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Hyers-Ulam Stability Of Mean Value Points For Conformable Fractional Differentiable Functions*

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Abstract

We will first give the proof of the existences of Flett's mean value points and Sahoo-Riedel's points for conformable fractional differentiable functions. Then we will prove the Hyers-Ulam stability of Lagrange's mean value points, Flett's mean value points and Sahoo-Riedel's points for conformable fractional differentiable functions.

1 Introduction

In 1940, Ulam [1] posed a question:"When is it true that by changing 'a little' the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or 'approximately' true?" We now call it Hyers-Ulam stability question.

Later on, Hyers [3] worked on the stability of the equation $f(x+y) = f(x) + f(y)$, where $f : X \to Y$ is a map and X, Y are Banach space. He proved that: corresponding to any $\varepsilon > 0$, there exists a $\delta > 0$, such that if $\|f(x+y) - f(x) - f(y)\| < \delta$, then there exists a addictive map $l(x)$ with $\|f(x) - l(x)\| \leq \varepsilon$.

In 1986, Rassias[4] gave a more general theorem of the Hyers-Ulam stability of addictive map.

Hyers-Ulam stability questions can infiltrate many fields, such as differential equation, homomorphism between metric groups, mean value points and so on. Especially for differential functions, there are lots of relevant results, including linear, nonlinear and fractional differential equations [12]–[17]. It is driving lots of people to study it. In 2010, Pasc, Jung and Li [6] proved the Hyers-Ulam stability of Lagrange's mean value points and Flett's mean value points. In 2009, Lee, Xu and Ye [8] proved the Hyers-Ulam stability of Sahoo-Riedel's points.

In this paper we will give corresponding results for conformable fractional differentiable functions.

Now, we will introduce the Lagrange's mean value points (Theorem 1), Flett's mean value points (Theorem 2) and Sahoo-Riedel's points (Corollary 1) [10] separately.

Theorem 1 *Let $f : [a,b] \to \mathbb{R}$ is a function on a finite closed interval. Suppose that f is continuous on $[a,b]$ and differentiable on (a,b), then there exists a point $\xi \in (a,b)$ such that*

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

Theorem 2 *If $f(x)$ is differentiable in $[a,b]$ and $f'(a) = f'(b)$, then there exists a point ξ in (a,b) such that*

$$f'(\xi) = \frac{f(\xi) - f(a)}{\xi - a}.$$

Corollary 1 *If $f : [a,b] \to \mathbb{R}$ is a differentiable function, then there exists a point $\eta \in (a,b)$ such that*

$$f(\eta) - f(a) = (\eta - a)f'(\eta) - \frac{1}{2}\frac{f'(b) - f'(a)}{b - a}(\eta - a)^2.$$

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Now let's introduce the definition of "conformable fractional derivative" posed by Khalil, Horani, Yousel and Sababheh [7].

Definition 1 *Given a function $f:[0,\infty)\longrightarrow\mathbb{R}$. The conformable fractional derivative of f of order α is defined by*

$$T_\alpha(f)(t)=\lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f(t)}{\varepsilon}$$

for all $t>0$, $\alpha\in(0,1)$. Sometimes we write $T_\alpha(f)(t)$ as $f^{(\alpha)}(t)$. If f is α-differentiable in some $(0,a), a>0$ and $\lim_{t\to 0^+}f^{(\alpha)}(t)$ exists, then define

$$f^{(\alpha)}(0)=\lim_{t\to 0^+}f^{(\alpha)}(t).$$

They also gave the Rolle's theorem (Theorem 3) and the mean value theorem for conformable fractional differentiable functions (Theorem 4). The results are similar to familiar Lagrange's mean value theorem with only slightly different in form (see [7] for more results and details).

Theorem 3 *Let $a>0$ and $f:[a,b]\longrightarrow\mathbb{R}$ be a given function satisfying:*

(1) *f is continuous on $[a,b]$.*

(2) *f is α-differentiable for some $\alpha\in(0,1)$.*

(3) *$f(a)=f(b)$.*

Then there exists $c\in(a,b)$ satisfying $f^{(\alpha)}(c)=0$. We call c the α-order Rolle's point.

