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A Modified Mann Algorithm For Solving Convex Minimization And Fixed Point Problems With Composed Nonlinear Operators*

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Abstract

The main aim of this paper is to introduce and study an iterative algorithm, which is based on the Krasnoselskii-Mann iterative method and the gradient-projection algorithm for solving a constrained convex minimization problem and fixed point problem with quasi-nonexpansive and firmly nonexpansive mappings in a real Hilbert space. Finally, we apply our main result for finding a common solution of convex minimization problem, fixed point problem and equilibrium problem. Essentially, a new approach for solving some nonlinear problems is provided.

1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|.\|$. Let K be a nonempty, closed and convex subset of H. Consider the following constrained convex minimization problem:

$$\min_{y \in K} g(y), \tag{1}$$

where $g : K \to \mathbb{R}$ is a convex function. Assume that (1) is consistent (i.e., it has a solution) and we use Ω to denote its solution set. It is well known that the gradient-projection algorithm (GPA, for short) is usually applied to solve the minimization problem (1). This algorithm generates a sequence $\{x_n\}$ through the recursion:

$$x_{n+1} = P_K(x_n - \lambda_n \nabla g(x_n)), \ \ n \geq 0, \tag{2}$$

where the initial guess $x_0 \in K$ is chosen arbitrarily and $\{\lambda_n\}$ is a sequence of stepsizes which may be chosen in different ways. GPA (2) has well been studied in the case of constant stepsizes $\lambda_n = \lambda$ for all n (see the books [27, 28]). A fundamental convergence result for GPA (2) is the following one which can be found in literature (cf. [[28], Theorem 6.1] with constant stepsize).

Theorem 1 ([28]) *Let $\{x_n\}$ be the sequence generated by GPA (2). Assume*

(i) g is continuously differentiable and its gradient is Lipschitz continuous:

$$\|\nabla g(x) - \nabla g(y)\| \leq L\|x - y\|, \ \forall x, y \in \mathbb{R}^n,$$

where $L \geq 0$ is a constant;

(ii) the set $K_0 := \{x \in K : g(x) \leq g(x_0)\}$ is bounded;

(iii) the sequence $\{\lambda_n\}$ satisfies the condition:

$$0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{2}{L}.$$

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Then, the sequence $\{x_n\}$ *generated by the gradient-projection algorithm* (2) *converges weakly to a solution of* (1).

However, Xu [33] constructed a counterexample which shows that algorithm (2) does not converge in norm in an infinite-dimensional space, and also presented two modifications of gradient-projection algorithms which are shown to have strong convergence.

In 2012, H. Iiduka [19] introduced the following algorithm for solving problem

Algorithm 1 Step 0. *Choose* $x_0 \in H$ *arbitrarily, set* $\lambda_0 \subset (0,1)$, $\alpha_0 \subset (0,1)$ *and* $d_0 = -\nabla g(x_0)$ *arbitrarily and let* $n := 0$.

Step 1. *Given.* $x_n \in H$ *and* $d_n \in H$, *choose* $\lambda_n \subset (0,1)$, $\alpha_n \subset (0,1)$ *and compute* $x_{n+1} \in K$ *as*

$$\begin{cases} y_n = T(x_n + \lambda_n d_n), \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n) y_n. \end{cases}$$

Step 2. *Choose* $\beta_{n+1} \in (0,1]$ *and compute the direction* $d_{n+1} \in H$, *by*

$$d_{n+1} = -\nabla g(x_n) + \beta_{n+1} d_n.$$

Update $n := n+1$ and go to Step 1.

Under suitable conditions, he proved that $\{x_n\}_{n\in\mathbb{N}}$ in Algorithm 1 weakly converges to a unique solution to Problem (1). Recently, studies on solutions of the minization problem (1) were extensively carried out in Hilbert spaces and in certain Banach spaces; see, for example, [18, 8, 17, 20, 5, 30, 3, 6] and the references therein.

Recall that a mapping $T : K \to H$ is called L-Lipschitzian if for all $x, y \in K$,

$$\|Tx - Ty\| \leq L\|x - y\|,$$

where $L \geq 0$ is a constant. In particular, if $L \in [0,1)$ then T is called a contraction mapping; if $L = 1$ then T is called a nonexpansive mapping. A point $x \in K$ is called a fixed point of T if $Tx = x$. We denote the set of all fixed points of T by $Fix(T)$. A mapping T is said to be

(1) quasi-nonexpansive if $Fix(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\|, \;\; x \in K, \;\; p \in Fix(T);$$

(2) firmly nonexpansive if for all $x, y \in K$, we have

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle.$$

For nonexpansive mappings with fixed points, Mann iterative method [23] is a valuable tool to study them. Mann's scheme is defined by:

$$\begin{cases} x_0 \in K, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) T x_n, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. But Mann's iteration process has only weak convergence, even in Hilbert space setting. Hence the modification is necessary in order to guarantee the strong convergence of Mann's method. Lot of works have been done for the modification of the Mann's iteration so that strong convergence is guaranteed. See, e.g., [31, 35, 36, 16, 25, 22, 11, 12, 29, 13, 26] and the reference therein.

