Applied Mathematics E-Notes, 20(2020), 204-214 © ISSN 1607-2510
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Extended Quasi b-Metric-Like Spaces And Some Fixed Point Theorems For Contractive Mappings*

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Received 14 November 2019

Abstract

In this paper, we introduce the structure of extended quasi b-metric like spaces as a generalization of both quasi metric like spaces and quasi b-metric like spaces. Also, we present the notion of JSR-contractive mappings in the setup of extended quasi b-metric like spaces and investigate the existence of fixed point for such mappings. We also provide examples to illustrate the results presented herein.

1 Introduction

Because of the importance of the concept of a distance between two abstract objects of an underlying universe, there are several generalizations of the notion of a distance function defined on a nonempty set. Some of the most important generalizations of metric space are b-metric space in [3] (see also [4]), partial metric space in [9], metric-like space in [2], dislocated metric space in [5], b-metric-like space in [1] (see also [6]), etc.

An extended b-metric or p-metric was introduced by Parvaneh and Ghoncheh [10] which is an extension of the concept of a b-metric. Subsequently, Parvaneh and Kadelburg [11] extended this concept to a partial p-metric space. The notion of a p-metric-like space was then introduced in [12].

Introducing the concept of a quasi b-metric, Chen *et al.* [15] generalized the concepts of quasi b-metric and b-metric-like spaces. In this paper, we introduce the notion of a quasi p-metric-like space to generalize and unify all the concepts mentioned above. We also obtain the existence of fixed point of JSR-contractive type mappings in such spaces. Our results generalize and improve the main results in [12].

2 Mathematical Background

Let $\Upsilon = \{\Omega : \mathbb{R}_0^+ \to \mathbb{R}_0^+ : \Omega$ is a strictly increasing continuous function satisfying $\Omega^{-1}(t) \le t \le \Omega(t)\}$.

Definition 1 ([10]) *Let Λ be a nonempty set. A function $\widetilde{d} : \Lambda \times \Lambda \to \mathbb{R}_0^+$ is said to be an extended b-metric or a p-metric if there exists $\Omega \in \Upsilon$ such that for any $\eta_1, \eta_2, \eta_3 \in \Lambda$, the following conditions hold:*

(p_1) $\widetilde{d}(\eta_1, \eta_2) = 0$ *iff* $\eta_1 = \eta_2$,

(p_2) $\widetilde{d}(\eta_1, \eta_2) = \widetilde{d}(\eta_2, \eta_1)$,

(p_3) $\widetilde{d}(\eta_1, \eta_3) \le \Omega(\widetilde{d}(\eta_1, \eta_2) + \widetilde{d}(\eta_2, \eta_3))$.

The pair $(\Lambda, \widetilde{d})$ is called an extended b-metric space or a p-metric space.

*Mathematics Subject Classifications: 47H10, 54H25.

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Note that the class of p-metric spaces is considerably larger than the class of b-metric spaces. Indeed, if we define $\Omega(t) = st$, $s \geq 1$, then a p-metric becomes a b-metric. Also, if $\Omega(t) = t$ then a p-metric is a metric.

Definition 2 ([11]) *Let Λ be a nonempty set and $\Omega \in \Upsilon$. A function $p_p : \Lambda \times \Lambda \to \mathbb{R}_0^+$ is called an extended partial b -metric, or a partial p-metric if for any $\eta_1, \eta_2, \eta_3 \in \Lambda$, the following conditions are satisfied:*

(p_p1) $\eta_1 = \eta_2 \Longleftrightarrow p_p(\eta_1, \eta_1) = p_p(\eta_1, \eta_2) = p_p(\eta_2, \eta_2)$,

(p_p2) $p_p(\eta_1, \eta_1) \leq p_p(\eta_1, \eta_2)$,

(p_p3) $p_p(\eta_1, \eta_2) = p_p(\eta_2, \eta_1)$,

(p_p4) $p_p(\eta_1, \eta_2) - p_p(\eta_1, \eta_1) \leq \Omega(p_p(\eta_1, \eta_3) + p_p(\eta_3, \eta_2) - p_p(\eta_3, \eta_3) - p_p(\eta_1, \eta_1))$.

The pair (Λ, p_p) is called a partial p-metric space, or an extended partial b-metric space.

Definition 3 ([2]) *Let Λ be a nonempty set. A function $\sigma : \Lambda \times \Lambda \to \mathbb{R}_0^+$ is said to be a metric-like on Λ if for any $\eta_1, \eta_2, \eta_3 \in \Lambda$, the following conditions hold:*

$(\sigma 1)$ $\sigma(\eta_1, \eta_2) = 0$ implies $\eta_1 = \eta_2$,

$(\sigma 2)$ $\sigma(\eta_1, \eta_2) = \sigma(\eta_2, \eta_1)$,

$(\sigma 3)$ $\sigma(\eta_1, \eta_2) \leq \sigma(\eta_1, \eta_3) + \sigma(\eta_3, \eta_2)$.

The pair (Λ, σ) is called a metric-like space.

Every metric space is a metric-like space. Following are some examples of metric-like spaces.

Example 1 ([14]) *Let $b \in \Lambda = \mathbb{R}$ and $a \geq 0$. The mapping $\sigma_i : \Lambda \times \Lambda \longrightarrow \mathbb{R}_0^+$ for each $i \in \{1, 2, 3\}$ defined by*

$$\sigma_1(\eta_1, \eta_2) = |\eta_1| + |\eta_2| + a,$$
$$\sigma_2(\eta_1, \eta_2) = |\eta_1 - b| + |\eta_2 - b|,$$
$$\sigma_3(\eta_1, \eta_2) = \eta_1^2 + \eta_2^2,$$

are some examples of metric-like on Λ.

