Applied Mathematics E-Notes, 20(2020), 422-431 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/~amen/

Some Applications Of The Boundary Schwarz Lemma For Polynomials With Restricted Zeros*

Nisar Ahmad Rather†, Ishfaq Ahmad Dar‡

Received 19 November 2019

Abstract

By using the boundary Schwarz lemma of Osserman, certain inequalities for the derivatives of the polynomials with restricted zeros are obtained. These estimates strengthen some well known inequalities for polynomial due to Turán, Dubinin and others.

1 Introduction

Polynomials permeate mathematics, and much that is attractive in mathematics is related to polynomials. Almost every branch of mathematics, from algebraic number theory and algebraic geometry to applied analysis, Fourier analysis, numerical analysis and computer sciences, has its corpus of theory arising from study of polynomials. Historically, the question relating to polynomials, for example, the solution of polynomial equations and the approximation by polynomials, gave rise to some of the most important problems of the day. The concept of best approximation was introduced in mathematical analysis mainly by the work of the famous mathematician Chebyshev(1821-1894), who studied some properties of polynomial with least deviation from given continuous function. He introduced the polynomial known today as Chebyshev polynomial of first kind, which appear prominently in various extremal problems with polynomial. Extremal problems of Markov and Bernstein (see[17]) for polynomial such as inequalities for the derivative of a polynomial are very important in polynomial approximation theory. The first result in this area was connected with some investigation of well-known Russain chemist Mendelveev [13]. In fact, Mendeleev's problem was to know how large is the modulus of the derivative of a polynomial on a given interval? A.A. Markov [12] provided solution to this problem for polynomial of degree n. An analogue of Markov's theorem for the unit disk in the complex plane instead of the interval [-1,1] was formulated by Bernstein (see [14]). Inequalities of Markov and Bernstein-type are fundamental for the proofs of many inverse theorems in polynomial approximation theory (see Ivanov [9], Lorentz [10], Telyakovskii [19]). For instance, Telyakovskii [19] writes: Among those that are fundamental in approximation theory are the extremal problems connected with inequalities for the derivatives of polynomials. The use of inequalities of this kind is a fundamental method in proofs of inverse problems of approximation theory (as can be seen in [6, p. 241]). As such further progress in inverse theorems has depended on first obtaining a corresponding generalization or analog of Markov's and Bernstein's inequalities and therefore, it is of interest to obtain refinements and generalizations of polynomial inequalities.

Let $\mathcal{P}_n$ denote the space of all algebraic polynomials of the form $P(z)=\sum\limits_{\nu=0}^{n} a_\nu z^\nu$ of degree n and let $P'(z)$ be the derivative of $P(z)$. Then concerning the maximum of $|P'(z)|$ in terms of maximum of $|P(z)|$ on $|z|=1$, Turán [20] showed that if $P\in\mathcal{P}_n$ and $P(z)$ has all zeros in $|z|\leq 1$, then

$$\max_{|z|=1}|P'(z)|\geq \frac{n}{2}\max_{|z|=1}|P(z)|. \tag{1}$$

*Mathematics Subject Classifications: 26D10, 41A17, 30C15.

†Department of Mathematics, University of Kashmir, Srinagar-190006, India
‡Department of Mathematics, University of Kashmir, Srinagar-190006, India

Equality in (1) holds for those polynomials $P \in \mathcal{P}_n$ which have all their zeros on $|z| = 1$. As an extension of (1), Govil [8] proved that if $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \leq k, k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k^n} \max_{|z|=1} |P(z)|. \tag{2}$$

In literature, there exist several generalizations and extensions of (1) and (2) (see [1], [2], [4], [5], [18]). Dubinin [7] used the boundary Schwarz lemma due to Osserman [15] to obtain an interesting refinement of (1), in fact, proved that if all the zeros of $P \in \mathcal{P}_n$ lie in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{1}{2}\left(n + \frac{|a_n| - |a_0|}{|a_n| + |a_0|}\right) \max_{|z|=1} |P(z)|. \tag{3}$$

The polar derivative $D_\alpha P(z)$ of $P \in \mathcal{P}_n$ with respect to the point $\alpha \in \mathbb{C}$ is defined by

$$D_\alpha P(z) := nP(z) + (\alpha - z)P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative $P'(z)$ of $P(z)$ in the sense that

$$\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z) \tag{4}$$

uniformly for $|z| \leq R, R > 0$.