If T_1 and T_2 are self-mappings on K, a point $x \in K$ is called a common fixed point of $T_i (i = 1, 2)$ if $x \in Fix(T_1) \cap Fix(T_2)$. To find a solution of the common fixed point problems, several iterative approximation methods were introduced and studied. This problem can be applied in solving solutions of various problems in science and applied science, see [15, 21, 9] for instance. For almost all the results on common fixed point of nonlinear mappings in Hilbert spaces, commuting assumptions are needed on the operators.

Now, we introduce two minimization problems coupled with fixed point problems, firstly, we consider the constrained convex minimization problem coupled with fixed point problem involving two mappings, namely, find an x^* with the property:

$$x^* \in \Omega \cap Fix(T_1) \cap Fix(T_2). \tag{3}$$

On the other hand, we consider the constrained convex minimization problem coupled with fixed point problem involving composed mapping, namely, find an x^* with the property:

$$x^* \in \Omega \cap Fix(T_1 \circ T_2), \tag{4}$$

where T_1 and T_2 be quasi-nonexpansive and firmly nonexpansive mappings on K, respectively.

Remark 1 *Easily, we obtain the following conclusions:*

(i) $Fix(T_1) \cap Fix(T_2) \subset Fix(T_1 \circ T_2)$;

(ii) Problem of finding an element of $\Omega \cap Fix(T_1 \circ T_2)$ *is more general and more complex than the problem of finding an element of* $\Omega \cap Fix(T_1) \cap Fix(T_2)$.

Above discussion suggests the following questions.

Question 1: Is it always true that the set of solutions of problem (3) coincides with the set of solutions of problem (4) without commuting assumptions?

Question 2: Could we construct an explicit algorithm based on a modified Mann iterative method and the gradient-projection algorithm such that it converges strongly to a solution of problem (4) without compactness assumption?

The purpose of this paper is to give affirmative answers to these questions mentioned above. Applications are also considered.

2 Preliminaries

Recall that a map $A : H \to H$, the domain of A, $D(A)$, the image of a subset S of H, $A(S)$ the range of A, $R(A)$ and the graph of A, $G(A)$ are defined as follows:

$$D(A) := \{x \in H : Ax \neq \emptyset\}, \ A(S) := \cup\{Ax : x \in S\},$$

$$R(A) := A(H), \ G(A) := \{[x, u] : x \in D(A), \ u \in Ax\}.$$

Let K be a nonempty, closed and convex subset of H. An operator $A : K \to H$ is called *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall\, x, y \in K.$$

An operator $A : K \to H$ is said *α-inverse strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, \ x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall\, x, y \in K.$$

The demiclosedness of a nonlinear operator T usually plays an important role in dealing with the convergence of fixed point iterative algorithms.

Definition 1 *Let H be a real Hilbert space and $T : D(T) \subset H \to H$ be a mapping. $I - T$ is said to be demiclosed at 0 if for any sequence $\{x_n\} \subset D(T)$ such that $\{x_n\}$ converges weakly to p and $\|x_n - Tx_n\|$ converges to zero, then $p \in Fix(T)$.*

Lemma 2 ([4]) *Let H be a real Hilbert space, K be a closed convex subset of H, and $T : K \to K$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Then $I - T$ is demiclosed.*

Lemma 3 ([10]) *Let H be a real Hilbert space. Then for any $x, y \in H$, the following inequalities hold:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle,$$
$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - (1-\lambda)\lambda\|x - y\|^2, \;\; \lambda \in (0,1).$$

Lemma 4 ([34]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n$ for all $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that*

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(b) $\limsup_{n\to\infty} \sigma_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\sigma_n\alpha_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 5 ([24]) *Let t_n be a sequence of real numbers that does not decrease at infinity in a sense that there exists a subsequence t_{n_i} of t_n such that t_{n_i} such that $t_{n_i} \leq t_{n_{i+1}}$ for all $i \geq 0$. For sufficiently large numbers $n \in \mathbb{N}$, an integer sequence $\{\tau(n)\}$ is defined as follows:*

$$\tau(n) = \max\{k \leq n : \;\; t_k \leq t_{k+1}\}.$$

Then, $\tau(n) \to \infty$ as $n \to \infty$ and

$$\max\{t_{\tau(n)}, \;\; t_n\} \leq t_{\tau(n)+1}.$$

Lemma 6 *Let H be a real Hilbert space and K be a nonempty, closed convex subset of H. Let $A : K \to H$ be an α-inverse strongly monotone mapping. Then, $I - \theta A$ is nonexpansive mapping for all $x, y \in K$ and $\theta \in [0, 2\alpha]$.*

