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A Note On Loxodromes On Helicoidal Surfaces In Euclidean n-Space*

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Abstract

We first give the definition and parametrization of a helicoidal surface in Euclidean n-space. Afterwards, we investigate the parametrizations of loxodromes on the helicoidal surfaces as well as the rotational surfaces in this space.

1 Introduction

Loxodromes correspond to the curves which intersect all meridians at the same angle on the Earth's surface. Loxodromes are often used in navigation because of the fact that they don't need a change of course. Noble [1] found the equations of loxodromes on the rotational surfaces in Euclidean 3-space. Also, Babaarslan and Yayli [3] investigated the equations of loxodromes on the helicoidal surfaces which are a natural generalization of rotational surfaces in Euclidean 3-space. In Euclidean 4-space the loxodromes on the helicoidal surfaces were studied by Babaarslan [4].

The aim of the present paper is to find the parametrizations of loxodromes on the helicoidal surfaces as well as the rotational surfaces in Euclidean n-space by using similar differential geometry methods used in previous papers.

2 Preliminaries

Euclidean n-space is denoted by the symbol $\mathbb{R}^n$ and it is defined as the set of all n-tuples of real numbers written by $x = (x_1, \ldots, x_n)$. $\mathbb{R}^n$ is a $\mathbb{R}$-vector space and the Euclidean inner product is defined as

$$< x, y >= x_1y_1 + \ldots + x_ny_n. \tag{1}$$

This allows us to define the length of vectors in $\mathbb{R}^n$ by the norm

$$\|x\| = \sqrt{\langle x, x\rangle}, \tag{2}$$

as well as introducing the angle φ between two vectors x, $y \neq 0$ is given by the formula

$$\cos\varphi = \frac{\langle x, y\rangle}{\|x\|\,\|y\|}. \tag{3}$$

Let $\alpha : I \to \mathbb{R}^n$ be a regular curve in $\mathbb{R}^n$, that is $\alpha'(t) \neq 0$ holds everywhere. The arc-length parameter of α is introduced by

$$s(t) = \int_{t_0}^{t} \|\alpha'(t)\|\, dt. \tag{4}$$

Also, α is a unit speed curve if $\|\alpha'(s)\| = 1$ for all $s \in I \subset \mathbb{R}$ (for more details, see [5]).

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The definition of a helicoidal surface in $\mathbb{R}^n$ can be given as follows.

Let $\alpha : I \to H$ be a regular curve in a hyperplane H in $\mathbb{R}^n$ and P be a $(n-2)$-plane in H. If α is rotated about P (an abstract object), then the resulting surface is a rotational surface in $\mathbb{R}^n$. Also, when α is rotated about P, it is simultaneously translated along a line l which is parallel to P so that the speed of the translation is proportional to the speed of rotation, then the resulting surface is a helicoidal surface in $\mathbb{R}^n$ (for the definition in $\mathbb{R}^4$, see [2]).

Let $(x_1, \ldots, x_n)$ be the coordinate system and $\{e_1, \ldots, e_n\}$ be the standard orthonormal base of $\mathbb{R}^n$. We consider the hyperplane $H = span\{e_1, e_3, \ldots, e_n\}$, the $(n-2)$-plane $P = span\{e_3, \ldots, e_n\}$ and the axis $l = span\{e_n\}$. Thus, the rotation which leaves P invariant is given by the following rotational matrix

$$\begin{bmatrix} \cos v & -\sin v & 0 & \ldots & 0 \\ \sin v & \cos v & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}. \tag{5}$$

Now, we consider the profile curve $\alpha(u) = (x_1(u), 0, x_3(u), \ldots, x_n(u))$ in H, where $u \in I$ and $x_1(u) > 0$. As a result, the parametrization of a helicoidal surface M is given by

$$x(u,v) = \begin{bmatrix} \cos v & -\sin v & 0 & \ldots & 0 \\ \sin v & \cos v & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix} \begin{bmatrix} x_1(u) \\ 0 \\ x_3(u) \\ \vdots \\ x_n(u) \end{bmatrix} + \lambda v \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

that is

$$x(u,v) = (x_1(u)\cos v, x_1(u)\sin v, x_3(u), \ldots, x_{n-1}(u), x_n(u) + \lambda v), \tag{6}$$

where $\lambda \in \mathbb{R}^+$. When x_n is a constant function, the helicoidal surface is called right helicoidal surface. Also, if we take $\lambda = 0$, then we have the rotational surfaces in $\mathbb{R}^n$.