Definition 4 ([1]) *Let Λ be a nonempty set and $s \geq 1$ be a given real number. A function $\sigma_b : \Lambda \times \Lambda \to \mathbb{R}_0^+$ is said to be a b-metric-like if for any $\eta_1, \eta_2, \eta_3 \in \Lambda$, the following conditions are satisfied:*

$(\sigma_b 1)$ $\sigma_b(\eta_1, \eta_2) = 0$ implies $\eta_1 = \eta_2$,

$(\sigma_b 2)$ $\sigma_b(\eta_1, \eta_2) = \sigma_b(\eta_2, \eta_1)$,

$(\sigma_b 3)$ $\sigma_b(\eta_1, \eta_2) \leq s[\sigma_b(\eta_1, \eta_3) + \sigma_b(\eta_3, \eta_2)]$.

The pair (Λ, σ_b) is called a b-metric-like space with parameter s.

Definition 5 ([15]) *Let Λ be a nonempty set and $s \geq 1$ be a given real number. A function $\sigma_{qb} : \Lambda \times \Lambda \to \mathbb{R}_0^+$ is called a quasi b-metric-like if for any $\eta_1, \eta_2, \eta_3 \in \Lambda$, the following conditions are satisfied:*

$(\sigma_{qb} 1)$ $\sigma_{qb}(\eta_1, \eta_2) = 0$ implies $\eta_1 = \eta_2$,

$(\sigma_{qb} 2)$ $\sigma_{qb}(\eta_1, \eta_2) \leq s[\sigma_{qb}(\eta_1, \eta_3) + \sigma_{qb}(\eta_3, \eta_2)]$.

The pair (Λ, σ_{qb}) is called a quasi b-metric-like space with parameter s.

Definition 6 ([12]) *Let Λ be a nonempty set and $\Omega \in \Upsilon$. A function $\sigma_p : \Lambda \times \Lambda \to \mathbb{R}_0^+$ is called a p-metric-like if for any $\eta_1, \eta_2, \eta_3 \in \Lambda$, the following conditions are satisfied:*

(σ_p1) $\sigma_p(\eta_1, \eta_2) = 0$ *implies that* $\eta_1 = \eta_2$,

(σ_p2) $\sigma_p(\eta_1, \eta_2) = \sigma_p(\eta_2, \eta_1)$,

(σ_p3) $\sigma_p(\eta_1, \eta_2) \le \Omega[\sigma_p(\eta_1, \eta_3) + \sigma_p(\eta_3, \eta_2)]$.

The pair (Λ, σ_p) is called a p-metric-like space or an extended b-metric-like space.

Definition 7 *Let Λ be a nonempty set and $\Omega \in \Upsilon$. A function $\sigma_{qp} : \Lambda \times \Lambda \to \mathbb{R}_0^+$ is called quasi p-metric-like if for any $\eta_1, \eta_2, \eta_3 \in \Lambda$, the following conditions are satisfied:*

(σ_{qp}1) $\sigma_{qp}(\eta_1, \eta_2) = 0$ *implies* $\eta_1 = \eta_2$,

(σ_{qp}2) $\sigma_{qp}(\eta_1, \eta_2) \le \Omega[\sigma_{qp}(\eta_1, \eta_3) + \sigma_{qp}(\eta_3, \eta_2)]$.

The pair (Λ, σ_{qp}) is called a quasi p-metric-like space.

Note that every metric-like space is a p-metric-like space, every p-metric space is also a p-metric-like space and every p-metric like space is also a quasi p-metric-like space. However, the reverse implications do not hold in general.

Definition 8 *Let (Λ, σ_{qp}) be a quasi p-metric-like space (QPMLS) and $\eta \in \Lambda$. A sequence $\{\eta_n\}$ in Λ is said to be:*

(i) σ_{qp}*-convergent to* η, *if*
$$\lim_{n\to\infty} \sigma_{qp}(\eta_n, \eta) = \lim_{n\to\infty} \sigma_{qp}(\eta, \eta_n) = \sigma_{qp}(\eta, \eta).$$

(ii) a right σ_{qp}-Cauchy sequence in (Λ, σ_{qp}) if $\lim_{n>m\to\infty} \sigma_{qp}(\eta_m, \eta_n)$ exists and is finite.

(iii) a left σ_{qp}-Cauchy sequence in (Λ, σ_{qp}) if $\lim_{m>n\to\infty} \sigma_{qp}(\eta_m, \eta_n)$ exists and is finite.

Definition 9 *Let (Λ, σ_{qp}) be a quasi p-metric-like space (QPMLS) and $\alpha : \Lambda \times \Lambda \to \mathbb{R}_0^+$. Then (Λ, σ_{qp}) is said to be*

(i) right α-complete quasi p-metric-like space if for every right σ_{qp}-Cauchy sequence $\{\eta_n\}$ in Λ with $\alpha(\eta_n, \eta_{n+1}) \ge 1$, there exists $\eta \in \Lambda$ such that
$$\lim_{n>m\to\infty} \sigma_p(\eta_m, \eta_n) = \lim_{n\to\infty} \sigma_p(\eta_n, \eta) = \lim_{n\to\infty} \sigma_p(\eta, \eta_n) = \sigma_p(\eta, \eta),$$

(ii) left α-complete quasi p-metric-like space if for every left σ_{qp}-Cauchy sequence $\{\eta_n\}$ in Λ with $\alpha(\eta_{n+1}, \eta_n) \ge 1$ there exists $\eta \in \Lambda$ such that
$$\lim_{m>n\to\infty} \sigma_p(\eta_m, \eta_n) = \lim_{n\to\infty} \sigma_p(\eta_n, \eta) = \lim_{n\to\infty} \sigma_p(\eta, \eta_n) = \sigma_p(\eta, \eta).$$

Here, we present an example to show that a QPML is not QbML in general.