Proof. For all $x, y \in K$, we have

$$\begin{aligned} \|(I-\theta A)x - (I-\theta A)y\|^2 &= \|(x-y) - \theta(Ax - Ay)\|^2 \\ &= \|x-y\|^2 - 2\theta\langle Ax - Ay, x - y\rangle + \theta^2\|Ax - Ay\|^2 \end{aligned}$$

By using property of A and $\theta \in [0, 2\alpha]$, we have

$$\|(I-\theta A)x - (I-\theta A)y\|^2 = \|x-y\|^2 + \theta(\theta - 2\alpha)\|Ax - Ay\|^2 \leq \|x-y\|^2.$$

This shows that $I - \theta A$ is nonexpansive. ■

Lemma 7 ([2]) *Let H be a real Hilbert space, g a continuously Fréchet differentiable, convex functional on H and ∇g the gradient of g. If ∇g is $\frac{1}{\alpha}$-Lipschitz continuous, then ∇g is α-inverse strongly monotone.*

Lemma 8 *Let H be a real Hilbert space and K be a nonempty, closed convex subset of H. Let $g : K \to \mathbb{R}$ be a continuously Fréchet differentiable, convex functional on K with a $\dfrac{1}{\alpha}$-Lipschitz continuous ∇g. Then, $I - \theta\nabla g$ is nonexpansive mapping for all $x, y \in K$ and $\theta \in [0, 2\alpha]$.*

Proof. The proof follows Lemmas 6 and 7. ■

Remark 2 *A necessary condition of optimality for a point $x^* \in \Omega$ is that $x^* \in VI(\nabla g, K)$, where*

$$VI(\nabla g, K) := \{x^* \in K, \;\; \langle\nabla g(x^*), x - x^*\rangle \geq 0, \;\; \forall x \in K\}.$$

Lemma 9 ([18]) *Let K be a nonempty closed convex of a real Hilbert H. Let $g : K \to \mathbb{R}$ be a continuously Fréchet differentiable, convex functional on K with a $\dfrac{1}{\alpha}$-Lipschitz continuous ∇g. Then for all $\lambda > 0$,*

$$VI(\nabla g, K) = Fix(P_K(I - \lambda\nabla g)).$$

3 Main Results

We start by the following result.

Lemma 10 *Let H be a real Hilbert space and let K be a nonempty closed convex subset of H. Let $T_1 : K \to K$ be a quasi-nonexpansive mapping and $T_2 : K \to K$ be a firmly nonexpansive mapping. Then, $Fix(T_1) \cap Fix(T_2) = Fix(T_1 \circ T_2)$ and $T_1 \circ T_2$ is a quasi-nonexpansive mapping on K.*

Proof. We split the proof into two steps.
Step 1: First, we show that $Fix(T_1) \cap Fix(T_2) = Fix(T_1 \circ T_2)$. We note that

$$Fix(T_1) \cap Fix(T_2) \subset Fix(T_1 \circ T_2).$$

Thus, we only need to show that $Fix(T_1 \circ T_2) \subseteq Fix(T_1) \cap Fix(T_2)$. Let $p \in Fix(T_1) \cap Fix(T_2)$ and $q \in Fix(T_1 \circ T_2)$. By using properties of T_1 and T_2, we have

$$\|q - p\|^2 = \|T_1 \circ T_2 q - T_1 p\|^2 \leq \|T_2 q - p\|^2. \tag{5}$$

Using the fact that T_2 is firmly nonexpansive, we have

$$\begin{aligned} \|T_2 q - p\|^2 &\leq \langle T_2 q - p, q - p\rangle \\ &= \frac{1}{2}(\|T_2 q - p\|^2 + \|q - p\|^2 - \|T_2 q - q\|^2. \end{aligned} \tag{6}$$

By virtue of (6), we can infer that

$$\|T_2 q - p\|^2 \leq \|q - p\|^2 - \|T_2 q - q\|^2. \tag{7}$$

Using (5) implies that (7) becomes

$$\|T_2 q - p\|^2 \leq \|q - p\|^2 - \|T_2 q - q\|^2 \leq \|T_2 q - p\|^2 - \|T_2 q - q\|^2.$$

Clearly, $\|T_2 q - q\| = 0$ which implies that

$$q = T_2 q.$$

Keeping in mind that $T_1 \circ T_2 q = q$, we have

$$q = T_1 \circ T_2 q = T_1 q.$$

Thus, $q \in Fix(T_1) \cap Fix(T_2)$. Hence, $Fix(T_1) \cap Fix(T_2) = Fix(T_1 \circ T_2)$.
Step 2: We show $T_1 \circ T_2$ is a quasi-nonexpansive mapping on K. Let $x \in K$ and $p \in Fix(T_1 \circ T_2)$. Then, $p \in Fix(T_1) \cap Fix(T_2)$ by step 1. We observe that,

$$\|T_1 \circ T_2 x - p\| = \|T_1 \circ T_2 x - T_1 p\| \leq \|T_2 x - p\| \leq \|x - p\|.$$

This completes the proof. ■

We now prove the following theorem.