3 The Parametrizations of Loxodromes

We consider the helicoidal surface M which is parametrized by (6). We assume that the profile curve α is a unit speed curve. The tangent plane to M at the point $p = x(u,v)$ can be defined by $span\{x_u, x_v\}$.

Thus, the coefficients of first fundamental form of M are given by

$$E = 1, \ F = \lambda x_n'(u) \text{ and } G = x_1^2(u) + \lambda^2. \tag{7}$$

Notice that the constant parameter curves of M are orthogonal if and only if M is either a right helicoidal surface or a rotational surface.

Suppose that $\lambda^2 \left(1 - x_n'^2(u)\right) + x_1^2(u) > 0$, that is, M is a regular surface.

The first fundamental form (or line element) of M is written as

$$ds^2 = du^2 + 2\lambda x_n'(u) du dv + \left(\lambda^2 + x_1^2(u)\right) dv^2. \tag{8}$$

Also, the arc-length of any curve on M between u_1 and u_2 is given by

$$s = \left| \int_{u_1}^{u_2} \sqrt{1 + 2\lambda x_n'(u) \frac{dv}{du} + \left(\lambda^2 + x_1^2(u)\right) \left(\frac{dv}{du}\right)^2} du \right|. \tag{9}$$

We now consider the curve $\sigma(t) = x\,(u(t), v(t))$ which lies on M. By using (3), at the point p, where the loxodrome intersects the meridians at a constant angle φ, we have the following formula

$$\cos\varphi \quad = \quad \frac{du + \lambda x_n'(u)dv}{\sqrt{du^2 + 2\lambda x_n'(u)dudv + \left(\lambda^2 + x_1^2(u)\right)dv^2}}. \tag{10}$$

If we rearrange this formula, then we obtain the following differential equation of loxodrome on the helicoidal surface in $\mathbb{R}^n$

$$\left(\cos^2\varphi\left(\lambda^2 + x_1^2(u)\right) - \lambda^2 x_n'^2(u)\right)\left(\frac{dv}{du}\right)^2 - 2\lambda\sin^2\varphi x_n'(u)\frac{dv}{du} = \sin^2\varphi, \tag{11}$$

whose general solution is

$$v(u) = \int_{u_0}^{u} \frac{2\lambda\sin^2\varphi x_n'(u) + \varepsilon\sqrt{\sin^2 2\varphi\left(x_1^2(u) - \lambda^2(x_n'^2(u) - 1)\right)}}{2\cos^2\varphi\left(\lambda^2 + x_1^2(u)\right) - 2\lambda^2 x_n'^2(u)}du, \tag{12}$$

where $\varepsilon \in \{-1, 1\}$.

Thus, the parametrizations of loxodromes on the helicoidal surfaces as well as the rotational surfaces in $\mathbb{R}^n$ can be given by the following theorems:

Theorem 1 *The parametrization of loxodrome on a helicoidal surface in $\mathbb{R}^n$ is*

$$\sigma(u) = \left(x_1(u)\cos v(u), x_1(u)\sin v(u), x_3(u), \ldots, x_{n-1}(u), x_n(u) + \lambda v(u)\right), \tag{13}$$

where $v(u)$ is given by (12).

Theorem 2 *The parametrization of loxodrome on a rotational surface in $\mathbb{R}^n$ is*

$$\gamma(u) = \left(x_1(u)\cos v(u), x_1(u)\sin v(u), x_3(u), \ldots, x_{n-1}(u), x_n(u)\right), \tag{14}$$

where $v(u) = \varepsilon\tan\varphi\int_{u_0}^{u}\frac{du}{x_1(u)}$.

Also, we have

Corollary 3 *The arc-length of the loxodrome on a right helicoidal surface or a rotational surface in $\mathbb{R}^n$ is given by*

$$s = \left|\frac{u_2 - u_1}{\cos\varphi}\right|. \tag{15}$$

We note that the notable applications of loxodromes on the helicoidal surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$ respectively can be found in [3] and [4].

4 Conclusion

We investigate the parametrizations of loxodromes on the helicoidal surfaces as well as the rotational surfaces in Euclidean n-space. The next time, we will study space-like and time-like loxodromes on the non-degenerate helicoidal surfaces in Minkowski n-space.

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