Theorem 4 *Let $a>0$ and $f:[a,b]\longrightarrow\mathbb{R}$ be a given function satisfying*

(1) *f is continuous on $[a,b]$.*

(2) *f is α-differentiable for some $\alpha\in(0,1)$.*

Then there exists $c\in(a,b)$ such that $f^{(\alpha)}(c)=\frac{f(b)-f(a)}{\frac{1}{\alpha}b^\alpha-\frac{1}{\alpha}a^\alpha}$. We call c the α-order Lagrange's mean value point.

Remark 1 *If $\alpha=1$, then the theorem is the classical Lagrange theorem.*

At the end of this section, we will introduce two Hyers-Ulam stability results that we will use later.

Theorem 5 *Let $f(x)$ be a function with n^{th} derivative in a neighborhood N of the point $x=\eta$. If $f^{(n)}(\eta)=0$ and $f^{(n)}(x)$ changes sign at $x=\eta$, then corresponding to each $\varepsilon>0$, there exists $\delta>0$ such that, for each function $g(x)$ with an n^{th} derivative in N and satisfying the inequality $|g(x)-f(x)|<\delta$ in N, there exists a point $x=\xi$ such that $g^{(n)}(\xi)=0$ and $|\xi-\eta|<\varepsilon$.*

Theorem 6 *Let X be a complex Banach space and let $I=(a,b)$ be an open interval, where $a,b\in\mathbb{R}\cup\{\pm\infty\}$ are arbitrarily given with $a<b$. Assume that $g:I\to\mathbb{C}$ and $h:I\to X$ are continuous functions such that $g(x)$ and $\exp\left\{\int_b^x g(u)\mathrm{d}u\right\}h(x)$ are integrable on (c,b) for every $c\in I$. Moreover, suppose $\varphi:I\to[0,\infty)$ is a function such that $\varphi(x)\exp\left\{\Re\left(\int_b^x g(u)\mathrm{d}u\right)\right\}$ is integrable on I. If a continuously differentiable function $p:I\to X$ satisfies the differential inequality*

$$\|p'(x)+g(x)p(x)+h(x)\|\le\varphi(x)$$

for all $x\in I$, then there exists a unique $z\in X$ such that

$$\begin{aligned}&\left\|p(x)-\exp\left\{-\int_b^x g(u)\mathrm{d}u\right\}\left(z-\int_b^x\exp\left\{\int_b^v g(u)\mathrm{d}u\right\}h(v)\mathrm{d}v\right)\right\|\\ \le\ &\exp\left\{-\Re\left(\int_b^x g(u)\mathrm{d}u\right)\right\}\int_a^x\varphi(v)\exp\left\{\Re\left(\int_b^v g(u)\mathrm{d}u\right)\right\}\mathrm{d}v.\end{aligned}$$

Here $\Re(z)$ denotes the real part of complex numbers z.

2 Main Results

2.1 Hyers-Ulam stability of α-Order Lagrange's Mean Value Points

First, we will prove two lemmas.

Lemma 1 *Assume function $f : (0,\infty) \longrightarrow \mathbb{R}$ and $\alpha \in (0,1)$. Then f is α-differentiable at $t > 0$ if and only if f is differentiable at t. Moreover, we have $f^{(\alpha)}(t) = f'(t)t^{1-\alpha}$ (or $f'(t) = f^{(\alpha)}(t)t^{\alpha-1}$).*

Proof. Sufficiency. If f is differentiable at $t > 0$, let $\varepsilon = ht^{\alpha-1}$, then

$$\begin{aligned} T_\alpha(f)(t) &= \lim_{\varepsilon\to 0} \frac{f\left(t+\varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon} = \lim_{h\to 0} \frac{f(t+h)-f(t)}{ht^{\alpha-1}} \\ &= t^{1-\alpha} \lim_{h\to 0} \frac{f(t+h)-f(t)}{h} = t^{1-\alpha}\frac{df}{dt}(t). \end{aligned}$$