A. Aziz [1], Aziz and Rather ([4], [5]) obtained several sharp estimates for maximum modulus of $D_\alpha P(z)$ on $|z| = 1$ and among other things they extended inequality (2) to the polar derivative of a polynomial by showing that if $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \leq k, k \geq 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$

$$\max_{|z|=1} |D_\alpha P(z)| \geq \frac{n(|\alpha| - k)}{1+k^n} \max_{|z|=1} |P(z)|. \tag{5}$$

2 Lemmas

For the proof of theorems we require the following lemmas. The first Lemma is a simple deduction from Maximum Modulus Principle (see [14] or [16]).

Lemma 1 *If $P \in \mathcal{P}_n$, then for $R \geq 1$,*

$$\max_{|z|=R} |P(z)| \leq R^n \max_{|z|=1} |P(z)|.$$

The next Lemma is due to Aziz [1].

Lemma 2 *If $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq k$ where $k \geq 1$, then*

$$\max_{|z|=k} |P(z)| \geq \frac{2k^n}{1+k^n} |P(z)| \text{ for } |z| = 1.$$

From Lemma 2, we deduce:

Lemma 3 *If $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq k$ where $k \geq 1$, then*

$$\max_{|z|=k} |P(z)| \geq \frac{2k^n}{1+k^n} \max_{|z|=1} |P(z)| + \frac{k^n - 1}{k^n + 1} \min_{|z|=k} |P(z)|.$$

Proof. Let $m = \min_{|z|=k} |P(z)|$. Then $m \leq |P(z)|\, for\, |z| = k$. If $m = 0$, then the result follows from *Lemma* 2 so we assume $m > 0$. By Rouche's theorem, it follows that all the zeros of the polynomial $f(z) = P(z) + \lambda m$ lie in $|z| < k$ where $k \geq 1$ for every λ with $|\lambda| < 1$. Applying Lemma 2 to the polynomial $f(z)$, we get

$$\max_{|z|=k} |P(z) + \lambda m| \geq \frac{2k^n}{k^n + 1} |P(z) + \lambda m| \qquad \text{for} \quad |z| = 1. \tag{6}$$

Choosing argument of λ in the right hand side of (6) such that

$$|P(z) + \lambda m| = |P(z)| + |\lambda| m,$$

we obtain,

$$\max_{|z|=k} |P(z)| + |\lambda| m \geq \frac{2k^n}{k^n + 1} \left\{ |P(z)| + |\lambda| m \right\} \qquad for \quad |z| = 1.$$

Hence,

$$\max_{|z|=k} |P(z)| \geq \frac{2k^n}{1 + k^n} \max_{|z|=1} |P(z)| + \left(\frac{k^n - 1}{k^n + 1} \right) lm.$$

where $\quad m = \min_{|z|=k} |P(z)|$ and $0 \leq l < 1$. This proves Lemma. ■

The next lemma is special case of a result due to Aziz and Rather [3, 4].

Lemma 4 *If $P \in \mathcal{P}_n$ and $P(z)$ has its all zeros in $|z| \leq 1$, then for $|z| = 1$,*

$$|Q'(z)| \leq |P'(z)|$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

Lemma 5 *If all the zeros of $P \in \mathcal{P}_n$ lie in a circular region $\mathcal{C}$ and w is any zero of $D_\alpha P(z)$, the polar derivative of $P(z)$, then at most one of the points w and α may lie outside $\mathcal{C}$.*

The above lemma is due to Laguerre (see [11]). Finally we need the following lemma due to R.Osserman [15], known as boundary Schwarz lemma.

Lemma 6 *If*

(a) $T(z)$ is analytic for $|z| < 1$,

(b) $|T(z)| < 1$ for $|z| < 1$,

(c) $T(0) = 0$,

(d) for some b with $|b| = 1, T(z)$ extends continuously to b, $|T(b)| = 1$ and $T'(b)$ exists.

Then

$$|T'(b)| \geq \frac{2}{1 + |T'(0)|}.$$

3 Main Results

In this paper we present certain refinements and generalizations of inequalities (1), (2), (3) and (5). We first present the following result.

Theorem 7 *If $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \leq k$, $k \geq 1$, then*

$$\max_{|z|=1} |P'(z)| \geq \frac{1}{1 + k^n} \left(n + \frac{k^n |a_n| - |a_0|}{k^n |a_n| + |a_0|} \right) \max_{|z|=1} |P(z)|. \tag{7}$$

The result is sharp and equality holds for $P(z) = z^n + k^n$.