Theorem 11 *Let H be a real Hilbert space and K a nonempty, closed convex cone of H. Let $g : K \to \mathbb{R}$ be a continuously Fréchet differentiable, convex functional on K with a $\frac{1}{\alpha}$-Lipschitz continuous ∇g. Let $T_1 : K \to K$ be a quasi-nonexpansive mapping and $T_2 : K \to K$ be a firmly nonexpansive mapping such that $\Gamma := \Omega \cap Fix(T_1) \cap Fix(T_2) \neq \emptyset$. Assume that $I - T_1 \circ T_2$ is demiclosed at origin and $\lambda \in (0, 2\alpha)$. Let $\{x_n\}$ be a sequence generated iteratively from arbitrary $x_0 \in K$ by*

$$\begin{cases} z_n = \theta_n x_n + (1 - \theta_n) T_1 \circ T_2 x_n, \quad y_n = \beta_n z_n + (1 - \beta_n) P_K (I - \lambda \nabla g) z_n, \quad x_{n+1} = \alpha_n (\lambda_n x_n) + (1 - \alpha_n) y_n, \\ y_n = \beta_n z_n + (1 - \beta_n) P_K (I - \lambda \nabla g) z_n, \qquad x_{n+1} = \alpha_n (\lambda_n x_n) + (1 - \alpha_n) y_n, \\ x_{n+1} = \alpha_n (\lambda_n x_n) + (1 - \alpha_n) y_n, \end{cases}$$

where $\{\alpha_n\}$, $\{\theta_n\}$, $\{\lambda_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0, \lim_{n\to\infty} \lambda_n = 1$ *and* $\sum_{n=0}^{\infty}(1-\lambda_n)\alpha_n = \infty$,

(ii) $\liminf_{n\to\infty} \theta_n(1-\theta_n) > 0$,

(iii) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$.

Then, the sequence $\{x_n\}$ *converges strongly to* $x^* \in \Gamma$, *where* $x^* = P_\Gamma(0)$.

Proof. First of all, we prove that the sequence $\{x_n\}$ is bounded. Indeed, if we let $p \in \Gamma$, by using Lemma 3, we get

$$\begin{aligned}\|z_n - p\|^2 &= \Big\|\theta_n(x_n - p) + (1-\theta_n)(T_1 \circ T_2 x_n - p)\Big\|^2 \\ &= \theta_n\|x_n - p\|^2 + (1-\theta_n)\|T_1 \circ T_2 x_n - p\|^2 - \theta_n(1-\theta_n)\|T_1 \circ T_2 x_n - x_n\|^2 \\ &\leq \|x_n - p\|^2 - \theta_n(1-\theta_n)\|T_1 \circ T_2 x_n - x_n\|^2.\end{aligned}$$

Since $\theta_n \in (0,1)$, we get that

$$\|z_n - p\|^2 \leq \|x_n - p\|^2. \tag{8}$$

By using the definition of $\{x_n\}$ and Lemma 8, it follows that

$$\begin{aligned}\|y_n - p\| &= \|\beta_n z_n + (1-\beta_n)P_K(I - \lambda\nabla g)z_n - p\| \\ &\leq \beta_n\|z_n - p\| + (1-\beta_n)\|P_K(I - \lambda\nabla g)z_n - p\| \\ &\leq \|z_n - p\|.\end{aligned} \tag{9}$$

From (8) and (9), we have

$$\|y_n - p\| \leq \|z_n - p\| \leq \|x_n - p\|. \tag{10}$$

From (10), we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n - p\| \\ &\leq \alpha_n\lambda_n\|x_n - p\| + (1-\alpha_n)\|y_n - p\| + (1-\lambda_n)\alpha_n\|p\| \\ &\leq \alpha_n\lambda_n\|x_n - p\| + (1-\alpha_n)\|x_n - p\| + (1-\lambda_n)\alpha_n\|p\| \\ &\leq [1 - (1-\lambda_n)\alpha_n]\|x_n - p\| + (1-\lambda_n)\alpha_n\|p\| \\ &\leq \max\{\|x_n - p\|,\ \|p\|\}.\end{aligned}$$

By induction, we get

$$\|x_n - p\| \leq \max\{\|x_0 - p\|,\ \|p\|\}, \quad n \geq 1.$$

Then, we obtain that $\{x_n\}$ is bounded, and so are $\{y_n\}$, $\{z_n\}$. By (8) and convexity of $\|.\|^2$, we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq \|\alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n - p\|^2 \\ &\leq \alpha_n\|(\lambda_n x_n) - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\ &\leq \alpha_n\|(\lambda_n x_n) - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 \\ &\leq \alpha_n\|(\lambda_n x_n) - p\|^2 + (1-\alpha_n)\Big[\|x_n - p\|^2 \\ &\quad -\theta_n(1-\theta_n)\|T_1 \circ T_2 x_n - x_n\|^2\Big].\end{aligned}$$