Necessity. If f is α-differentiable at $t > 0$, let $h = \varepsilon t^{1-\alpha}$, then

$$f'(t) = \lim_{h\to 0}\frac{f(t+h)-f(t)}{h} = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon t^{1-\alpha}} = t^{\alpha-1}T_\alpha(f)(t).$$

■

Lemma 2 *Let $f : (0,\infty) \longrightarrow \mathbb{R}$ be α-differentiable in a neighborhood N of the point η, where $\alpha \in (0,1)$. Suppose that $f^{(\alpha)}(\eta) = 0$ and $f^{(\alpha)}(x)$ changes sign at η. Then, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for every function $g : \mathbb{R} \to \mathbb{R}$ which is α-differentiable in N and satisfies $|f(x) - g(x)| < \delta$ for any $x \in N$, there exists a point $\xi \in N$ with $g^{(\alpha)}(\xi) = 0$ and $|\xi - \eta| < \varepsilon$.*

Proof. Utilize Theorem 5 in the case $n = 1$, if $f^{(\alpha)}(\eta) = \eta^{1-\alpha}f'(\eta) = 0$, since $\eta > 0$. We get $f'(\eta) = 0$ and $f^{(\alpha)}(x)$ changes sign at η means $f'(x)$ changes sign at η. So there exists $\delta > 0$ such that for every function $g : \mathbb{R} \to \mathbb{R}$ which is differentiable in N and satisfies $|f(x) - g(x)| < \delta$ for any $x \in N$, there exists a point $\xi \in N$ with $g'(\xi) = 0$ and $|\xi - \eta| < \varepsilon$. Note that $g^{(\alpha)}(\xi) = \xi^{1-\alpha}g'(\xi) = 0$, which completes the proof. ■

Now, let's prove the Hyers-Ulam stability of α-order Lagrange's mean value points for conformable fractional differentiable functions.

Theorem 7 *Let a, b, η be real numbers satisfying $0 < a < \eta < b$ and $\alpha \in (0,1)$. Assume that $f : (0,\infty) \to \mathbb{R}$ is twice continuously α-differentiable and η is the unique α-order Lagranges mean value point of f in open interval (a,b) and moreover that $[T_\alpha(T_\alpha)f](\eta) \neq 0$. Suppose $g : (0,\infty) \to \mathbb{R}$ is α-differentiable function. Then for any given $\varepsilon > 0$, there exists $\delta > 0$ such that if $|f(x) - g(x)| < \delta$ for all $x \in [a,b]$, then there is an α-order Lagrange's mean value point $\xi \in (a,b)$ of g with $|\xi - \eta| < \varepsilon$.*

Proof. Consider the function:

$$H_f(x) = f(x) - \frac{f(b)-f(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}\left(\frac{1}{\alpha}x^\alpha - \frac{1}{\alpha}a^\alpha\right).$$

We find H_f is also twice continuously α-differentiable and $H_f(a) = H_f(b) = f(a)$. According to Theorem 3, there exists a $\eta^* \in (a,b)$ satisfying $H_f^{(\alpha)}(\eta^*) = 0$, that is

$$f^{(\alpha)}(\eta^*) - \frac{f(b)-f(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha} = 0.$$

Suppose α-order Lagrange's mean value point of f is η ,by uniqueness, we get $\eta^* = \eta$. Since

$$H_f^{(\alpha)}(x) = f^{(\alpha)}(x) - \frac{f(b)-f(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha},$$

we see that

$$\begin{aligned}(H_f^{(\alpha)}(x))' &= (f^{(\alpha)}(x))' \\ &= (x^{1-\alpha}f'(x))' \\ &= (1-\alpha)x^{-\alpha}f'(x) + x^{1-\alpha}f''(x).\end{aligned}$$

Moreover, since

$$[T_\alpha(T_\alpha)f](\eta) = (\eta^{1-\alpha}f''(\eta) - (\alpha-1)\eta^{-\alpha}f'(\eta))\left(\eta^{1-\alpha}\right) \neq 0,$$

we have

$$f''(\eta) \neq \frac{(\alpha-1)f'(\eta)}{\eta}.$$

Thus $(H_f^{(\alpha)}(\eta))' \neq 0$, which means $H_f^{(\alpha)}$ changes sign at η, since $H_f^{(\alpha)}(\eta) = 0$ and $(H_f^{(\alpha)}(\eta))'$ is continuous.