Proof. By hypothesis $P \in \mathcal{P}_n$ and $P(z)$ has all zeros in $|z| \leq k, k \geq 1$. If $f(z) = P(kz)$, then $f \in \mathcal{P}_n$ and $f(z)$ has all zeros in $|z| \leq 1$ and hence all the zeros of $z^n \overline{f(1/\overline{z})}$ lie in $|z| \geq 1$.

Now consider the function

$$G(z) = \frac{f(z)}{z^{n-1}\overline{f(1/\overline{z})}},$$

which gives for $|z| = 1$,

$$\frac{zG'(z)}{G(z)} = 1 - n + \frac{zf'(z)}{f(z)} + \overline{\left(\frac{zf'(z)}{f(z)}\right)}$$

so that

$$Re\left(\frac{zG'(z)}{G(z)}\right) = 1 - n + 2Re\left(\frac{zf'(z)}{f(z)}\right). \tag{8}$$

Using the fact that

$$\frac{zG'(z)}{G(z)} = |G'(z)| \qquad for \quad |z| = 1,$$

we get from (8), for points z on $|z| = 1$ with $f(z) \neq 0$,

$$1 - n + 2Re\left(\frac{zf'(z)}{f(z)}\right) = |G'(z)|.$$

Applying lemma 6 to $G(z)$ we obtain for all points on $|z| = 1$ with $f(z) \neq 0$,

$$1 - n + 2Re\left(\frac{zf'(z)}{f(z)}\right) \geq \frac{2}{1 + |G'(0)|},$$

that is, for $|z| = 1$ with $f(z) \neq 0$,

$$Re\left(\frac{zf'(z)}{f(z)}\right) \geq \frac{1}{2}\left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right).$$

This implies

$$\left|\frac{zf'(z)}{f(z)}\right| \geq \frac{1}{2}\left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right)$$

and hence,

$$|f'(z)| \geq \frac{1}{2}\left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right)|f(z)|$$

Replacing $f(z)$ by $P(kz)$, we get for $|z| = 1$,

$$k|P'(kz)| \geq \frac{1}{2}\left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right)|P(kz)|,$$

or equivalently,

$$\max_{|z|=k}|P'(z)| \geq \frac{1}{2k}\left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right)\max_{|z|=k}|P(z)|. \tag{9}$$

Since $P'(z)$ is a polynomial of degree $n-1$, by Lemma 1 with $R = k \geq 1$, we have

$$\max_{|z|=k}|P'(z)| \leq k^{n-1}\max_{|z|=1}|P'(z)|.$$

Using this inequality and Lemma 2 in inequality (9), we get

$$\max_{|z|=1}|P'(z)| \geq \frac{1}{1 + k^n}\left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right)\max_{|z|=1}|P(z)|.$$

This completes the proof of Theorem. ■

Remark 1 Since all the zeros of $P(z)$ lie in $|z| \le k$, therefore, $|a_0| \le k^n|a_n|$ and hence (7) refines (2).

Example 1 *Consider the polynomial $P(z) = z^3 + 6z^2 + 11z + 6$, clearly it satisfies all the conditions of Theorem 7 with $k = 3, \max_{|z|=1}|P(z)| = 24$ and $\max_{|z|=1}|P'(z)| = 26$. On substituting these one can easily see that the conclusion of Theorem 7 holds.*

Theorem 8 *If all the zeros of $P \in \mathcal{P}_n$ lie in $|z| \le k, k \ge 1$, then for $0 \le l < 1$,*

$$\max_{|z|=1}|P'(z)| \ge \frac{n}{(1+k^n)}\left(\max_{|z|=1}|P(z)|+lm\right) + \frac{1}{k^n(1+k^n)}\left(\frac{k^n|a_n| - lm - |a_0|}{k^n|a_n| - lm + |a_0|}\right)\left\{k^n \max_{|z|=1}|P(z)| - lm\right\}, \quad (10)$$

where $m = \min_{|z|=k}|P(z)|$. The result is sharp and equality holds for $P(z) = z^n + k^n$.

