|^2 \leq \mathbb{L}_*\|x - y\|^2.$$

(H_5)

$$\Gamma = \mathcal{K}^2\vartheta^2 \max\{1, \vartheta^2\}(T-\tau)^2\left[\mathbb{L}_0 + \mathbb{L}_1 + \frac{\mathbb{L}_*}{(T-\tau)^2}\right] < 1.$$

4 Existence and Uniqueness

We discuss the existence and uniqueness of the mild solution for the system (4).

Theorem 1 *Assume that the hypothesis (H_1)–(H_5) hold. Then the system (4) has a unique mild solution in $\mathbb{B}$.*

Proof. Let T be an arbitrary number $T \leq +\infty$. First we define the nonlinear operator $F : \mathbb{B} \to \mathbb{B}$ as follows

$$\begin{aligned} Fx(t) &= \sum_{k=0}^{+\infty}\bigg[\prod_{i=1}^k b_i(\tau_i)S(t-t_0)(x_0 - g(x)) + \sum_{i=1}^k\prod_{j=i}^k b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i} S(t-s)f(s, x(s))ds \\ &+ \int_{\xi_k}^t S(t-s)f(s, x(s))ds + \sum_{i=1}^k\prod_{j=i}^k b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i} S(t-s)\bigg\{\int_0^T f_1(\eta, x(s+\eta))d\eta\bigg\}ds \\ &+ \int_{\xi_k}^t S(t-s)\bigg\{\int_0^T f_1(\eta, x(s+\eta))d\eta\bigg\}ds\bigg]I_{[\xi_k, \xi_{k+1})}(t), \qquad t \in [\tau, T]. \end{aligned}$$

We can prove the continuity of F easily. Next we will show that $\mathbb{B}$ is mapped into $\mathbb{B}$ under F.

$$\begin{aligned} \|Fx(t)\|^2 &\leq \bigg[\sum_{k=0}^{+\infty}\bigg[\|\prod_{i=1}^k b_i(\tau_i)\|\|S(t-t_0)\|\|x_0 - g(x)\| \\ &+ \sum_{i=1}^k\|\prod_{j=i}^k b_j(\tau_j)\|\int_{\xi_{i-1}}^{\xi_i}\|S(t-s)f(s, x(s))\|ds \\ &+ \int_{\xi_k}^t\|S(t-s)f(s, x(s))\|ds + \sum_{i=1}^k\|\prod_{j=i}^k b_j(\tau_j)\|\int_{\xi_{i-1}}^{\xi_i} S(t-s)\bigg\{\int_0^T f_1(\eta, x(s+\eta))d\eta\bigg\}\|ds \\ &+ \int_{\xi_k}^t\|S(t-s)\bigg\{\int_0^T f_1(\eta, x(s+\eta))d\eta\bigg\}\|ds\bigg]I_{[\xi_k, \xi_{k+1})}(t)\bigg]^2 \end{aligned}$$