Now let $g(x)$ be an α-differentiable function, satisfying $|f(x)-g(x)| < \frac{\delta}{3}$ for all $x \in [a,b]$, we consider an α-differentiable function $H_g(x)$:

$$H_g(x) = g(x) - \frac{g(b)-g(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}\left(\frac{1}{\alpha}x^\alpha - \frac{1}{\alpha}a^\alpha\right).$$

Then for all $x \in [a,b]$, we get

$$\begin{aligned}|H_f(x) - H_g(x)| &\leq |f(x)-g(x)| + \frac{x^\alpha - a^\alpha}{b^\alpha - a^\alpha}|f(a)-g(a)| + \frac{x^\alpha - a^\alpha}{b^\alpha - a^\alpha}|f(b)-g(b)| \\ &\leq |f(x)-g(x)| + |f(a)-g(a)| + |f(b)-g(b)| \\ &\leq \delta.\end{aligned}$$

Utilize Lemma 2, there exists a point $\xi \in (a,b)$ with $H_g^{(\alpha)}(\xi) = 0$ and $|\xi - \eta| < \varepsilon$, note that the ξ is exactly α-order Lagrange's mean value point of function g. ■

Theorem 8 *Let a, b, ξ be real numbers satisfying $0 < a < \xi < b$ and $\alpha \in (0,1)$. Assume that $f : [a,b] \to \mathbb{R}$ is twice continuously α-differentiable. Moreover, suppose that either $[T_\alpha(T_\alpha f)](x) > 0$ for all $x \in [a,b]$, or $[T_\alpha(T_\alpha f)](x) < 0$ for all $x \in [a,b]$. Then, if*

$$\left|f^{(\alpha)}(\xi) - \frac{f(b)-f(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}\right| < \delta \tag{1}$$

for some $\delta > 0$, then there exists an α-order Lagrange's mean value point η of f on (a,b) satisfying

$$\left|\frac{1}{\alpha}\eta^\alpha - \frac{1}{\alpha}\xi^\alpha\right| \leq \frac{\delta}{\min_{x\in[a,b]}|[T_\alpha(T_\alpha f)](x)|}.$$

Proof. Due to α-order Lagrange's mean value theorem(Theorem 4), there exists a Lagrange's mean value point $\eta \in (a,b)$,such that

$$f^{(\alpha)}(\eta) = \frac{f(b)-f(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}.$$

So from (1), we have

$$\left|f^{(\alpha)}(\xi) - f^{(\alpha)}(\eta)\right| \leq \delta.$$

If $\xi = \eta$, then our assertion is true. Otherwise, without loss of generality, we assume that $a < \eta < \xi < b$. Since f is twice continuously α-differentiable, use Theorem 4 on $f^{(\alpha)}$ on interval $[\eta, \xi]$, we know that there exists a point $\theta \in (\eta, \xi)$ with

$$[T_\alpha(T_\alpha f)](\theta) = \frac{f^{(\alpha)}(\xi) - f^{(\alpha)}(\eta)}{\frac{1}{\alpha}\xi^\alpha - \frac{1}{\alpha}\eta^\alpha}.$$

So

$$\left|\frac{1}{\alpha}\xi^\alpha - \frac{1}{\alpha}\eta^\alpha\right| = \left|\frac{f^{(\alpha)}(\xi) - f^{(\alpha)}(\eta)}{T_\alpha(T_\alpha f)(\theta)}\right| \leq \frac{\delta}{\min_{x\in[a,b]}|T_\alpha(T_\alpha f)(x)|}.$$

■

2.2 Hyers-Ulam Stability of α-Order Flett's Mean Value Points And α-Order Sahoo-Riedel's Points

In [7], Khalil, Horani, Yousel and Sababheh proved the α-order Rolle's points theorem and α-order Lagrange mean value points theorem for conformable fractional differentiable functions. However, they didn't prove the same results of α-order Flett's mean value points and α-order Sahoo-Riedel's points.