Example 2 *Let (Λ, σ_{qb}) be a QbMS (with parameter s) and $\rho(\eta_1, \eta_2) = \sinh[\sigma_{qb}(\eta_1, \eta_2)]$. We show that ρ is a QPML with $\Omega(t) = \sinh(st)$ for all $t \ge 0$. Obviously, condition (σ_{qp}1) of Definition 7 is satisfied. For each $\eta_1, \eta_2, \eta_3 \in \Lambda$, we have*
$$\begin{aligned}\rho(\eta_1, \eta_2) &= \sinh(\sigma_{qb}(\eta_1, \eta_2)) \le \sinh(s \cdot \sinh(\sigma_{qb}(\eta_1, \eta_2)) + s \cdot \sinh(\sigma_{qb}(\eta_1, \eta_2))) \\ &= \sinh(s \cdot \rho(\eta_1, \eta_3) + s \cdot \rho(\eta_3, \eta_2)) \\ &= \Omega(\rho(\eta_1, \eta_3) + \rho(\eta_3, \eta_2)).\end{aligned}$$

Thus, condition ($\sigma_{qp}2$) of Definition 7 is satisfied and hence ρ is a QPML. Note that $\sinh[|\eta_1 - \eta_2| + |\eta_1|]$ *is not a QML on* $\mathbb{R}$*. Indeed*

$$\begin{aligned} \sinh[|5-0|+5] &= 11013.2328747 \\ &\nleq 548.316123273 + 201.71315737 \\ &= \sinh[2+5] + \sinh[3+3]. \end{aligned}$$

Also,

$$d(\eta_1, \eta_2) = (\eta_1 - \eta_2)^2 + \eta_1^2$$

is a QbML on $\mathbb{R}$ *with* $s = 2$*. There is no* $s \geq 1$ *such that* $\rho(\eta_1, \eta_2) = \sinh[(\eta_1 - \eta_2)^2 + \eta_1^2]$ *is a QbML with parameter* s*. Indeed, for* $y = 0$ *and* $\eta_3 = 1$ *(with arbitrary* η_1*)*

$$\sinh 2\eta_1^2 \leq s(\sinh[(\eta_1 - 1)^2 + \eta_1^2] + \sinh 2)$$

which does not hold for any fixed s *and* η_1 *sufficiently large.*

In general, we have the following proposition.

Proposition 1 *Let* (Λ, σ_{qb}) *be a QbML with coefficient* $s \geq 1$ *and* $\rho(\eta_1, \eta_2) = \xi(d(\eta_1, \eta_2))$*, where* $\xi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *is a strictly increasing function with* $\eta_1 \leq \xi(\eta_1)$ *and* $0 = \xi(0)$*. We show that* ρ *is a QPML with* $\Omega(t) = \xi(s \cdot t)$*. For each* $\eta_1, \eta_2, \eta_3 \in \Lambda$*, we have*

$$\begin{aligned} \rho(\eta_1, \eta_2) = \xi(\sigma_{qb}(\eta_1, \eta_2)) &\leq \xi(s[\sigma_{qb}(\eta_1, \eta_3) + \sigma_{qb}(\eta_3, \eta_2)]) \\ &\leq \xi(s[\xi(\sigma_{qb}(\eta_1, \eta_3)) + \xi(\sigma_{qb}(\eta_3, \eta_2)]) \\ &= \Omega(s[\rho(\eta_1, \eta_3) + \rho(\eta_3, \eta_2)]). \end{aligned}$$

So, ρ *is a QPML.*

With the help of the above proposition, we construct the following example:

Example 3 *Let* (Λ, σ_{qb}) *be a QbML and* $\rho(\eta_1, \eta_2) = e^{\sigma_{qb}(\eta_1, \eta_2)} \sec^{-1}(e^{\sigma_{qb}(\eta_1, \eta_2)})$*. Then* ρ *is a QPML with* $\Omega(t) = e^{s \cdot t} \sec^{-1}(e^{s \cdot t})$*, where* s *is the parameter of QbML space* (Λ, σ_{qb})*.*

The concept of α-admissible mapping was introduced by Samet et al. in 2012.

Definition 10 ([13]) *Let* Λ *be a non-empty set and* $\Gamma : \Lambda \to \Lambda$ *and* $\alpha : \Lambda \times \Lambda \to \mathbb{R}_0^+$ *be given mappings.* Γ *is said to be* α-admissible, *if for all* $\eta_1, \eta_2 \in \Lambda$, $\alpha(\eta_1, \eta_2) \geq 1$ *implies that* $\alpha(\Gamma\eta_1, \Gamma\eta_2) \geq 1$*.*

Definition 11 ([8]) *A map* $\Gamma : \Lambda \to \Lambda$ *is said to be triangular* α*-admissible, if*

($\Gamma1$) Γ *is* α*-admissible,*

($\Gamma2$) $\alpha(\eta_1, \eta_2) \geq 1$ *and* $\alpha(\eta_2, \eta_3) \geq 1$ *imply* $\alpha(\eta_1, \eta_3) \geq 1$ *for all* $\eta_1, \eta_2, \eta_3 \in \Lambda$*.*

3 Main Results

Motivated by the work in [7], Δ_θ denotes the set of all functions $\theta : \mathbb{R}_0^+ \to [1, \infty)$ satisfying the following conditions:

($\theta1$) θ is strictly increasing;

($\theta2$) θ is continuous;

($\theta3$) for each sequence $\{t_n\} \subseteq (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$.

Definition 12 *Let (Λ, σ_{qp}) be a quasi p-metric-like space (QPMLS), and $\alpha : \Lambda \times \Lambda \to \mathbb{R}_0^+$. A self mapping Γ on Λ is said to be a right Jleli-Samet-Reich (JSR) contraction, if for any $\eta_1, \eta_2 \in \Lambda$ with $1 \leq \alpha(\eta_1, \eta_2)$ and $\Gamma\eta_1 \neq \Gamma\eta_2$, we have*

$$\theta(\Omega^2(\sigma_{qp}(\Gamma\eta_1, \Gamma\eta_2))) \leq [\theta(\lambda_1\sigma_{qp}(\eta_1, \eta_2) + \lambda_2\sigma_{qp}(\eta_1, \Gamma\eta_1) + \lambda_3\sigma_{qp}(\eta_2, \Gamma\eta_2)]^{\lambda}, \tag{1}$$

where $\theta \in \Delta_\theta$, $\lambda, \lambda_i \in [0, 1)$ and $\lambda_1 + \lambda_2 + \lambda_3 < 1$.