$$
\begin{aligned}
&\leq 2\Big[\sum_{k=0}^{+\infty}\Big[\|\prod_{i=1}^{k}b_i(\tau_i)\|^2\|S(t-t_0)\|^2\|x_0-g(x)\|^2 I_{[\xi_k,\xi_{k+1})}(t)\Big]\\
&+\Big[\sum_{k=0}^{+\infty}\Big[\sum_{i=1}^{k}\|\prod_{j=i}^{k}b_j(\tau_j)\|\int_{\xi_{i-1}}^{\xi_i}\|S(t-s)f(s,x(s))\|ds\\
&+\int_{\xi_k}^{t}\|S(t-s)f(s,x(s))\|ds\Big]I_{[\xi_k,\xi_{k+1})}(t)\Big]^2\Big]\\
&+\Big[\sum_{k=0}^{+\infty}\Big[\sum_{i=1}^{k}\|\prod_{j=i}^{k}b_j(\tau_j)\|\int_{\xi_{i-1}}^{\xi_i}\|S(t-s)\Big\{\int_0^T f_1(\eta,x(s+\eta))d\eta\Big\}\|ds\\
&+\int_{\xi_k}^{t}\|S(t-s)\Big\{\int_0^T f_1(\eta,x(s+\eta))d\eta\Big\}\|ds\Big]I_{[\xi_k,\xi_{k+1})}(t)\Big]^2\\
&\leq 2\mathcal{K}^2\max_k\Big\{\|\prod_{i=1}^{k}b_i(\tau_i)\|^2\Big\}\|x_0-g(x)\|^2\\
&+2\mathcal{K}^2\Big[\max_{i,k}\Big\{1,\|\prod_{j=i}^{k}b_i(\tau_j)\|^2\Big\}\Big]^2\Big\{\int_{t_0}^{t}\|f(s,x(s))\|ds I_{[\xi_k,\xi_{k+1})}(t)\Big\}^2\\
&+2\mathcal{K}^2\Big[\max_{i,k}\Big\{1,\|\prod_{j=i}^{k}b_i(\tau_j)\|\Big\}\Big]^2\Big\{\int_{t_0}^{t}\|\int_0^T f_1(\eta,x(s+\eta))d\eta\|ds I_{[\xi_k,\xi_{k+1})}(t)\Big\}^2\\
&\leq 2\mathcal{K}^2\vartheta^2\|x_0-g(x)\|^2+2\mathcal{K}^2\max\{1,\vartheta^2\}\Big\{\int_{t_0}^{t}\|f(s,x(s))\|ds\Big\}^2\\
&+2\mathcal{K}^2\max\{1,\vartheta^2\}\Big\{\int_{t_0}^{t}\int_0^T\|f_1(\eta,x(s+\eta))\|d\eta ds\Big\}^2\\
&\leq 2\mathcal{K}^2\vartheta^2\|x_0-g(x)\|^2+2\mathcal{K}^2\max\{1,\vartheta^2\}(t-t_0)\int_{t_0}^{t}\|f(s,x(s))\|^2ds\\
&+2\mathcal{K}^2\max\{1,\vartheta^2\}(t-t_0)\int_{t_0}^{t}\Big\{\|\int_0^T f_1(\eta,x(s+\eta))d\eta\|ds\Big\}^2.
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
E\|Fx(t)\|^2 &\leq 2\mathcal{K}^2\vartheta^2\|x_0-g(x)\|^2+2\mathcal{K}^2\max\{1,\vartheta^2\}(T-\tau)\int_{t_0}^{t}E\|f(s,x(s))\|^2ds\\
&+2\mathcal{K}^2\max\{1,\vartheta^2\}(T-\tau)\int_{t_0}^{t}E\|\int_0^T f_1(\eta,x(s+\eta))\|^2d\eta ds\\
&\leq 2\mathcal{K}^2\vartheta^2\|x_0-g(x)\|^2+4\mathcal{K}^2\max\{1,\vartheta^2\}(T-\tau)\mathbb{L}_0\int_{t_0}^{t}E\|x(s)\|^2ds\\
&+4\mathcal{K}^2\max\{1,\vartheta^2\}(T-\tau)^2\mathbb{M}_0+4\mathcal{K}^2\max\{1,\vartheta^2\}(T-\tau)^2\mathbb{M}_1\\
&+4\mathcal{K}^2\max\{1,\vartheta^2\}(T-\tau)\mathbb{L}_1\int_{t_0}^{t}E\|x(s+\eta)\|^2ds.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sup_{t\in[\tau,T]}E\|Fx(t)\|^2 &\leq 2\mathcal{K}^2\vartheta^2\|x_0-g(x)\|^2+4\mathcal{K}^2\max\{1,\vartheta^2\}(T-\tau)\mathbb{L}_0\int_{t_0}^{t}\sup_{t\in[\tau,T]}E\|x(s)\|^2ds\\
&+4\mathcal{K}^2\max\{1,\vartheta^2\}(T-\tau)^2\mathbb{M}_0+4\mathcal{K}^2\max\{1,\vartheta^2\}(T-\tau)^2\mathbb{M}_1
\end{aligned}
$$

$$+4\mathcal{K}^2\max\{1,\vartheta^2\}(T-\tau)\mathbb{L}_1\int_{t_0}^{t}\sup_{t\in[\tau,T]}E\|x(s+\eta)\|^2ds$$

for every t, $\tau\le t\le T$, Hence F maps $\mathbb{B}$ into $\mathbb{B}$.