Thus,

$$(1-\alpha_n)(1-\theta_n)\theta_n\|T_1 \circ T_2 x_n - x_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n\|(\lambda_n x_n) - p\|^2.$$

Hence,

$$(1-\alpha_n)(1-\theta_n)\theta_n\|T_1 \circ T_2 x_n - x_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n\|(\lambda_n x_n) - p\|^2. \tag{11}$$

Now we divide the rest of the proof into two cases.

Case I. Assume that there is $n_0 \in N$ such that $\{\|x_n - p\|\}$ is decreasing for all $n \geq n_0$. Since $\{\|x_n - p\|\}$ is monotonic and bounded, $\{\|x_n - p\|\}$ is convergent. Clearly, we have

$$\lim_{n\to\infty} \left[\|x_n - p\|^2 - \|x_{n+1} - p\|^2\right] = 0.$$

It then implies from (11) that

$$\lim_{n\to\infty} (1-\theta_n)\theta_n \|T_1 \circ T_2 x_n - x_n\|^2 = 0. \tag{12}$$

Since $\theta_n \in (0,1)$ and $\liminf_{n\to\infty} \theta_n(1-\theta_n) > 0$, we have

$$\lim_{n\to\infty} \left\|x_n - T_1 \circ T_2 x_n\right\| = 0. \tag{13}$$

Now, we observe that,

$$\begin{aligned} \|z_n - x_n\| &= \|(1-\theta_n)x_n + \theta_n T_1 \circ T_2 x_n - x_n\| \\ &= \|(1-\theta_n)x_n + \theta_n T_1 \circ T_2 x_n - \theta_n x_n - (1-\theta_n)x_n\| \\ &\leq \|T_1 \circ T_2 x_n - x_n\|. \end{aligned}$$

Therefore, from (13) we have

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{14}$$

Then from Lemma 11, inequality (8) and the fact that $P_K(I - \lambda\nabla g)$ is nonexpasive, we have

$$\begin{aligned} \|y_n - p\|^2 &= \|\beta_n z_n + (1-\beta_n)P_K(I-\lambda\nabla g)z_n - p\|^2 \\ &= \beta_n\|z_n - p\|^2 + (1-\beta_n)\|P_K(I-\lambda\nabla g)z_n - p\|^2 \\ &\quad -\beta_n(1-\beta_n)\|P_K(I-\lambda\nabla g)z_n - z_n\|^2 \\ &\leq \|x_n - p\|^2 - \beta_n(1-\beta_n)\|P_K(I-\lambda\nabla g)z_n - z_n\|^2. \end{aligned} \tag{15}$$

By defintion of $\{x_n\}$ and the above inequality, we get

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n - p\|^2 \\ &\leq \alpha_n\|(\lambda_n x_n) - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\ &\leq \alpha_n\|(\lambda_n x_n) - p\|^2 + (1-\alpha_n)(\|x_n - p\|^2 \\ &\quad -\beta_n(1-\beta_n)\|P_K(I-\lambda\nabla g)z_n - z_n\|^2). \end{aligned}$$

Thus,

$$\begin{aligned} (1-\alpha_n)\beta_n(1-\beta_n)\|P_K(I-\lambda\nabla g)z_n - z_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\quad +\alpha_n\|(\lambda_n x_n) - p\|^2. \end{aligned}$$

Since $\beta_n \in (0,1)$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, we have

$$\lim_{n\to\infty} \|P_K(I-\lambda\nabla g)z_n - z_n\| = 0. \tag{16}$$

Since H is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that x_{n_k} converges weakly to a in K and

$$\limsup_{n\to+\infty} \langle x^*, x^* - x_n\rangle = \lim_{k\to+\infty} \langle x^*, x^* - x_{n_k}\rangle.$$

From (13) and $I - T_1 \circ T_2$ is demiclosed, we obtain $a \in Fix(T_1 \circ T_2)$. Using Lemma 10, we have $a \in Fix(T_2) \cap Fix(T_1)$. It follows from (16) and Lemma 2, we obtain $a \in Fix(P_K(I - \lambda\nabla g))$. By Lemma 9, we have $a \in VI(\nabla g, K)$. Therefore, $a \in \Gamma$. On other hand, using property of x^* ($x^* = P_\Gamma(0)$), we then have

$$\begin{aligned}\limsup_{n\to+\infty}\langle x^*, x^* - x_n\rangle &= \lim_{k\to+\infty}\langle x^*, x^* - x_{n_k}\rangle \\ &= \langle x^*, x^* - a\rangle \le 0.\end{aligned}$$