Theorem 1 *Let (Λ, σ_{qp}) be a right α-complete QPMLS. Suppose that $\Gamma : \Lambda \to \Lambda$ is continuous triangular α -admissible and a right JSR-contraction. If there exists $\eta_0 \in \Lambda$ such that $\alpha(\eta_0, \Gamma\eta_0) \geq 1$, then Γ has a fixed point.*

Proof. Let $\{\eta_n\}$ be the sequence generated by Picard iterative algorithm starting with a given point η_0, that is, $\eta_n = \Gamma^n\eta_0 = \Gamma\eta_{n-1}$. Since Γ is an α-admissible mapping and $\alpha(\eta_0, \Gamma\eta_0) = \alpha(\eta_0, \eta_1) \geq 1$, therefore $\alpha(\Gamma\eta_0, \Gamma\eta_1) = \alpha(\eta_1, \eta_2) \geq 1$. Continuing this process, we have $\alpha(\eta_{n-1}, \eta_n) \geq 1$ for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $\eta_{n_0} = \eta_{n_0+1}$, then η_{n_0} is a fixed point of Γ and hence the result has been obtained.

Now, we assume that $\eta_n \neq \eta_{n+1}$ for all $n \in \mathbb{N}$. Thus, $\sigma_{qp}(\Gamma\eta_{n-1}, \Gamma\eta_n) > 0$ for all $n \in \mathbb{N}$. Since Γ is a right JSR-contraction, it follows that

$$\begin{aligned}\theta\left(\sigma_{qp}(\eta_n, \eta_{n+1})\right) &= \theta\big(\sigma_{qp}(\Gamma\eta_{n-1}, \Gamma\eta_n)\big) \\ &\leq \theta\big(\lambda_1\sigma_{qp}(\eta_{n-1}, \eta_n) + \lambda_2\sigma_{qp}(\eta_{n-1}, \Gamma\eta_{n-1}) + \lambda_3\sigma_{qp}(\eta_n, \Gamma\eta_n)\big)^{\lambda} \\ &= \theta\big(\lambda_1\sigma_{qp}(\eta_{n-1}, \eta_n) + \lambda_2\sigma_{qp}(\eta_{n-1}, \eta_n) + \lambda_3\sigma_{qp}(\eta_n, \eta_{n+1}))\big)^{\lambda}.\end{aligned} \tag{2}$$

As θ is strictly increasing and $\lambda < 1$, we obtain that

$$\sigma_{qp}(\eta_n, \eta_{n+1}) \leq \lambda_1\sigma_{qp}(\eta_{n-1}, \eta_n) + \lambda_2\sigma_{qp}(\eta_{n-1}, \eta_n) + \lambda_3\sigma_{qp}(\eta_n, \eta_{n+1}).$$

If there exists $n > 0$ such that $\sigma_{qp}(\eta_n, \eta_{n-1}) \leq \sigma_{qp}(\eta_n, \eta_{n+1})$, then

$$\begin{aligned}\sigma_{qp}(\eta_n, \eta_{n+1}) &\leq \lambda_1\sigma_{qp}(\eta_n, \eta_{n+1}) + \lambda_2\sigma_{qp}(\eta_n, \eta_{n+1}) + \lambda_3\sigma_{qp}(\eta_n, \eta_{n+1}), \\ \sigma_{qp}(\eta_n, \eta_{n+1}) &> 0 \text{ and } \lambda_1 + \lambda_2 + \lambda_3 < 1,\end{aligned}$$

imply that $\sigma_{qp}(\eta_n, \eta_{n+1}) < \sigma_{qp}(\eta_n, \eta_{n+1})$, a contradiction. Thus, $\{\sigma_{qp}(\eta_n, \eta_{n+1})\}$ is a decreasing and bounded below sequence. Consequently, there exists $r \geq 0$ such that $\lim_{n\to\infty} \sigma_{qp}(\eta_n, \eta_{n+1}) = r$. From (2), we have

$$\theta\big(\sigma_{qp}(\eta_n, \eta_{n+1})\big) \leq \theta\big(\sigma_{qp}(\eta_{n-1}, \eta_n)\big)^{\lambda} \leq \theta\big(\sigma_{qp}(\eta_{n-2}, \eta_{n-1})\big)^{\lambda^2}.$$

Thus,

$$1 \leq \theta\big(\sigma_{qp}(\eta_{n-1}, \eta_n)\big)^{\lambda} \leq \theta\big(\sigma_{qp}(\eta_{n-2}, \eta_{n-1})\big)^{\lambda^2} \leq \cdots \leq \theta\big(\sigma_{qp}(\eta_0, \eta_1)\big)^{\lambda^n}. \tag{3}$$

On taking limit as $n \to \infty$ on both sides of (3), we have

$$\lim_{n\to\infty} \theta\big(\sigma_{qp}(\eta_{n-1}, \eta_n)\big) = 1,$$

which further implies that

$$r = \lim_{n\to\infty} \sigma_{qp}(\eta_{n-1}, \eta_n) = 0. \tag{4}$$

Now, we show that $\{\eta_n\}$ is a right σ_{qp}-Cauchy sequence in Λ . That is, $\lim\limits_{n>m\to\infty} \sigma_{qp}(\eta_m, \eta_n) = 0$. If not, then there exists $\varepsilon > 0$ such that we may find two subsequences $\{\eta_{m_k}\}$ and $\{\eta_{n_k}\}$ of $\{\eta_n\}$ with n_k the smallest index for which $n_k > m_k > k$ and

$$\sigma_{qp}(\eta_{m_k}, \eta_{n_k}) \geq \varepsilon \quad \text{and} \quad \sigma_{qp}(\eta_{m_k}, \eta_{n_{k-1}}) < \varepsilon. \tag{5}$$