Next we will show that F is a contraction mapping:

$$\begin{aligned}
&\|Fx(t)-Fy(t)\|^2\\
\le\ &\Big[\sum_{k=0}^{+\infty}\prod_{i=1}^{k}\|b_i(\tau_i)\|\|S(t-t_0)\|\|(g(x)-g(y))\|I_{[\xi_k,\xi_{k+1})}(t)\Big]^2\\
&+\Big[\sum_{k=0}^{+\infty}\Big\{\sum_{i=1}^{k}\prod_{j=i}^{k}\|b_i(\tau_j)\|\int_{\xi_{i-1}}^{\xi_i}\|S(t-s)\|\|f(s,x(s))-f(s,y(s))\|ds\\
&+\int_{\xi_k}^{t}\|S(t-s)\|\|f(s,x(s))-f(s,y(s))\|ds\Big\}I_{\xi_k,\xi_{k+1}}(t)\Big]^2\\
&+\Big[\sum_{k=0}^{+\infty}\Big[\sum_{i=1}^{k}\prod_{j=i}^{k}\|b_j(\tau_j)\|\int_{\xi_{i-1}}^{\xi_i}\|S(t-s)\|\int_0^T\|f_1(\eta,x(s+\eta))-f_1(\eta,y(s+\eta))d\eta\|ds\\
&+\int_{\xi_k}^{t}\|S(t-s)\|\|\int_0^T f_1(\eta,x(s+\eta))-f_1(\eta,y(s+\eta))d\eta\|ds\Big]I_{[\xi_k,\xi_{k+1})}(t)\Big]^2\\
\le\ &\mathcal{K}^2\Big[\max_k\Big\{\prod_{i=1}^{k}\|b_i(\tau_i)\|^2\Big\}\|g(x)-g(y)\|^2\Big]\\
&+\mathcal{K}^2\Big[\max_{i,k}\Big\{1,\prod_{j=i}^{k}\|b_j(\tau_j)\|\Big\}\Big]^2\Big\{\int_{t_0}^{t}\|f(s,x(s))-f(s,y(s))\|dsI_{[\xi_k,\xi_{k+1})}(t)\Big\}^2\\
&+\mathcal{K}^2\Big[\max_{i,k}\Big\{1,\prod_{j=i}^{k}\|b_j(\tau_j)\|\Big\}\Big]^2\Big\{\int_{t_0}^{t}\|\int_0^T f_1(\eta,x(s+\eta))-f_1(\eta,y(s+\eta))d\eta\|dsI_{[\xi_k,\xi_{k+1})}(t)\Big\}^2\\
\le\ &\mathcal{K}^2\vartheta^2\|g(x)-g(y)\|^2+\mathcal{K}^2\max\{1,\vartheta^2\}(t-t_0)\int_{t_0}^{t}\|f(s,x(s))-f(s,y(s))\|^2ds\\
&+\mathcal{K}^2\max\{1,\vartheta^2\}(t-t_0)\int_{t_0}^{t}\|\int_0^T f_1(\eta,x(s+\eta))-f_1(\eta,y(s+\eta))d\eta\|^2ds.
\end{aligned}$$

$$\begin{aligned}
E\|Fx(t)-Fy(t)\|^2\ \le\ &\mathcal{K}^2\vartheta^2E\|g(x)-g(y)\|^2\\
&+\mathcal{K}^2\max\{1,\vartheta^2\}(T-t_0)\int_{t_0}^{t}E\|f(s,x(s))-f(s,y(s))\|^2ds\\
&+\mathcal{K}^2\max\{1,\vartheta^2\}(T-t_0)\int_{t_0}^{t}E\|\int_0^T f_1(\eta,x(s+\eta))-f_1(\eta,y(s+\eta))d\eta\|^2ds\\
\le\ &\mathcal{K}^2\vartheta^2E\|g(x)-g(y)\|^2\\
&+\mathcal{K}^2\max\{1,\vartheta^2\}(T-\tau)\mathbb{L}_0\int_{t_0}^{t}E\|x(s)-y(s)\|^2ds\\
&+\mathcal{K}^2\max\{1,\vartheta^2\}(T-\tau)\mathbb{L}_1\int_{t_0}^{t}E\|x(s+\eta)-y(s+\eta)\|^2ds.
\end{aligned}$$