Finally, we show that $x_n \to x^*$. Applying Lemma 3, we get

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \langle x_{n+1} - x^*, x_{n+1} - x^*\rangle = \alpha_n\lambda_n\langle x_n - x^*, x_{n+1} - x^*\rangle \\ &\quad +(1-\lambda_n)\alpha_n\langle x^*, x^* - x_{n+1}\rangle + (1-\alpha_n)\langle y_n - x^*, x_{n+1} - x^*\rangle \\ &\le \alpha_n\lambda_n\langle x_n - x^*, x_{n+1} - x^*\rangle + (1-\lambda_n)\alpha_n\langle x^*, x^* - x_{n+1}\rangle \\ &\quad +(1-\alpha_n)\|y_n - x^*\|\|x_{n+1} - x^*\| \\ &\le \alpha_n\lambda_n\|x_n - x^*\|\|x_{n+1} - x^*\| + (1-\lambda_n)\alpha_n\langle x^*, x^* - x_{n+1}\rangle \\ &\quad +(1-\alpha_n)\|x_n - x^*\|\|x_{n+1} - x^*\| \\ &\le [1-(1-\lambda_n)\alpha_n]\|x_n - x^*\|\|x_{n+1} - x^*\| \\ &\quad +(1-\lambda_n)\alpha_n\langle x^*, x^* - x_{n+1}\rangle \\ &\le \frac{1-(1-\lambda_n)\alpha_n}{2}(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\ &\quad +(1-\lambda_n)\alpha_n\langle x^*, x^* - x_{n+1}\rangle,\end{aligned}$$

which implies that

$$\|x_{n+1} - x^*\|^2 \le [1-(1-\lambda_n)\alpha_n]\|x_n - x^*\| + 2(1-\lambda_n)\alpha_n\langle x^*, x^* - x_{n+1}\rangle.$$

We can check that all the assumptions of Lemma 4 are satisfied. Therefore, we deduce $x_n \to x^*$.

Case II. Assume that the sequence $\{\|x_n - x^*\|\}$ is not monotonically decreasing. Set $B_n = \|x_n - x^*\|$ and $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \ge$ (for some n_0 large enough) by $\tau(n) = \max\{k \in \mathbb{N} : k \le n,\ \ B_k \le B_{k+1}\}$. We have τ is a non-decreasing such that $\tau(n) \to \infty$ as $n \to \infty$ and $B_{\tau(n)} \le B_{\tau(n)+1}$ for $n \ge n_0$. From (11), we have

$$(1-\alpha_{\tau(n)})(1-\theta_{\tau(n)})\theta_{\tau(n)}\|x_{\tau(n)} - T_1 \circ T_2 x_{\tau(n)}\|^2 \le \alpha_{\tau(n)}\|\lambda_{\tau(n)}x_{\tau(n)} - x^*\|^2 \to 0 \text{ as } n \to \infty.$$

Since $\theta_n \in]0,1[$ and $\liminf_{n\to\infty}(1-\theta_{\tau(n)})\theta_{\tau(n)} > 0$, we can deduce

$$\lim_{n\to\infty}\|x_{\tau(n)} - T_1 \circ T_2 x_{\tau(n)}\| = 0. \tag{17}$$

By a similar argument as in case 1, we can show that $x_{\tau(n)}$ converges weakly in H and $\limsup_{n\to+\infty}\langle x^*, x^* - x_{\tau(n)}\rangle \le 0$. We have for all $n \ge n_0$,

$$0 \le \|x_{\tau(n)+1} - x^*\|^2 - \|x_{\tau(n)} - x^*\|^2 \le \Big(1 - \lambda_{\tau(n)}\Big)\alpha_{\tau(n)}[-\|x_{\tau(n)} - x^*\|^2 + 2\langle x^*, x^* - x_{\tau(n)+1}\rangle],$$

which implies that

$$\|x_{\tau(n)} - x^*\|^2 \le 2\langle x^*, x^* - x_{\tau(n)+1}\rangle.$$

Then, we have

$$\lim_{n\to\infty}\|x_{\tau(n)} - x^*\|^2 = 0.$$

Therefore,

$$\lim_{n\to\infty} B_{\tau(n)} = \lim_{n\to\infty} B_{\tau(n)+1} = 0.$$

Thus, by Lemma 5, we conclude that

$$0 \le B_n \le \max\{B_{\tau(n)},\ B_{\tau(n)+1}\} = B_{\tau(n)+1}.$$

Hence, $\lim_{n\to\infty} B_n = 0$, that is $\{x_n\}$ converges strongly to x^*. This completes the proof. ■

We now apply Theorem 11 when T_1 is a nonexpansive mapping. In this case demiclosedness assumption ($I - T_1 \circ T_2$ is demiclosed at origin) is not necessary.