From (5), we obtain that

$$\varepsilon \leq \sigma_{qp}(\eta_{m_k}, \eta_{n_k}) \leq \Omega[\sigma_{qp}(\eta_{m_k}, \eta_{m_k+1}) + \sigma_{qp}(\eta_{m_k+1}, \eta_{n_k})].$$

On taking the upper limit as $k \to \infty$, we get

$$\Omega^{-1}(\varepsilon) \leq \limsup_{k\to\infty} \sigma_{qp}(\eta_{m_k+1}, \eta_{n_k}). \tag{6}$$

Also,

$$\sigma_{qp}(\eta_{m_k}, \eta_{n_k}) \leq \Omega[\sigma_{qp}(\eta_{m_k}, \eta_{n_k-1}) + \sigma_{qp}(\eta_{n_k-1}, \eta_{n_k})].$$

From (4) and (5), we have

$$\limsup_{k\to\infty} \sigma_{qp}(\eta_{m_k}, \eta_{n_k}) \leq \Omega(\varepsilon). \tag{7}$$

As $\alpha(\eta_{m_k}, \eta_{n_k}) \geq 1$, so we have

$$\begin{aligned}\theta\big(\Omega^2(\sigma_{qp}(\eta_{m_k+1}, \eta_{n_k}))\big) &= \theta\big(\Omega^2(\sigma_{qp}(\Gamma\eta_{m_k}, \Gamma\eta_{n_k-1}))\big) \\ &\leq \theta\big(\lambda_1\sigma_{qp}(\eta_{m_k}, \eta_{n_k-1}) + \lambda_2\sigma_{qp}(\eta_{m_k}, \eta_{m_k+1}) + \lambda_3\sigma_{qp}(\eta_{n_k-1}, \eta_{n_k})\big)^{\lambda}.\end{aligned}$$

On taking the upper limit as $k \to \infty$ on both sides of the above inequality, we have

$$\begin{aligned}1 &< \theta\big(\Omega(\varepsilon)\big) = \theta\big(\Omega^2(\Omega^{-1}(\varepsilon)\big) \\ &\leq \theta\big(\Omega^2(\limsup_{i\to\infty} \sigma_{qp}(\eta_{m_k+1}, \eta_{n_k}))\big) \\ &\leq \theta\big(\limsup_{i\to\infty}[\lambda_1\sigma_{qp}(\eta_{m_k}, \eta_{n_k-1}) + \lambda_2\sigma_{qp}(\eta_{m_k}, \eta_{m_k+1}) + \lambda_3\sigma_{qp}(\eta_{n_k-1}, \eta_{n_k})]\big)^{\lambda} \\ &\leq \theta\big(\lambda_1\Omega(\varepsilon)\big)^{\lambda} \\ &< \theta\big(\Omega(\varepsilon)\big)^{\lambda},\end{aligned}$$

which is a contradiction. Hence, $\{\eta_n\}$ is a right σ_{qp}-Cauchy sequence in the (QPMLS) (Λ, σ_{qp}). Since (Λ, σ_{qp}) is right σ_{qp}-complete, the sequence $\{\eta_n\}$ σ_{qp}-converges to some $\varrho \in \Lambda$, that is,

$$\lim_{n>m\to\infty} \sigma_{qp}(\eta_m, \eta_n) = \lim_{n\to\infty} \sigma_{qp}(\eta_n, \varrho) = \lim_{n\to\infty} \sigma_{qp}(\varrho, \eta_n) = \sigma_{qp}(\varrho, \varrho) = 0.$$

As Γ is continuous, $\eta_{n+1} = \Gamma\eta_n \to \Gamma\varrho$ when $n \to \infty$. Thus

$$\sigma_{qp}(\varrho, \Gamma\varrho) \leq \Omega(\sigma_{qp}(\varrho, \Gamma\eta_n) + \sigma_{qp}(\Gamma\eta_n, \Gamma\varrho)).$$

On taking limit as $n \to \infty$ on both sides of the above inequality, we obtain that

$$\sigma_{qp}(\varrho, \Gamma\varrho) \leq \Omega(\lim_{n\to\infty} \sigma_{qp}(\varrho, \Gamma\eta_n) + \lim_{n\to\infty} \sigma_{qp}(\Gamma\eta_n, \Gamma\varrho)) = \Omega(\sigma_{qp}(\varrho, \varrho) + \sigma_{qp}(\Gamma\varrho, \Gamma\varrho)).$$

From (1), we have

$$\begin{aligned}\theta\big(\sigma_{qp}(\varrho, \Gamma\varrho)\big) \leq \theta\big(\Omega[\sigma_{qp}(\Gamma\varrho, \Gamma\varrho)]\big) &\leq \theta\big(\Omega^2[\sigma_p(\Gamma\varrho, \Gamma\varrho)]\big) \\ &\leq \theta\big(\lambda_1\sigma_{qp}(\varrho, \varrho) + \lambda_2\sigma_{qp}(\varrho, \Gamma\varrho) + \lambda_3\sigma_{qp}(\varrho, \Gamma\varrho)) \\ &\leq \theta\big(\lambda_2\sigma_{qp}(\varrho, \Gamma\varrho) + \lambda_3\sigma_{qp}(\varrho, \Gamma\varrho)\big)^{\lambda} \\ &\leq \theta\big(\sigma_{qp}(\varrho, \Gamma\varrho)\big)^{\lambda},\end{aligned}$$

which is not impossible unless we have $\Gamma\varrho = \varrho$. ■

In the following theorem, we omit the continuity of the mapping Γ.