Proof. By hypothesis $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \le k$, $k \ge 1$. If $P(z)$ has a zero on $|z| = k$, then $m = \min_{|z|=k}|P(z)| = 0$ and result follows from theorem 7. Henceforth, we assume that $P(z)$ has all its zeros in $|z| < k$, $k \ge 1$ so that $m > 0$. Now if $f(z) = P(kz)$, then $f \in \mathcal{P}_n$ has all its zeros in $|z| < 1$ and and $m = \min_{|z|=k}|P(z)| = \min_{|z|=1}|f(z)|$. This implies, $m \le |f(z)|$ for $|z| = 1$, hence for every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$, we have

$$|m\lambda z^n| < |f(z)| \qquad \text{for} \quad |z| = 1.$$

By Rouche's theorem it follows that $g(z) = f(z) + \lambda m z^n$ has all its zeros in $|z| < 1$. Now proceeding similarly as in the proof of Theorem 7 (with $f(z)$ replacing by $g(z)$), we obtain

$$|g'(z)| \ge \frac{1}{2}\left(n + \frac{|k^n a_n + \lambda m| - |a_0|}{|k^n a_n + \lambda m| + |a_0|}\right)|g(z)| \qquad \text{for } |z| = 1.$$

Using the fact that the function $t(x) = \frac{x-|a|}{x+|a|}$ is non-decreasing function of x and $|k^n a_n + \lambda m| \ge k^n|a_n| - |\lambda m|$, we get for every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ and $|z| = 1$,

$$|g'(z)| \ge \frac{1}{2}\left(n + \frac{k^n|a_n| - |\lambda m| - |a_0|}{k^n|a_n| - |\lambda m| + |a_0|}\right)|g(z)|. \quad (11)$$

Equivalently for $|z| = 1$ and $|\lambda| < 1$,

$$|f'(z) + nm\lambda z^{n-1}| \ge \frac{1}{2}\left(n + \frac{k^n|a_n| - |\lambda m| - |a_0|}{k^n|a_n| - |\lambda m| + |a_0|}\right)(|f(z)| - m|\lambda|). \quad (12)$$

Since all the zeros of $g(z) = f(z) + \lambda m z^n$ lie in $|z| < 1$, by Guass Lucas Theorem it follows that all the zeros of $g'(z) = f'(z) + \lambda n m z^{n-1}$ lie in $|z| < 1$. This implies

$$|f'(z)| \ge nm \qquad \text{for } |z| \ge 1. \quad (13)$$

Choosing argument of λ in the left hand side of (12) such that

$$|f'(z) + nm\lambda z^{n-1}| = |f'(z)| - nm|\lambda| \qquad \text{for } |z| = 1,$$

which is possible by (13), we get

$$|f'(z)| - nm|\lambda| \ge \frac{1}{2}\left(n + \frac{k^n|a_n| - |\lambda m| - |a_0|}{k^n|a_n| - |\lambda m| + |a_0|}\right)(|f(z)| - m|\lambda|),$$

that is,

$$|f'(z)| \ge \frac{1}{2}\left(n + \frac{k^n|a_n| - |\lambda m| - |a_0|}{k^n|a_n| - |\lambda m| + |a_0|}\right)|f(z)| + \frac{1}{2}\left(n - \frac{k^n|a_n| - |\lambda m| - |a_0|}{k^n|a_n| - |\lambda m| + |a_0|}\right)|\lambda| m.$$

Replacing $f(z)$ by $P(kz)$, we get

$$k\max_{|z|=k}|P'(z)|\geq\frac{1}{2}\left(n+\frac{k^n|a_n|-|\lambda m|-|a_0|}{k^n|a_n|-|\lambda m|+|a_0|}\right)\max_{|z|=k}|P(z)|+\frac{1}{2}\left(n-\frac{k^n|a_n|-|\lambda m|-|a_0|}{k^n|a_n|-|\lambda m|+|a_0|}\right)|\lambda|m.$$