Taking supremum over t, it follows that

$$\|Fx-Fy\|^2\ \le\ \mathcal{K}^2\vartheta^2\mathbb{L}_*\|x-y\|^2+\mathcal{K}^2\max\{1,\vartheta^2\}(T-\tau)^2\mathbb{L}_0\|x-y\|^2$$

$$
\begin{aligned}
&+\mathcal{K}^2 \max\{1,\vartheta^2\}(T-\tau)^2\mathbb{L}_1\|x-y\|^2\\
\leq\ & \left[\mathcal{K}^2\vartheta^2 \max\{1,\vartheta^2\}(T-\tau)^2 L\right]\|x-y\|^2\\
\leq\ & \Gamma\|x-y\|^2,
\end{aligned}
$$

where $\Gamma=\mathcal{K}^2\vartheta^2\max\{1,\vartheta^2\}(T-\tau)^2L$ and $L=\mathbb{L}_0+\mathbb{L}_1+\dfrac{\mathbb{L}_*}{(T-\tau)^2}$. From H_5 and $0<\Gamma<1$, we get that F is a contraction mapping. Thus using Banach fixed point theorem we get F has a unique fixed point on $\mathbb{B}$. Hence (4) has a unique mild solution. ■

Remark 1 *Let $f:\Re_\tau\times X\to X$, $f_1:\Re_\tau\times X\to X$ and $g:X\to X$ satisfy the assumptions (H_1)–(H_5). Then there exists a unique, global, continuous solution x to (4) for any initial value (t_0,x_0) with $t_0\geq 0$ and $x_0\in\mathbb{B}$.*

Remark 2 *The above theorem is an extension of [19, Theorem 3.1]. Theorem 1 gives existence and uniqueness of random impulsive integro-differential equations with nonlocal condition. This solution is practically more useful than the solution of random impulsive differential equations.*

Remark 3 *Assume that all hypotheses hold. Then the mild solution for the system (7) without existence of nonlocal condition and the solution is*

$$
\begin{aligned}
x(t)\ =\ & \sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}b_i(\tau_i)S(t-t_0)x_0+\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}S(t-s)f(s,x(s))ds\\
&+\int_{\xi_k}^{t}S(t-s)f(s,x(s))ds+\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}S(t-s)\int_0^T f_1(\eta,x(s+\eta))d\eta ds\\
&+\int_{\xi_k}^{t}S(t-s)\int_0^T f_1(\eta,x(s+\eta))d\eta ds\Bigg]I_{[\xi_k,\xi_{k+1})}(t),\qquad t\in[\tau,T].
\end{aligned}
$$

5 Stability

Theorem 2 *Let $x(t)$ and $\widehat{x}(t)$ be solutions of the system (4) with initial value $x_0-g(x)$ and $\widehat{x}_0-g(\widehat{x})$ respectively. If the assumptions (H_1)–(H_4) of Theorem 1 are satisfied, then the system (4) is stable in the mean square.*

Proof. From assumptions, $x(t)$ and $\widehat{x}(t)$ are two solutions of the system (4) for every $t\in[\tau,T]$. Then

$$
\begin{aligned}
x(t)-\widehat{x}(t)\ =\ & \sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}b_i(\tau_i)S(t-t_0)(x_0-\widehat{x}_0)+\sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}b_i(\tau_i)S(t-t_0)(g(x)-g(\widehat{x}))\\
&+\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}S(t-s)\big\{f(s,x(s))-f(s,\widehat{x}(s))\big\}ds\\
&+\int_{\xi_k}^{t}S(t-s)\big[f(s,x(s))-f(s,\widehat{x}(s))\big]ds\\
&+\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}S(t-s)\Bigg\{\int_0^T\big[f_1(\eta,x(s+\eta))-f_1(\eta,\widehat{x}(s+\eta))\big]d\eta\Bigg\}ds\\
&+\int_{\xi_k}^{t}S(t-s)\Bigg\{\int_0^T\big[f_1(\eta,x(s+\eta))-f_1(\eta,\widehat{x}(s+\eta))\big]d\eta\Bigg\}ds\Bigg]\Bigg]I_{[\xi_k,\xi_{k+1})}(t).
\end{aligned}
$$