Theorem 2 *Let (Λ, σ_{qp}) be a right α-complete (QPMLS). Suppose that $\Gamma : \Lambda \to \Lambda$ is a triangular α-admissible and a right JSR-contraction. If there exists $\eta_0 \in \Lambda$ such that $\alpha(\eta_0, \Gamma\eta_0) \geq 1$, then, Γ has a fixed point provided that for any $\{\eta_n\}$ in Λ with $\alpha(\eta_n, \eta_{n+1}) \geq 1$ and $\eta_n \to \eta$ as $n \to \infty$, we have $1 \leq \alpha(\eta_n, \eta)$ for all $n \in \mathbb{N}$.*

Proof. Following arguments similar to those in the proof of Theorem 1, we obtain a sequence $\{\eta_n\}$ such that

$$\alpha(\eta_n, \eta_{n+1}) \geq 1 \quad \text{and} \quad \eta_n \to \varrho \quad \text{as } n \to \infty,$$

where $\eta_{n+1} = \Gamma\eta_n$ and $\sigma_p(\varrho, \varrho) = 0$. By given assumption, we have $1 \leq \alpha(\eta_n, \varrho)$ for all $n \in \mathbb{N}$. Assume that $\sigma_{qp}(\varrho, \Gamma\varrho) > 0$. Note that

$$\Omega^{-1}(\sigma_p(\varrho, \Gamma\varrho)) \leq \limsup_{n\to\infty} \sigma_{qp}(\Gamma\eta_n, \Gamma\varrho)$$

and

$$\limsup_{n\to\infty} \sigma_p(\eta_n, \Gamma\varrho) \leq \Omega(\sigma_{qp}(\varrho, \Gamma\varrho)).$$

Now, from (1), we have

$$\begin{aligned}
\theta\big(\sigma_{qp}(\varrho, \Gamma\varrho)\big) &\leq \theta\big(\Omega(\Omega^{-1}(\sigma_{qp}(\varrho, \Gamma\varrho))\big) \leq \theta\big(\Omega(\limsup_{n\to\infty} \sigma_{qp}(\Gamma\eta_n, \Gamma\varrho))\big) \\
&\leq \big[\limsup_{n\to\infty} \theta\big(\lambda_1\sigma_{qp}(\eta_n, \varrho) + \lambda_2\sigma_{qp}(\eta_n, \Gamma\eta_n) + \lambda_3\sigma_{qp}(\varrho, \Gamma\varrho)\big]^\lambda \\
&\leq \big[\theta\big(\lambda_3\sigma_{qp}(\varrho, \Gamma\varrho)\big)\big]^\lambda \\
&< \big[\theta\big(\sigma_{qp}(\varrho, \Gamma\varrho)\big)\big]^\lambda,
\end{aligned}$$

a contradiction. Thus $\sigma_{qp}(\varrho, \Gamma\varrho) = 0$. ■

Example 4 *Let $\Lambda = [0, 1]$. Define the mapping $\sigma_{qp} : \Lambda \times \Lambda \to \mathbb{R}_0^+$ by*

$$\sigma_{qp}(\eta_1, \eta_2) = e^{[\eta_1^2+\eta_2^2]^2+\eta_1^2} - 1.$$

Define $\alpha : \Lambda \times \Lambda \to \mathbb{R}_0^+$ by

$$\alpha(\eta_1, \eta_2) = \begin{cases} 1, & \text{if } \eta_1 \geq \eta_2, \\ \frac{1}{9}, & \text{otherwise.} \end{cases}$$

Then (Λ, σ_{qp}) is a right α-complete (QPMLS) with $\Omega(t) = e^t - 1$. Let $\lambda = \frac{1}{\sqrt{2}}$ and $\theta(t) = e^{te^t}$. Define $\Gamma : \Lambda \to \Lambda$ by

$$\Gamma\eta_1 = \ln(1 + \frac{\eta_1}{16}).$$

Note that Γ is an α-admissible and continuous self map on Λ and $\alpha(1, \Gamma1) \geq 1$. Also, we have

$$\begin{aligned}
\sigma_{qp}(\Gamma\eta_1, \Gamma\eta_2) &= e^{[(\Gamma\eta_1)^2+(\Gamma\eta_2)^2]^2+(\Gamma\eta_1)^2} - 1 \\
&= e^{\left[\left(\ln(1+\frac{\eta_1}{16})\right)^2+\left(\ln(1+\frac{\eta_2}{16})\right)^2\right]^2+\left(\ln(1+\frac{\eta_1}{16})\right)^2} - 1 \\
&\leq e^{\left[\left(\frac{\eta_1}{16}\right)^2+\left(\frac{\eta_2}{16}\right)^2\right]^2+\left(\frac{\eta_1}{16}\right)^2} - 1 \\
&\leq e^{\frac{1}{256}\left([\eta_1^2+\eta_2^2]^2+\eta_1^2\right)} - 1 \\
&\leq \frac{1}{256}\sigma_{qp}(\eta_1, \eta_2).
\end{aligned}$$