Using Lemma 1 with $R=k\geq 1$ and Lemma 3, we obtain

$$\begin{aligned}k^n\max_{|z|=1}|P'(z)| \quad &\geq \quad \frac{1}{2}\left(n+\frac{k^n|a_n|-|\lambda m|-|a_0|}{k^n|a_n|-|\lambda m|+|a_0|}\right)\left\{\frac{2k^n}{1+k^n}\max_{|z|=1}|P(z)|+\left(\frac{k^n-1}{k^n+1}\right)|\lambda|m\right\}\\ &\quad +\frac{1}{2}\left(n-\frac{k^n|a_n|-|\lambda m|-|a_0|}{k^n|a_n|-|\lambda m|+|a_0|}\right)|\lambda|m,\end{aligned}$$

which on simplification yields,

$$\max_{|z|=1}|P'(z)|\geq\frac{n}{(1+k^n)}\left(\max_{|z|=1}|P(z)|+|\lambda|m\right)+\frac{1}{k^n(1+k^n)}\left(\frac{k^n|a_n|-|\lambda|m-|a_0|}{k^n|a_n|-|\lambda|m+|a_0|}\right)\left\{k^n\max_{|z|=1}|P(z)|-|\lambda|m\right\}.$$

The above inequality is equivalent to (10) and thus completes the proof of Theorem. ■

Remark 2 For $l=0$, Theorem 8 reduces to Theorem 7 and for $k=1$, inequality (10) refines inequality (3). Further as in the case of Remark 1, it can be easily seen that Theorem 8 is refinement of Theorem 7.

Example 2 *Consider the polynomial* $P(z)=z^2+4z+13$, *clearly it satisfies all the conditions of Theorem 8 with* $k=4$, $\max_{|z|=1}|P(z)|=18$, $\max_{|z|=1}|P'(z)|=6$ *and* $\min_{|z|=4}|P(z)|=13$. *On substituting these one can easily see that the conclusion of Theorem 8 holds.*

Next we shall extend Theorem 7 and Theorem 8 to the polar derivatives.

Theorem 9 *If* $P\in\mathcal{P}_n$ *and* $P(z)$ *has all its zeros in* $|z|\leq k$, $k\geq 1$, *then for every* $\alpha\in\mathbb{C}$ *with* $|\alpha|\geq k$

$$\max_{|z|=1}|D_\alpha P(z)|\geq\frac{|\alpha|-k}{1+k^n}\left(n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\max_{|z|=1}|P(z)|. \tag{14}$$

In view of (4), *the result is sharp in limiting case when* $|\alpha|\to\infty$ *as shown by polynomial* $P(z)=z^n+k^n$.

Proof. Let $f(z)=P(kz)$. Since $P\in\mathcal{P}_n$ and $P(z)$ has all its zeros in $|z|\leq k$ where $k\geq 1$, therefore, $f\in\mathcal{P}_n$ and $f(z)$ has all its zeros in $|z|\leq 1$. If $Q(z)=z^n\overline{f(1/\overline{z})}$, then it is easy to verify that

$$|Q'(z)|=|nf(z)-zf'(z)|\qquad\text{for}\quad |z|=1. \tag{15}$$

Combining (15) with Lemma 4, we get

$$|f'(z)|\geq|nf(z)-zf'(z)|\qquad\text{for}\quad |z|=1. \tag{16}$$

Now for every $\alpha\in\mathbf{C}$ with $|\alpha|\geq k$, we have for $|z|=1$,

$$\left|D_{\alpha/k}f(z)\right|=|nf(z)+(\alpha/k-z)f'(z)|\geq|\alpha/k||f'(z)|-|nf(z)-zf'(z)|,$$

which gives with the help of (16),

$$|D_{\alpha/k}f(z)|\geq\left(\frac{|\alpha|-k}{k}\right)|f'(z)|, \tag{17}$$

consequently,

$$\max_{|z|=k}|D_\alpha P(z)|\geq(|\alpha|-k)\max_{|z|=k}|P'(z)|.$$

Since $D_\alpha P(z)$ is a polynomial of degree at most $n-1$ and $k \geq 1$, it follow from Lemma 1 that

$$k^{n-1}\max_{|z|=1}|D_\alpha P(z)| \geq (|\alpha|-k)\max_{|z|=k}|P'(z)|. \tag{18}$$

Further since all the zeros of $P(z)$ lie in $|z| \leq k$, $k \geq 1$, therefore, using (9) in (18), we get

$$k^{n-1}\max_{|z|=1}|D_\alpha P(z)| \geq \frac{(|\alpha|-k)}{2k}\left(n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\max_{|z|=k}|P(z)|.$$

Using Lemma 2, we get for $|z|=1$,

$$\max_{|z|=1}|D_\alpha P(z)| \geq \frac{|\alpha|-k}{1+k^n}\left(n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\max_{|z|=1}|P(z)|.$$

This completes the proof of Theorem. ■

Remark 3 As in the case of Remark 1, it can be easily seen that inequality (14) refines inequality (5). Further, if we divide the two sides of (14) by $|\alpha|$ and let $|\alpha| \to \infty$, we get Theorem 7.