Theorem 12 *Let H be a real Hilbert space and K a nonempty, closed convex cone of H. Let $g : K \to \mathbb{R}$ be a continuously Fréchet differentiable, convex functional on K with a $\frac{1}{\alpha}$-Lipschitz continuous ∇g. Let $T_1 : K \to K$ be a nonexpansive mapping and $T_2 : K \to K$ be a firmly nonexpansive mapping such that $\Gamma := \Omega \cap Fix(T_1) \cap Fix(T_2) \neq \emptyset$. Assume that $\lambda \in (0, 2\alpha)$. Let $\{x_n\}$ be a sequence generated iteratively from arbitrary $x_0 \in K$ by*

$$\begin{cases} z_n = \theta_n x_n + (1-\theta_n)T_1 \circ T_2 x_n, \quad y_n = \beta_n z_n + (1-\beta_n)P_K(I - \lambda\nabla g)z_n, \quad x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n, \\ y_n = \beta_n z_n + (1-\beta_n)P_K(I - \lambda\nabla g)z_n, \quad x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n, \\ x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n, \end{cases}$$

where $\{\alpha_n\}$, $\{\theta_n\}$, $\{\lambda_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0, \lim_{n\to\infty} \lambda_n = 1$ *and* $\sum_{n=0}^{\infty}(1-\lambda_n)\alpha_n = \infty$,

(ii) $\liminf_{n\to\infty} \theta_n(1-\theta_n) > 0$,

(iii) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$.

Then, the sequence $\{x_n\}$ converges strongly to $x^* \in \Gamma$ where $x^* = P_\Gamma(0)$.

Proof. We have $T_1 \circ T_2$ is nonexpansive mapping, then, the proof follows Lemma 2 and Theorem 11. ■

If $T_i \equiv I$, for $i = 1, 2$, then Theorem 11 is reduced to the following:

Theorem 13 *Let H be a real Hilbert space and K a nonempty, closed convex cone of H. Let $g : K \to \mathbb{R}$ be a continuously Fréchet differentiable, convex functional on K with a $\frac{1}{\alpha}$-Lipschitz continuous ∇g. Suppose that the minimization problem (1) is consistent and $\lambda \in (0, 2\alpha)$. Let $\{x_n\}$ be a sequence generated iteratively from arbitrary $x_0 \in K$ by:*

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)P_K(I - \lambda\nabla g)x_n, \quad x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n, \\ x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n, \end{cases}$$

where $\{\alpha_n\}$, $\{\lambda_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0, \lim_{n\to\infty} \lambda_n = 1$ *and* $\sum_{n=0}^{\infty}(1-\lambda_n)\alpha_n = \infty$,

(ii) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$.

Then, the sequence $\{x_n\}$ converges strongly to a minimizer of g.

If $g \equiv 0$, then Theorem 11 is reduced to the following:

Theorem 14 *Let H be a real Hilbert space and K a nonempty, closed convex cone of H. Let $T_1 : K \to K$ be a nonexpansive mapping and $T_2 : K \to K$ be a firmly nonexpansive mapping such that $Fix(T_1) \cap Fix(T_2) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined as follows:*

$$\begin{cases} x_0 \in K, \text{ choosenarbitrarily}, \quad z_n = \theta_n x_n + (1-\theta_n) T_1 \circ T_2 x_n, \quad x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n) z_n, \\ z_n = \theta_n x_n + (1-\theta_n) T_1 \circ T_2 x_n, \quad x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n) z_n, \\ x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n) z_n, \end{cases}$$

where $\{\alpha_n\}$, $\{\lambda_n\}$ and $\{\theta_n\}$ be sequences in $(0,1)$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0, \lim_{n\to\infty} \lambda_n = 1$ *and* $\sum_{n=0}^{\infty}(1-\lambda_n)\alpha_n = \infty$,

(ii) $\liminf_{n\to\infty} \theta_n(1-\theta_n) > 0$.

Then, the sequence $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2.

Remark 3 *In our theorems, we assume that K is a cone. But, in some cases, for example, if K is the closed unit ball, we can weaken this assumption to the following: $\lambda x \in K$ for all $\lambda \in (0,1)$ and $x \in K$. Therefore, our results can be used to approximate a common solution of convex minimization problem and fixed point problem involving composed operators from the closed unit ball to itself.*

4 Application to Some Nonlinear Problems

In this section, we apply our main results for finding a common solution of fixed points problem, convex minimization problem and equlibrium problem.

Problem 1 *Let K be a nonempty, closed convex subset of a real Hilbert space H. We consider the following minimization problem :*

$$\min_{x\in K} g(x), \tag{18}$$

where g be a continuously Fréchet differentiable, convex functional on K.

We denote the set of solutions of Problem 1 by Ω_1.

Problem 2 *Let K be a nonempty, closed convex subset of a real Hilbert space H. We consider the following fixed point problem :*

$$\text{find}\, x \in K \text{ such that}\, x = Tx, \tag{19}$$

where $T : K \to K$ be a quasi-nonexpansive mapping.