By using (H_1)–(H_4), we get

$$
\begin{aligned}
&\|x(t)-\widehat{x}(t)\|^2\\
\leq\ & 2\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}\|b_i(\tau_i)\|^2\|S(t-t_0)\|^2E\|x_0-\widehat{x}_0\|^2 I_{[\xi_k,\xi_{k+1})}(t)\Big]\\
&+2\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}\|b_i(\tau_i)\|^2\|S(t-t_0)\|^2E\|g(x)-g(\widehat{x})\|^2 I_{[\xi_k,\xi_{k+1})}(t)\Big]\\
&+2E\Big[\sum_{k=0}^{+\infty}\Big[\sum_{i=1}^{k}\prod_{j=i}^{k}\|b_j(\tau_j)\|\int_{\xi_{i-1}}^{\xi_i}\|S(t-s)\|\|f(s,x(s))-f(s,\widehat{x}(s))\|ds\\
&+\int_{\xi_k}^{t}\|S(t-s)\|\|f(s,x(s))-f(s,\widehat{x}(s))\|ds\Big]I_{[\xi_k,\xi_{k+1})}(t)\Big]^2\\
&+2E\Big[\sum_{k=0}^{+\infty}\Big[\sum_{i=1}^{k}\prod_{j=i}^{k}\|b_j(\tau_j)\|\int_{\xi_{i-1}}^{\xi_i}\|S(t-s)\|\|\int_0^T\big[f_1(\eta,x(s+\eta))-f_1(\eta,\widehat{x}(s+\eta))\big]d\eta\|ds\\
&+\int_{\xi_k}^{t}\|S(t-s)\|\|\int_0^T\big[f_1(\eta,x(s+\eta))-f_1(\eta,\widehat{x}(s+\eta))\big]d\eta\|ds\Big]I_{[\xi_k,\xi_{k+1})}(t)\Big]^2\\
\leq\ & 2\mathcal{K}^2\max_k\Big\{\|\prod_{i=1}^{k}b_i(\tau_i)\|^2\Big\}E\|(x_0-\widehat{x}_0)\|^2+2\mathcal{K}^2\max_k\Big\{\|\prod_{i=1}^{k}b_i(\tau_i)\|^2\Big\}E\|g(x)-g(\widehat{x})\|^2\\
&+2\mathcal{K}^2\Big[\max_{i,k}\Big\{1,\prod_{j=i}^{k}\|b_j(\tau_j)\|\Big\}\Big]^2E\Big\{\int_{t_0}^{t}\|f(s,x(s))-f(s,\widehat{x}(s))\|dsI_{[\xi_k,\xi_{k+1})}(t)\Big\}^2\\
&+2\mathcal{K}^2\Big[\max_{i,k}\Big\{1,\prod_{j=i}^{k}\|b_j(\tau_j)\|\Big\}\Big]^2E\Big\{\int_{t_0}^{t}\|\int_0^T\big[f_1(\eta,x(s+\eta))-f_1(\eta,\widehat{x}(s+\eta))\big]d\eta\|dsI_{[\xi_k,\xi_{k+1})}(t)\Big\}^2.
\end{aligned}
$$

Taking supremum over t, it follows that

$$
\begin{aligned}
\sup_{t\in[\tau,T]}\|x(t)-\widehat{x}(t)\|^2 \leq\ & 2\mathcal{K}^2\vartheta^2E\|x_0-\widehat{x}_0\|^2+2\mathcal{K}^2\vartheta^2E\|g(x)-g(\widehat{x})\|^2\\
&+2\mathcal{K}^2\max\{1,\vartheta^2\}(T-\tau)\mathbb{L}_0\int_{t_0}^{t}\sup_{s\in[\tau,T]}E\|x(t)-\widehat{x}(t)\|^2d\eta\\
&+2\mathcal{K}^2\max\{1,\vartheta^2\}(T-\tau)\mathbb{L}_1\int_{t_0}^{t}\sup_{t\in[\tau,T]}E\|x(t+\eta)-\widehat{x}(t+\eta)\|^2d\eta.
\end{aligned}
$$

Using Grownwall inequality, we get

$$
\begin{aligned}
\sup_{t\in[\tau,T]}\|x(t)-\widehat{x}(t)\|^2 &\leq 2\mathcal{K}^2\vartheta^2E\|x_0-\widehat{x}_0\|^2\exp\big[2\mathcal{K}^2\max(1,\vartheta^2)(T-\tau)^2\big]L\\
&\leq \Gamma E\|x_0-\widehat{x}_0\|^2.
\end{aligned}
$$

where

$$\Gamma=2\mathcal{K}^2\vartheta^2\Big[\exp\big[2\mathcal{K}^2\max(1,\vartheta^2)(T-\tau)^2\big]L\Big]$$

and

$$L=\mathbb{L}_0+\mathbb{L}_1+\frac{\mathbb{L}_*}{(T-\tau)^2}.$$

Now given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{\Gamma}$ such that $E\|x_0 - \widehat{x}_0\|^2 < \delta$. Then

$$\sup_{t\in[\tau,T]} E\|x(t) - \widehat{x}(t)\| \leq \varepsilon.$$

■

Remark 4 *Random impulsive integro-differential equation with local initial condition is a special case of the system (7). So the random impulsive integro-differential equation with local initial condition is stable in mean square.*