Hence,

$$\Omega^2[\sigma_{qp}(\Gamma\eta_1,\Gamma\eta_2)] = e^{e^{([(\Gamma\eta_1)^2+(\Gamma\eta_2)^2]^2+(\Gamma\eta_1)^2)}-1} - 1 \le e^{\frac{1}{256}\sigma_{qp}(\eta_1,\eta_2)} - 1 \le \frac{1}{256}\sigma_{qp}(\eta_1,\eta_2),$$

and so

$$\begin{aligned}
\theta(\Omega^2[\sigma_{qp}(\Gamma\eta_1,\Gamma\eta_2)]) &= e^{\Omega^2[\sigma_{qp}(\Gamma\eta_1,\Gamma\eta_2)]e^{\Omega^2[\sigma_{qp}(\Gamma\eta_1,\Gamma\eta_2)]}} \\
&\le e^{\frac{1}{256}\sigma_{qp}(\eta_1,\eta_2)e^{\frac{1}{256}\sigma_{qp}(\eta_1,\eta_2)}} \\
&\le \left[e^{\frac{1}{16}\sigma_p(\eta_1,\eta_2)e^{\frac{1}{16}\sigma_p(\eta_1,\eta_2)}}\right]^{\frac{1}{\sqrt{2}}} \\
&= [\theta(\frac{1}{16}\sigma_{qp}(\eta_1,\eta_2))]^{\frac{1}{\sqrt{2}}} \\
&\le [\theta(\lambda_1\sigma_{qp}(\eta_1,\eta_2)+\lambda_2\sigma_{qp}(\eta_1,\Gamma\eta_1)+\lambda_3\Omega^{-1}\sigma_{qp}(\eta_2,\Gamma\eta_2)]^{\frac{1}{\sqrt{2}}}.
\end{aligned}$$

Thus, (1) is satisfied with $\lambda_1 = \frac{1}{16}$ and $\lambda_i = 0$ for $i \in \{2,3\}$. Moreover, 0 is a fixed point of Γ.

Now, we have the following definition.

Definition 13 *Let (Λ, σ_{qp}) be a (QPMLS) and $\alpha : \Lambda \times \Lambda \to \mathbb{R}_0^+$ be a given mapping. A mapping $\Gamma : \Lambda \to \Lambda$ is called left Jleli-Samet-Reich (JSR) contraction, if for any $\eta_1, \eta_2 \in \Lambda$ with $1 \le \alpha(\eta_1,\eta_2)$ and $\Gamma\eta_1 \ne \Gamma\eta_2$, we have*

$$\theta(\Omega^2(\sigma_{qp}(\Gamma\eta_1,\Gamma\eta_2))) \le [\theta(\lambda_1\sigma_{qp}(\eta_1,\eta_2)+\lambda_2\sigma_{qp}(\Gamma\eta_1,\eta_1)+\lambda_3\sigma_{qp}(\Gamma\eta_2,\eta_2))]^\lambda, \tag{8}$$

where $\theta \in \Delta_\theta$, $\lambda, \lambda_i \in [0,1)$ and $\lambda_1+\lambda_2+\lambda_3 < 1$.

Following arguments similar to those in Theorems 1 and 2, we have the following theorems.

Theorem 3 *Let (Λ, σ_{qp}) be a left α-complete (QPMLS). Suppose that $\Gamma : \Lambda \to \Lambda$ is a continuous triangular α -admissible and a left JSR-contraction. If there exists $\eta_0 \in \Lambda$ such that $\alpha(\eta_0, \Gamma\eta_0) \ge 1$, then Γ has a fixed point.*

Theorem 4 *Let (Λ, σ_{qp}) be a left α-complete (QPMLS). Suppose that $\Gamma : \Lambda \to \Lambda$ is a triangular α-admissible and a left JSR-contraction. If there exists $\eta_0 \in \Lambda$ such that $\alpha(\eta_0, \Gamma\eta_0) \ge 1$, then Γ has a fixed point provided that for any $\{\eta_n\}$ in Λ with $\alpha(\eta_{n+1},\eta_n) \ge 1$ and $\eta_n \to \eta$ as $n \to \infty$, we have $1 \le \alpha(\eta,\eta_n)$ for all $n \in \mathbb{N}$.*

4 Existence of a Solution for an Integral Equation

Consider the following integral equation

$$\eta(t) = \int_a^b G(t,r)K(t,r,\eta(r))\,dr, \quad t \in \mathcal{J} = [a,b], \tag{9}$$

where $K : \mathcal{J} \times \mathcal{J} \times \mathbb{R} \to \mathbb{R}$ and $G(t,r)$ is the Green function. The purpose of this section is to present an existence theorem for a solution to (9) that belongs to $\Lambda := \mathcal{C}(\mathcal{J}, \mathbb{R})$ (the set of continuous real functions defined on $\mathcal{J}$), by using the obtained result in Theorem 2.

Let $\Gamma : \Lambda \to \Lambda$ be the mapping defined by

$$\Gamma\eta(t) = \int_a^b G(t,r)K(t,r,\eta(r))\,dr, \tag{10}$$

for all $\eta \in \Lambda$ and $t \in \mathcal{J}$. Then the existence of a solution to (9) is equivalent to the existence of a fixed point of Γ.

Define $\sigma_{qp} : \Lambda \times \Lambda \to \mathbb{R}_0^+$ by

$$\sigma_{qp}(\eta_1, \eta_2) = \xi(\sigma(\eta_1, \eta_2)) \text{ for all } \eta_1, \eta_2 \in \Lambda,$$

where $\xi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a strictly increasing function with $t \leq \xi(t)$ and $\xi(0) = 0$, and

$$\sigma(\eta_1, \eta_2) = \max_{t \in \mathcal{J}} [\, |\eta_1(t) - \eta_2(t)|^p + |\eta_1(t)|^p \,].$$

Then (Λ, σ_{qp}) is a complete quasi p-metric-like space.

Now, we will prove the following result.