Example 3 *Consider the polynomial $P(z) = z^2 + \frac{3}{2}z + \frac{1}{2}$, clearly it satisfies all the conditions of Theorem 9 with $k = 1, \max_{|z|=1}|P(z)| = 3$ and $\max_{|z|=1}|D_\alpha P(z)| = 9.5$ with $\alpha = 2$. On substituting these one can easily see that the conclusion of Theorem 9 holds.*

Theorem 10 *If all the zeros of $P \in \mathcal{P}_n$ lie in $|z| \leq k, k \geq 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$, $0 \leq l < 1$,*

$$\begin{aligned}\max_{|z|=1}|D_\alpha P(z)| \quad &\geq \quad \frac{n}{1+k^n}\left\{(|\alpha|-k)\max_{|z|=1}|P(z)| + (|\alpha|+1/k^{n-1})lm\right\}\\ &\quad + \frac{(|\alpha|-k)}{k^n(k^n+1)}\left(\frac{k^n|a_n|-lm-|a_0|}{k^n|a_n|-lm+|a_0|}\right)\left(k^n\max_{|z|=1}|P(z)|-lm\right),\end{aligned} \tag{19}$$

where $m = \min_{|z|=k}|P(z)|$. In view of (4), the result is sharp in limiting case when $|\alpha| \to \infty$ as shown by polynomial $P(z) = z^n + k^n$.

Proof. By hypothesis $P \in \mathcal{P}_n$ and $P(z)$ has all zeros in $|z| \leq k$, $k \geq 1$, therefore, proceeding similarly as in the proof of Theorem 8, we conclude that the polynomial $g(z) = f(z) - \lambda m z^n$ has all zeros in $|z| < 1$ where $f(z) = P(kz)$, $m = \min_{|z|=k}|P(z)| = \min_{|z|=1}|f(z)|$ and $|\lambda| < 1$. As before applying inequality (17) to the polynomial $g(z)$, it follows for $|z| = 1$ and $|\alpha| \geq k$,

$$|D_{\alpha/k}g(z)| \geq \left(\frac{|\alpha|-k}{k}\right)|g'(z)|.$$

Using inequality (11), we obtain

$$|D_{\alpha/k}g(z)| \geq \frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(n+\frac{k^n|a_n|-|\lambda m|-|a_0|}{k^n|a_n|-|\lambda m|+|a_0|}\right)|g(z)|.$$

for $|z| = 1$ and $|\alpha| \geq k$. Replacing $g(z)$ by $f(z) - \lambda m z^n$, we get for $|z| = 1$ and $|\alpha| \geq k$,

$$\left|D_{\alpha/k}f(z) - \frac{nm\alpha\lambda}{k}z^{n-1}\right| \geq \frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(n+\frac{k^n|a_n|-|\lambda m|-|a_0|}{k^n|a_n|-|\lambda m|+|a_0|}\right)\left(|f(z)-\lambda m|\right). \tag{20}$$

Since all the zeros of $f(z) - \lambda m z^n = g(z)$ lie in $|z| < 1$ and $|\alpha/k| \geq 1$, it follows by Lemma 5 that all the zeros of

$$D_{\alpha/k}(f(z) - m\lambda z^n) = D_{\alpha/k}f(z) - \frac{nm\alpha\lambda}{k}z^{n-1}$$

lie in $|z| < 1$. This implies that

$$\left|D_{\alpha/k}f(z)\right| \geq \frac{nm|\alpha|}{k}|z|^{n-1} \qquad \text{for} \quad |z| \geq 1.$$

In view of this inequality, choosing argument λ in the left hand side of inequality (20) such that

$$\left|D_{\alpha/k}f(z) - \frac{nm\alpha\lambda}{k}z^{n-1}\right| = \left|D_{\alpha/k}f(z)\right| - \frac{nm|\alpha||\lambda|}{k} \quad \text{for} \quad |z| = 1,$$

we get for $|z| = 1$ and $|\alpha| \geq k$,

$$\left|D_{\alpha/k}f(z)\right| - \frac{nm|\alpha||\lambda|}{k} \geq \frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(n + \frac{k^n|a_n| - |\lambda m| - |a_0|}{k^n|a_n| - |\lambda m| + |a_0|}\right)\big(|f(z)| - |\lambda|m\big),$$