6 Example 1

Consider partial random impulsive differential equations

$$\begin{cases} z_t(t,x) = z_{xx}(t,x) + F_1(t, z(t,x)), & t \neq \xi_k,\ t \geq \tau, \\ z(x,\xi_k) = q(k)\tau_k z(x,\xi_k^-), & \text{as } x \in \widehat{\triangle}, \\ z(t,0) = z(t,\pi) = 0, \\ z(t_0,x) + \sum_{j=1}^{q} c_j z(p_j,x) = z_0(x), & 0 < p_1 < p_2 < \cdots < p_q < T, \quad x \in \partial\widehat{\triangle}. \end{cases} \tag{6}$$

Let $\widehat{\triangle} \subset \Re^n$ be a bounded domain with smooth boundary $\partial\widehat{\triangle}$, $X = L^2(\widehat{\triangle})$, τ_k be random variable defined on $D_k \equiv (0, d_k)$ for $k \in \mathbb{N}$, $d_k \in (0, +\infty)$. Also assume that τ_k follow Erlang distribution and if $i \neq j$ then τ_i and τ_j are independent with each other for $i, j = 1, 2, \ldots$. Here q is a function of k, $\xi_k = \xi_{k-1} + \tau_k$ for $k \in \mathbb{N}$, $t_0 \in \Re^+$.

Let A be an operator on X by $Az = \dfrac{\partial^2 z}{\partial x^2}$ with the domain

$$D(A) = \left\{ z \in X \mid z \text{ and } \frac{\partial z}{\partial x} \text{ are absolutely continuous, } \frac{\partial^2 z}{\partial x^2} \in X, z = 0, z = \pi \text{ on } \partial\widehat{\triangle} \right\}.$$

Thus A generates a strongly continuous semigroup $S(t)$ which is analytic, self adjoint and compact. Furthermore the operator A can be represented as

$$Az = \sum_{n=1}^{\infty} n^2 < z, z_n > z_n, \quad z \in D(A).$$

Here $z_n(\zeta) = \sqrt{\frac{2}{\pi}} Sin(n\zeta)$, $n = 1, 2, \ldots$, forms the orthonormal set of eigenvectors of A. Also for every $z \in X$, $S(t)z = \sum_{n=1}^{\infty} e^{(-n^2 t)} < z, z_n > z_n$, which holds $\|S(t)\| \leq e^{(-\pi^2(t-t_0))}$, $t \geq t_0$. Therefore $S(t)$ is a contraction semigroup.

Consider the following assumptions:

(i) $f : \Re_\tau \times X \to X$, is a continuous function defined by

$$f(t,z)(x) = F_1(t, z(x)), \quad \tau \leq t \leq T,\ 0 \leq x \leq \pi$$

and also function f satisfies the Lipschitz condition.

(ii) $g : X \to X$ is a continuous function defined by

$$g(u)(t) = x_0(t) - \sum_{j=1}^{q} c_j z(p_j, x) \qquad 0 < p_1 < p_2 < \cdots < p_q < b\ x \in [0,\pi]$$

where $x(s)(t) = z(s,t)$, $0 \leq t \leq \pi$.

(iii) $E\Big[\max_{i,k}\big\{\prod_{j=i}^{k}\|q(j)(\tau_j)\|\big\}\Big] < \infty.$

Under the conditions, we can define the function b_k by

$$b_k = q(k)\tau_k.$$

Assume that assumptions (i) and (ii) are satisfied, then the problem (6) becomes an abstract random impulsive differential equation.

Proposition 1 *Assume that (H_1)–(H_4) hold. Then there exists a unique mild solution of the system (6) respectively, provided*

$$\mathcal{K}^2\vartheta^2\max\{1,\vartheta^2\}(T-\tau)^2\big[\mathbb{L}_0 + \frac{\mathbb{L}_*}{(T-\tau)^2}\big] < 1$$

is satisfied.