We denote the set of solutions of Problem 2 by Ω_2.

Problem 3 *Let $G : K \times K \to \mathbb{R}$ be a bifunction where $\mathbb{R}$ is the set of real numbers. The equilibrium problem corresponding to G is to find $x^* \in K$ such that*

$$G(x^*, y) \geq 0, \forall\, y \in K. \tag{20}$$

The set of solutions of Problem 3 is denoted by $EP(G)$. Numerous problems in physics, optimization, and economics are reduced to find the solution of an equilibrium problem (e.g., see [18]). For solving the equilibrium problem we assume that the bifunction G satisfies the following conditions:

(A1) $G(x,x) = 0$ for all $x \in K$;

(A2) G is monotone, i.e., $G(x,y)+G(y,x)\leq 0$ for all $x,y\in K$;

(A3) for each $x,y,z\in K$,

$$\lim_{t\to 0} G(tz+(1-t)x,y)\leq G(x,y);$$

(A4) for each $x\in K$, $y\to G(x,y)$ is convex and lower semicontinuous.

For solving Problem 3, we introduce the following lemma.

Lemma 15 **([32])** *Assume that $G: K\times K\to\mathbb{R}$ satisfies (A1)-(A4). For $r>0$ and $x\in H$, define a mapping $T_r^G: H\to K$ as follows*

$$T_r^G(x)=\{z\in K,\ G(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0,\quad \forall y\in K\},$$

for all $x\in H$. Then, the following hold:

1. *T_r^G is single-valued;*
2. *T_r^G is firmly nonexpansive;*
3. *$Fix(T_r^G)=EP(G)$;*
4. *$EP(G)$ is closed and convex.*

Therefore, by Theorem 11, the following result is obtained.

Theorem 16 *Let H be a real Hilbert space and K a nonempty, closed convex cone of H. Let $g: K\to\mathbb{R}$ be a continuously Fréchet differentiable, convex functional on K with a $\frac{1}{\alpha}$-Lipschitz continuous ∇g such that $\lambda\in(0,2\alpha)$ and, let $T: K\to K$ be a quasi-nonexpansive mapping. Let G be a bifunction from $K\times K\to\mathbb{R}$ satisfies (A1)-(A4) Such that $\Omega_1\cap\Omega_2\cap EP(G)\neq\emptyset$ and $I-T\circ T_\lambda^G$ is demiclosed at origin. Let $\{x_n\}$ be a sequence defined as follows:*

$$\begin{cases} x_0\in K,\ choosenarbitrarily,\ \ z_n=\theta_n x_n+(1-\theta_n)T\circ T_\lambda^G x_n,\ \ y_n=\beta_n z_n+(1-\beta_n)P_K(I-\lambda\nabla g)z_n,\\ z_n=\theta_n x_n+(1-\theta_n)T\circ T_\lambda^G x_n,\ \ y_n=\beta_n z_n+(1-\beta_n)P_K(I-\lambda\nabla g)z_n,\ \ x_{n+1}=\alpha_n(\lambda_n x_n)+(1-\alpha_n)y_n,\\ y_n=\beta_n z_n+(1-\beta_n)P_K(I-\lambda\nabla g)z_n,\ \ x_{n+1}=\alpha_n(\lambda_n x_n)+(1-\alpha_n)y_n,\\ x_{n+1}=\alpha_n(\lambda_n x_n)+(1-\alpha_n)y_n,\end{cases}$$

where $\{\alpha_n\}$, $\{\theta_n\}$, $\{\lambda_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty}\alpha_n=0, \lim_{n\to\infty}\lambda_n=1$ *and* $\sum_{n=0}^{\infty}(1-\lambda_n)\alpha_n=\infty$,

(ii) $\liminf_{n\to\infty}\theta_n(1-\theta_n)>0$,

(iii) $\liminf_{n\to\infty}\beta_n(1-\beta_n)>0$.

Then, the sequences $\{x_n\}$ converges a strongly to common solution of Problem 1, Problem 2 and Problem 3.

5 Open Problems

In this paper, we have only shown that $Fix(T_1 \circ T_2) = Fix(T_1) \cap Fix(T_2)$ with T_1 and T_2 are quasi-nonexpansive and firmly nonexpansive mappings respectively. It is well known that there are other nonlinear mappings more general than firmly nonexpansive mappings and quasi-nonexpansive mappings. Therefore, the results of this paper open up many forthcoming results regarding convex minimization problem coupled with the fixed point problem studied in this paper. These following questions are open for researchers interested in this field:

(i) Can we extend Lemma 10 to mappings that are more general than firmly nonexpansive and quasi-nonexpansive mappings mappings ?

(ii) Do the results hold in the setting of a more general Banach space by using our algorithm defined in Theorem 11?

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