which on simplification leads to

$$\begin{aligned}\left|D_{\alpha/k}f(z)\right| \quad &\geq \quad \frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(n + \frac{k^n|a_n| - |\lambda m| - |a_0|}{k^n|a_n| - |\lambda m| + |a_0|}\right)|f(z)| + \frac{n}{2}\left(\frac{|\alpha|+k}{k}\right)|\lambda|m \\ &\quad -\frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(\frac{k^n|a_n| - |\lambda m| - |a_0|}{k^n|a_n| - |\lambda m| + |a_0|}\right)|\lambda|m.\end{aligned}$$

This implies for $|z| = 1$ and $|\alpha| \geq k$,

$$\begin{aligned}\max_{|z|=k}|D_\alpha P(z)| \quad &\geq \quad \frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(n + \frac{k^n|a_n| - |\lambda m| - |a_0|}{k^n|a_n| - |\lambda m| + |a_0|}\right)\max_{|z|=k}|P(z)| + \frac{n}{2}\left(\frac{|\alpha|+k}{k}\right)|\lambda|m \\ &\quad -\frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(\frac{k^n|a_n| - |\lambda m| - |a_0|}{k^n|a_n| - |\lambda m| + |a_0|}\right)|\lambda|m.\end{aligned}$$

As before, applying Lemma 1 and 3, we obtain for $|z| = 1$ and $|\alpha| \geq k$,

$$\begin{aligned}k^{n-1}\max_{|z|=1}|D_\alpha P(z)| \quad &\geq \quad \frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(n + \frac{k^n|a_n| - lm - |a_0|}{k^n|a_n| - lm + |a_0|}\right)\left(\frac{2k^n}{1+k^n}\max_{|z|=1}|P(z)| + \frac{k^n-1}{k^n+1}lm\right) \\ &\quad -\frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(\frac{k^n|a_n| - lm - |a_0|}{k^n|a_n| - lm + |a_0|}\right)lm + \frac{n}{2}\left(\frac{|\alpha|+k}{k}\right)lm,\end{aligned}$$

equivalently, we have for $|z| = 1$ and $|\alpha| \geq k$,

$$\begin{aligned}\max_{|z|=1}|D_\alpha P(z)| \quad &\geq \quad \frac{n}{1+k^n}\left\{(|\alpha|-k)\max_{|z|=1}|P(z)| + (|\alpha| + 1/k^{n-1})lm\right\} \\ &\quad + \frac{(|\alpha|-k)}{k^n(k^n+1)}\left(\frac{k^n|a_n| - lm - |a_0|}{k^n|a_n| - lm + |a_0|}\right)\big(k^n\max_{|z|=1}|P(z)| - lm\big).\end{aligned}$$

This completes the proof of Theorem. ∎

Remark 4 If we divide both sides of (19) by $|\alpha|$ and let $|\alpha| \to \infty$, we get Theorem 8.

Example 4 *Consider the polynomial* $P(z) = z^2 + 3z + \frac{9}{4}$, *clearly it satisfies all the conditions of Theorem 10 with* $k = 1.7, \max_{|z|=1}|P(z)| = 6.25, \min_{|z|=1.7}|P(z)| = 0.04$ *and* $\max_{|z|=1}|D_\alpha P(z)| = 22.5$ *with* $\alpha = 3$. *On substituting these one can easily see that the conclusion of Theorem 10 holds.*

Setting $k = 1$ in Theorem 10, we obtain:

Corollary 11 *If all the zeros of $P \in \mathcal{P}_n$ lie in $|z| \leq 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$, $0 \leq l < 1$,*

$$\max_{|z|=1} |D_\alpha P(z)| \geq \frac{n}{2}\left\{(|\alpha|-1)\max_{|z|=1}|P(z)| + (|\alpha|+1)lm_1\right\} + \frac{(|\alpha|-1)}{2}\left(\frac{|a_n|-lm_1-|a_0|}{|a_n|-lm_1+|a_0|}\right)\left(\max_{|z|=1}|P(z)| - lm_1\right). \tag{21}$$

where $m_1 = \min_{|z|=1}|P(z)|$. The result is sharp and equality holds for $P(z) = (z+1)^n$ with real $\alpha \geq 1$.