Proposition 2 *Assume that the conditions of Proposition 1 hold. Then the mild solution z of the system (6) is stable in the mean square.*

Example 2

Consider partial integro-random impulsive differential equations

$$\begin{cases} z_t(t,x) = z_{xx}(t,x) + F_1(t,z(t,x)) + \int_0^T F_2(\eta, z(tsin\eta, x))d\eta, \quad t \neq \xi_k,\ t \geq \tau, \\ z(x,\xi_k) = q(k)\tau_k z(x,\xi_k^-), \quad \text{as } x \in \widehat{\triangle}, \\ z(t,0) = z(t,\pi) = 0, \\ z(t_0,x) + \sum_{j=1}^{q} c_j \sqrt[3]{z(p_j,x)} = z_0(x), \quad 0 < p_1 < p_2 < \cdots < p_q < T,\ x \in \partial\widehat{\triangle}. \end{cases} \tag{7}$$

Let $\widehat{\triangle} \subset \Re^n$ be a bounded domain with smooth boundary $\partial\widehat{\triangle}$, $X = L^2(\widehat{\triangle})$, τ_k be random variable defined on $D_k \equiv (0,d_k)$ for $k \in \mathbb{N}$, $d_k \in (0,+\infty)$. Also assume that τ_k follow Erlang distribution and if $i \neq j$ then τ_i and τ_j are independent with each other for $i,j = 1,2,\ldots$. Here q is a function of k, $\xi_k = \xi_{k-1} + \tau_k$ for $k \in \mathbb{N}$, $t_0 \in \Re^+$.

Let A be an operator on X by $Az = \dfrac{\partial^2 z}{\partial x^2}$ with the domain

$$D(A) = \Big\{z \in X \mid z \text{ and } \frac{\partial z}{\partial x} \text{ are absolutely continuous, } \frac{\partial^2 z}{\partial x^2} \in X, z = 0, z = \pi \text{ on } \partial\widehat{\triangle}\Big\}.$$

Thus A generates a strongly continuous semigroup $S(t)$ which is analytic, self adjoint and compact. Furthermore the operator A can be represented as

$$Az = \sum_{n=1}^{\infty} n^2 < z, z_n > z_n, \quad z \in D(A).$$

Here $z_n(\zeta) = \sqrt{\frac{2}{\pi}}Sin(n\zeta)$, $n = 1,2,\ldots$, forms the orthonormal set of eigenvectors of A. Also for every $z \in X$, $S(t)z = \sum_{n=1}^{\infty} e^{(-n^2 t)} < z, z_n > z_n$, which holds $\|S(t)\| \leq e^{(-\pi^2(t-t_0))}$, $t \geq t_0$. Therefore $S(t)$ is a contraction semigroup.

Consider the following assumptions:

(i) $f : \Re_\tau \times X \to X, f_1 : \Re_\tau \times X \to X$ is a continuous function defined by

$$f(t,z)(x) = F_1(t,z(x)), \quad \tau \leq t \leq T,\ 0 \leq x \leq \pi,$$

$$f_1(\eta, x(t+\eta))d\eta = \int_0^T F_2(\eta, z(tsin\eta, x))d\eta,$$

and also function f and f_1 satisfies the Lipschitz condition.

(ii) $g : X \to X$ is a continuous function defined by

$$g(u)(t) = x_0(t) - \sum_{j=1}^{q} c_j \sqrt[3]{x(p_j, t)}, \quad 0 \le t \le \pi,$$

where $x(s)(t) = z(s,t), \quad 0 \le t \le \pi$.

(iii) $E\Big[\max_{i,k} \big\{ \prod_{j=i}^{k} \|q(j)(\tau_j)\| \big\} \Big] < \infty$.

Under the conditions, we can define the function f_1, b_k by

$$b_k = q(k)\tau_k \quad \text{and} \quad f_1(\eta, x(t+\eta))d\eta = \int_0^T F_2(\eta, z(tsin\eta, x))d\eta.$$

Assume that assumptions (i) and (ii) are satisfied. Then the problem (7) becomes an abstract random impulsive integro-differential equation (4).

Proposition 3 *Assume that* (H_1)–(H_4) *hold. Then there exists a unique mild solution of the system (7) respectively, provided*

$$\mathcal{K}^2 \vartheta^2 \max\{1, \vartheta^2\}(T-\tau)^2 \big[\mathbb{L}_0 + \mathbb{L}_1 + \frac{\mathbb{L}_*}{(T-\tau)^2}\big] < 1$$

is satisfied.