Theorem 5 *Suppose that the following hypotheses hold:*

(A) $K : \mathcal{J} \times \mathcal{J} \times \mathbb{R} \to \mathbb{R}$ *is continuous;*

(B) there exists a function $\mu : \Lambda \times \Lambda \to \mathbb{R}$ *such that if* $\mu(\eta_1, \eta_2) \geq 0$ *for all* $\eta_1, \eta_2 \in \Lambda$*, then we have*

$$\begin{aligned}
&\xi^2(\int_a^b [|K_1(t, r, \eta_1(r)) - K_2(t, r, \eta_2(r))| + |K_1(t, r, \eta_1(r))|]^p dr) \\
&\leq \frac{1}{\zeta}[\alpha[|\eta_1(r) - \eta_2(r)|^p + |\eta_1(r)|^p] + \beta[|\eta_1(r) - \Gamma\eta_1(r)|^p + |\eta_1(r)|^p] \\
&\quad + \gamma[|\eta_2(r) - \Gamma\eta_2(r)|^p + |\eta_2(r)|^p]],
\end{aligned}$$

where $\zeta > 1$ *and* $\alpha, \beta, \gamma \in [0, 1)$*, with* $\alpha + \beta + \gamma < 1$*;*

(C) for all $t \in \mathcal{J}$*, we have* $(\int_a^b |G(t, r)|^q dr)^{\frac{1}{q}} < 1$ *(note that* $\frac{1}{p} + \frac{1}{q} = 1$*);*

(D) there exists $\eta_0 \in \Lambda$ *such that* $\mu(\eta_0, \Gamma\eta_0) \geq 0$*;*

(E) for all $\eta_1, \eta_2 \in \Lambda$*,* $\mu(\eta_1, \eta_2) \geq 0$ *implies* $\mu(\Gamma\eta_1, \Gamma\eta_2) \geq 0$ *and for all* $\eta_1, \eta_2, \eta_3 \in \Lambda$*,* $\mu(\eta_1, \eta_2) \geq 0$ *and* $\mu(\eta_2, \eta_3) \geq 0$ *imply* $\mu(\eta_1, \eta_3) \geq 0$*;*

(F) if $\{\eta_n\}$ *is a sequence in* Λ *such that* $\eta_n \to \eta \in \Lambda$ *and* $\mu(\eta_n, \eta_{n+1}) \geq 0$ *for all* n*, then* $\mu(\eta_n, \eta) \geq 0$ *for all* n*.*

Then, the integral equation (9) has a solution $\eta \in \Lambda$*.*

Proof. Let $\eta_1, \eta_2 \in \Lambda$ such that $\mu(\eta_1(t), \eta_2(t)) \geq 0$ and $\Gamma\eta_1(t) \neq \Gamma\eta_2(t)$ for all $t \in \mathcal{J}$. Then since $\ln\theta(\lambda t) \leq \lambda \ln\theta(t)$, from (B), we deduce

$$\begin{aligned}
&\ln\theta\big(\xi^2(|\Gamma\eta_1(t) - \Gamma\eta_2(t)|^p + |\Gamma\eta_1(t)|^p)\big) \\
&\leq \ln\theta[\xi^2(\int_a^b |G(t, r)|[|K_1(t, r, \eta_1(r)) - K_2(t, r, \eta_2(r))| + |K_1(t, r, \eta_1(r))|]dr)^p] \\
&\leq \ln\theta[\xi^2(\int_a^b |G(t, r)|^q dr)^{\frac{1}{q}}(\int_a^b [|K_1(t, r, \eta_1(r)) - K_2(t, r, \eta_2(r))| + |K_1(t, r, \eta_1(r))|]^p dr)^{\frac{1}{p}})^p] \\
&\leq \ln\theta[\xi^2(\int_a^b [|K_1(t, r, \eta_1(r)) - K_2(t, r, \eta_2(r))| + |K_1(t, r, \eta_1(r))|]^p dr)] \\
&\leq \ln\theta[\frac{1}{\zeta}(\alpha[|\eta_1(r) - \eta_2(r)|^p + |\eta_1(r)|^p] + \beta[|\eta_1(r) - \Gamma\eta_1(r)|^p + |\eta_1(r)|^p] \\
&\quad + \gamma[|\eta_2(r) - \Gamma\eta_2(r)|^p + |\eta_2(r)|^p])] \\
&\leq \ln(\theta(\frac{M(\eta_1, \eta_2)}{\zeta}))) \\
&\leq \frac{1}{\zeta}(\ln(\theta(M(\eta_1, \eta_2)))),
\end{aligned}$$

which implies

$$\theta\big(\xi^2(\sigma_{qp}(\Gamma\eta_1,\Gamma\eta_2))\big) \leq \big[\theta\big(M(\eta_1,\eta_2)\big)\big]^{\frac{1}{\xi}},$$

for all $\eta_1, \eta_2 \in \Lambda$ with $\mu(\eta_1, \eta_2) \geq 0$ and $\Gamma\eta_1 \neq \Gamma\eta_2$, where

$$M(\eta_1,\eta_2) = \alpha\sigma_{qp}(\eta_1,\eta_2) + \beta\sigma_{qp}(\eta_1,\Gamma\eta_1) + \gamma\sigma_{qp}(\eta_2,\Gamma\eta_2).$$

Define the function $\alpha : \Lambda \times \Lambda \to \mathbb{R}_0^+$ by

$$\alpha(\eta_1,\eta_2) = \begin{cases} 1, & \text{if } \mu(\eta_1(t), \eta_2(t)) \geq 0, \ t \in \mathcal{J}, \\ 0, & \text{otherwise.} \end{cases}$$

Also, putting $\Omega = \xi$ and $\lambda = \frac{1}{\xi}$, we get

$$\theta\big(\Omega^2(\sigma_{qp}(\Gamma\eta_1,\Gamma\eta_2))\big) \leq \big[\theta\big(M(\eta_1,\eta_2)\big)\big]^{\lambda},$$

for all $\eta_1, \eta_2 \in \Lambda$ with $\alpha(\eta_1, \eta_2) \geq 1$ and $\Gamma\eta_1 \neq \Gamma\eta_2$.
It is easy to show that all the hypotheses of Theorem 2 are satisfied and hence the mapping Γ has a fixed point, that is, there exists a solution in $\Lambda = \mathcal{C}(\mathcal{J}, \mathbb{R})$ for the integral equation (9). ■

Acknowledgments. The authors are thankful to the learned referee and editor for their valuable remarks and suggestions to improve this work.

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