So first we give the existences of α-order Flett's mean value points and α-order Sahoo-Riedel's points for conformable fractional differentiable functions.

Theorem 9 *Suppose $0 < a < b < \infty$, $\alpha \in (0, 1)$ and $f(x)$ is α-differentiable in $[a, b]$, and $f^{(\alpha)}(a) = f^{(\alpha)}(b)$. Then there exists a point η in (a, b) such that*

$$f^{(\alpha)}(\eta) = \frac{f(\eta) - f(a)}{\frac{1}{\alpha}\eta^\alpha - \frac{1}{\alpha}a^\alpha}.$$

Remark 2 *If $\alpha = 1$, then the theorem is the Flett's theorem in the case of integral order derivative [9].*

Definition 2 *We call the point η the α-order Flett's point of f.*

Proof of Theorem 9.

1) Suppose $f^{(\alpha)}(a) = f^{(\alpha)}(b) = 0$ first. Define a function $\psi(x)$ on $[a, b]$:

$$\psi(x) = \begin{cases} \frac{f(x)-f(a)}{\frac{1}{\alpha}x^\alpha - \frac{1}{\alpha}a^\alpha}, & a < x \le b, \\ 0, & x = a. \end{cases}$$

Then $\psi(x)$ is α-differentiable on $(a, b]$ with

$$\psi^{(\alpha)}(x) = \frac{f^{(\alpha)}(x)}{\frac{1}{\alpha}x^\alpha - \frac{1}{\alpha}a^\alpha} - \frac{f(x) - f(a)}{(\frac{1}{\alpha}x^\alpha - \frac{1}{\alpha}a^\alpha)^2}$$

and

$$\psi^{(\alpha)}(b) = -\frac{\psi(b)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}$$

and continuous on $[a, b]$ with

$$\psi(a) = f^{(\alpha)}(a) = 0.$$

It's enough to show that $\exists\eta \in (a, b)$ such that $\psi^{(\alpha)}(\eta) = 0$. If $\psi(b) = 0$, then by Theorem 3, we know it's true. If $\psi(b) > 0$ ($\psi(b) < 0$ is similar), then $\psi'(b) = \psi^{(\alpha)}(b)b^{\alpha-1} < 0$. So there exists a point $x_1 \in (a, b)$ such that $\psi(x_1) > \psi(b)$. Thus $0 = \psi(a) < \psi(b) \le \psi(x_1)$, there is a point $x_2 \in (a, x_1)$ such that $\psi(x_2) = \psi(b)$. Then by Theorem 3 in the interval (x_2, b), we get: $\exists\eta \in (x_2, b)$ such that $\psi^{(\alpha)}(\eta) = 0$.

2) For general case of $f^{(\alpha)}(a) = f^{(\alpha)}(b)$, consider the function $g(x) = f(x) - \frac{1}{\alpha}x^\alpha f^{(\alpha)}(a)$. Then $g^{(\alpha)}(x) = f^{(\alpha)}(x) - f^{(\alpha)}(a)$, so $g^{(\alpha)}(a) = g^{(\alpha)}(b) = 0$. Thus there exists a point η in (a, b) such that

$$g^{(\alpha)}(\eta) = \frac{g(\eta) - g(a)}{\frac{1}{\alpha}\eta^\alpha - \frac{1}{\alpha}a^\alpha}.$$

So

$$f^{(\alpha)}(\eta) - f^{(\alpha)}(a) = \frac{f(\eta) - f(a) - (\frac{1}{\alpha}\eta^\alpha f^{(\alpha)}(a) - \frac{1}{\alpha}a^\alpha f^{(\alpha)}(a))}{\frac{1}{\alpha}\eta^\alpha - \frac{1}{\alpha}a^\alpha},$$

that is

$$f^{(\alpha)}(\eta) = \frac{f(\eta) - f(a)}{\frac{1}{\alpha}\eta^\alpha - \frac{1}{\alpha}a^\alpha}.$$