Proposition 4 *Assume that the conditions of Proposition 3 hold. Then the mild solution* z *of the system (7) is stable in the mean square.*

7 Conclusion

We have investigated the existence, uniqueness and stability of an integro-differential system with nonlocal conditions. Here we used contraction mapping principle for proving the existence and uniqueness. Finally some examples are given to show the importance of random implusive differential equations as well as inegro-differential equations with nonlocal conditions. In future we can extend this work to fractional differential equation and the results derived in this paper can be used to analyse the variation in the behaviour of the solution with respect to the variation in the complexity of the system of differential equation considered.

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References

[1] S. Abinaya and S. A. Jose, Neutral impulsive stochastic differential equations driven by fractional Brownian motion with poisson jumps and nonlocal conditions, Glob. J. Pure Appl. Math., 15(2019), 305–321.

[2] A. Anguraj and A. Vinodkumar, Existence, uniqueness and stability results of random impulsive semilinear differential systems, Nonlinear Anal. Hybrid Syst., 4(2010), 475–483.

[3] A. Anguraj, S. Wu and A. Vinodkumar, The existence and exponential stability of semilinear functional differential equations with random impulses under non-uniqueness, Nonlinear Anal., 74(2011), 331–342.

[4] D. D. Bainov and P. S. Simeonov, Impulsive Differential Equations: Periodic Solutions and Applications, Longman Scientific and Technical, New York, 1993.

[5] L. Byszewski, Theorems about the existence and uniqueness of a solution of a semilinear evolution nonlocal Cauchy problem, J. Math. Anal. Appl., 162(1991), 496–505.

[6] L. Byszewski, Existence of solutions of semilinear functional-differential evolution nonlocal problem, Nonlinear Anal., 34(1998), 65–72.

[7] D. J. Guo, Multiple positive solutions for first order nonlinear integro-differential equations in Banach spaces, Nonlinear Anal., 53(2000), 183–195.

[8] M. Gowrisankar, P. Mohankumar and A. Vinodkumar, Stability results of random impulsive semilinear differential equations, Acta Math. Sci. Ser. B, 34(2014), 1055–1071.

[9] D. J. Guo and X. Z. Liu, Extremal solutions of nonlinear impulsive integro-differential equations in Banach spaces, J. Math. Anal. Appl., 177(1993), 538–553.

[10] James H. Liu, Iterative method for solutions and coupled quasi-solutions of nonlinear integro-differential equations of mixed type in Banach space, Nonlinear Anal., 42(2000), 583–598.

[11] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, 1989.

[12] X. Z. Liu and D. J. Guo, Initial value problems for first order impulsive integro-differentioal equations in Banach spaces, Comm. Appl. Nonlinear Anal., 2(1995), 65–83.

[13] J. H. Liu, Nonlinear impulsive evolution equations, Dynam. Contin. Discrete Impuls. Systems, 6(1999), 77–85

[14] L. S. Liu, The solutions of nonlinear integro-differential equations of mixed type in Banch spaces, Acta Math. Sci., 38(1995), 721–731.

[15] A. Pazy, Semigroups of Linear Operators and Applications to to Partial Differential Equations, Springer-Verlag, New York, 1983.

[16] Y. Rogovchenko, Impulsive evolution systems: main results and new trends, Dyn. Contin. Discrete Impuls. Syst., 3(1997), 57–88.

[17] S. H. Saker, Oscillation and global attractivity of impulsive periodic delay respiratory dynamics model, Chin. Ann. Math., 26(2005), 511–522.

[18] Sayooj Aby Jose, Venkitesh Usha, Existence and uniqueness of solutions for special random impulsive differential equation, J. Appl. Sci. Comput., 5(2018), 14–23.

[19] Sayooj Aby Jose, Venkitesh Usha, Existence of solutions for random impulsive differential equation with nonlocal conditions, International Journal of Computer Science and Engineering, 6(2018), 549–554.

[20] S. J. Wu and D. Han, Exponential stability of functional differential systems with impulsive effect on random moments, Comput. Math. Appl., 50(2005), 321–328.

[21] S. J. Wu and X. Z. Meng, Boundedness of nonlinear differential systems with impulsive effect on random moments, Acta Math. Sci., 20(2004), 147–154.

[22] J. Zhu, X. Fan, S. Zhang, Fixed points of increasing operators and solutions of nonlinear impulsive integro-differential equations in Banach spaces, Nonlinear Anal., 42(2000), 599–611.