Remark 5 If we divide both sides of (21) by $|\alpha|$ and let $|\alpha| \to \infty$, we get a sharp refinement inequality 3.

Lastly its worth mentioning that if we use Lemma 3 instead of lemma 2 in the proof of Theorem 7 and Theorem 9, we get the following refined result.

Theorem 12 *If $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \leq k, k \geq 1$, then for $0 \leq l < 1$*

$$\max_{|z|=1}|P'(z)| \geq \frac{1}{2}\left(n + \frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\left\{\frac{2}{1+k^n}\max_{|z|=1}|P(z)| + l\frac{k^n-1}{k^n(k^n+1)}\min_{|z|=1}|P(z)|\right\} \tag{22}$$

and for $|\alpha| \geq 1$,

$$\max_{|z|=1}|D_\alpha P(z)| \geq \frac{|\alpha|-k}{2}\left(n + \frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\left\{\frac{2}{1+k^n}\max_{|z|=1}|P(z)| + l\frac{k^n-1}{k^n(k^n+1)}\min_{|z|=1}|P(z)|\right\}.$$

Inequality (22) is sharp and equality in (22) holds for $P(z) = z^n + k^n$.

Acknowledgment. The authors are highly grateful to the referee for his valuable suggestions and comments.

References

[1] A. Aziz, Inequalities for the derivative of a polynomial, Proc. Amer. Math. Soc., 89(1983), 259–266.

[2] A. Aziz and Q. M. Dawood, Inequalities for a polynomial and its derivatives, J. Approx. Theory, 54(1998), 306–313.

[3] A. Aziz and N. A. Rather, Some Zygmund type L^q inequalities for polynomials, J. Math. Anal. Appl., 289(2004), 14–29.

[4] A. Aziz and N. A. Rather, Inequalities for the polar derivative of a polynomial with restricted zeros, Math. Bulk., 17(2003), 15–28.

[5] A. Aziz and N. A. Rather, A refinement of a theorem of Paul Turán concerning polynomials, Math. Ineq. Appl., 1(1998), 231–238.

[6] P. Borwein and T. Erdélyi, Polynomials and Polynomial Inequalities, Springer-Verlag, New York, 1995.

[7] V. N. Dubinin, Applications of the Schwarz lemma to inequalities for entire functions with constraints on zeros, J. Math. Sci., 143(2007), 3069–3076.

[8] N. K. Govil, On the derivative of a polynomial, Proc. Amer. Math. Soc., 41(1973), 543–546.

[9] V. I. Ivanov, Some extremal properties of polynomials and inverse inequalities in approximatiom theory, Trudy Mat. Inst. Steklov, 145(1979), 79–110.

[10] G. G. Lorentz, Approximation of Functions, Holt, Rinehart and Winston, New York, 1966.

[11] M. Marden, Geometry of Polynomials, Math. Surveys, No. 3, Amer. Math. Soc. Providence, RI, 1966.

[12] A. A. Markov, on a problem of D. I. Mendeleev, Zap. Imp. Akad. Nauk, St. Petersburg, 62(1889), 1–24.

[13] D. Mendeleev, Investigation of Aqueous Solution Based on Specfic Gravity, St. petersberg, 1887(Russian).

[14] G. V. Milovanović, D. S. Mitrinović and Th. M. Rassias, Topics in Polynomials: Extremal Properties, Inequalities, Zeros, World scientific Publishing Co., Singapore, (1994).

[15] R. Osserman, A sharp Schwarz inequality on the boundary, Proc. Amer. Math. Soc., 128(2000), 3513–3517.

[16] G.Pólya and G. Szegö, Problems and Theorems in Analysis, II, Springer- Verlag, Berlin, New York, 1976.

[17] A. C. Schaeffer, Inequalities of A. Markoff and S. N. Bernstein for polynomial and related functions, Bull. Amer. Math. Soc., 47(1941), 565–579.

[18] N. A. Rather and Suhail Gulzar, Generalization of an inequality involving maximum moduli of a polynomial and its polar derivative, Non-linear Funct. Anal. and Appl., 19(2014), 213–221.

[19] S. A. Telyakovskii, Research in the theory of approximation of functions at the mathematical institute of the academy of sciences, Trudi Mat. Inst. Steklov, 182 (1988): English trans. in: Proc. Steklov Inst. Math., 1990, 141–197.

[20] P. Turán, Über die ableitung von polynomen, Compos. Math., 7(1939), 89–95.