■

Corollary 2 *Suppose $0<a<b<\infty$, $\alpha\in(0,1)$ and $f:[a,b]\to\mathbb{R}$ is an α-differentiable function. Then there exists a point $\eta\in(a,b)$ such that*

$$f(\eta)-f(a)=(\frac{1}{\alpha}\eta^{\alpha}-\frac{1}{\alpha}a^{\alpha})f^{(\alpha)}(\eta)-\frac{1}{2}\frac{f^{(\alpha)}(b)-f^{(\alpha)}(a)}{\frac{1}{\alpha}b^{\alpha}-\frac{1}{\alpha}a^{\alpha}}(\frac{1}{\alpha}\eta^{\alpha}-\frac{1}{\alpha}a^{\alpha})^2.$$

Remark 3 *If $\alpha=1$, then the theorem is the Sahoo-Riedel's theorem in the case of integral order derivative [10].*

Definition 3 *We call the point η the α-order Sahoo-Riedel's point of f.*

Proof of Corollary 2. Define an auxiliary function $\psi:[a,b]\to\mathbb{R}$ as

$$\psi(x)=f(x)-\frac{1}{2}\frac{f^{(\alpha)}(b)-f^{(\alpha)}(a)}{\frac{1}{\alpha}b^{\alpha}-\frac{1}{\alpha}a^{\alpha}}(\frac{1}{\alpha}x^{\alpha}-\frac{1}{\alpha}a^{\alpha})^2.$$

Then

$$\psi^{(\alpha)}(x)=f^{(\alpha)}(x)-\frac{f^{(\alpha)}(b)-f^{(\alpha)}(a)}{\frac{1}{\alpha}b^{\alpha}-\frac{1}{\alpha}a^{\alpha}}(\frac{1}{\alpha}x^{\alpha}-\frac{1}{\alpha}a^{\alpha}).$$

We see that $\psi^{(\alpha)}(a)=\psi^{(\alpha)}(b)=f^{(\alpha)}(a)$. Then there exists a point η in (a,b) such that

$$\psi^{(\alpha)}(\eta)=\frac{\psi(\eta)-\psi(a)}{\frac{1}{\alpha}\eta^{\alpha}-\frac{1}{\alpha}a^{\alpha}}.$$

That is

$$\begin{aligned}&f^{(\alpha)}(\eta)(\frac{1}{\alpha}\eta^{\alpha}-\frac{1}{\alpha}a^{\alpha})-\frac{f^{(\alpha)}(b)-f^{(\alpha)}(a)}{\frac{1}{\alpha}b^{\alpha}-\frac{1}{\alpha}a^{\alpha}}(\frac{1}{\alpha}\eta^{\alpha}-\frac{1}{\alpha}a^{\alpha})^2\\ =\;&f(\eta)-f(a)-\frac{1}{2}\frac{f^{(\alpha)}(b)-f^{(\alpha)}(a)}{\frac{1}{\alpha}b^{\alpha}-\frac{1}{\alpha}a^{\alpha}}(\frac{1}{\alpha}\eta^{\alpha}-\frac{1}{\alpha}a^{\alpha})^2,\end{aligned}$$

therefore,

$$f(\eta)-f(a)=(\frac{1}{\alpha}\eta^{\alpha}-\frac{1}{\alpha}a^{\alpha})f^{(\alpha)}(\eta)-\frac{1}{2}\frac{f^{(\alpha)}(b)-f^{(\alpha)}(a)}{\frac{1}{\alpha}b^{\alpha}-\frac{1}{\alpha}a^{\alpha}}(\frac{1}{\alpha}\eta^{\alpha}-\frac{1}{\alpha}a^{\alpha})^2.$$

■

Next, we will prove Hyers-Ulam stability of α-order Flett's mean value points and α-order Sahoo-Riedel's points.

Theorem 10 *Suppose $0<a<b<\infty$, $0<\alpha<1$. Let $f,h:[a,b]\to\mathbb{R}$ be α-differentiable and η be an α-order Sahoo-Riedel's point of f in (a,b). If f has 2nd α-order derivative at η and*

$$[T_\alpha\,(T_\alpha f)](\eta)\neq\frac{f^{(\alpha)}(b)-f^{(\alpha)}(a)}{\frac{1}{\alpha}b^{\alpha}-\frac{1}{\alpha}a^{\alpha}},$$

then corresponding to any $\varepsilon>0$ and any neighborhood $N\subset(a,b)$ of η, there exists $\delta>0$,such that for every h satisfying $|h(x)-h(a)-(f(x)-f(a))|<\delta$ for all x in N and $h^{(\alpha)}(b)-h^{(\alpha)}(a)=f^{(\alpha)}(b)-f^{(\alpha)}(a)$, and there exists a point $\xi\in N$ such that ξ is an α-order Sahoo-Riedel's point of h and $|\xi-\eta|<\varepsilon$.

Proof. Define a function $G_f(x)$ on $[a,b]$:

$$G_f(x)=\begin{cases}\frac{f(x)-f(a)}{\frac{1}{\alpha}x^{\alpha}-\frac{1}{\alpha}a^{\alpha}}-\frac{1}{2}\frac{f^{(\alpha)}(b)-f^{(\alpha)}(a)}{\frac{1}{\alpha}b^{\alpha}-\frac{1}{\alpha}a^{\alpha}}(\frac{1}{\alpha}x^{\alpha}-\frac{1}{\alpha}a^{\alpha}), & a<x\le b,\\ f^{(\alpha)}(a), & x=a.\end{cases}$$

1) First, it is easy to see that G_f is continuous in $[a,b]$ and α-differentiable in $(a,b]$. Moreover, we have

$$G_f^{(\alpha)}(x) = -\frac{f(x)-f(a)}{(\frac{1}{\alpha}x^\alpha - \frac{1}{\alpha}a^\alpha)^2} + \frac{f^{(\alpha)}(x)}{\frac{1}{\alpha}x^\alpha - \frac{1}{\alpha}a^\alpha} - \frac{f^{(\alpha)}(b)-f^{(\alpha)}(a)}{2(\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha)}, \quad x \in (a,b]$$

with $G_f^{(\alpha)}(\eta) = 0$, since η is an α-order Sahoo-Riedel's point of f.

2) Second, we prove $G_f^{(\alpha)}(x)$ changes sign at point η. By Taylor formula and expand the following four items at η, we have

$$\begin{aligned}
\frac{1}{(\frac{1}{\alpha}x^\alpha - \frac{1}{\alpha}a^\alpha)^2} &= \frac{1}{(\frac{1}{\alpha}\eta^\alpha - \frac{1}{\alpha}a^\alpha)^2} - \frac{2\eta^{\alpha-1}}{(\frac{1}{\alpha}\eta^\alpha - \frac{1}{\alpha}a^\alpha)^3}(x-\eta) + o(x-\eta),\\
\frac{1}{\frac{1}{\alpha}x^\alpha - \frac{1}{\alpha}a^\alpha} &= \frac{1}{\frac{1}{\alpha}\eta^\alpha - \frac{1}{\alpha}a^\alpha} - \frac{\eta^{\alpha-1}}{(\frac{1}{\alpha}\eta^\alpha - \frac{1}{\alpha}a^\alpha)^2}(x-\eta) + o(x-\eta),\\
f^{(\alpha)}(x) &= f^{(\alpha)}(\eta) + (f^{(\alpha)})'(\eta)(x-\eta) + o(x-\eta),\\
f(x) &= f(\eta) + f'(\eta)(x-\eta) + o(x-\eta).
\end{aligned}$$

Substituting into $G_f^{(\alpha)}(x)$, we obtain

$$\begin{aligned}
G_f^{(\alpha)}(x) = &-[f(\eta) - f(a) + f'(\eta)(x-\eta) + o(x-\eta)]\\
&\times \left[\frac{1}{(\frac{1}{\alpha}\eta^\alpha - \frac{1}{\alpha}a^\alpha)^2} - \frac{2\eta^{\alpha-1}}{(\frac{1}{\alpha}\eta^\alpha - \frac{1}{\alpha}a^\alpha)^3}(x-\eta) + o(x-\eta)\right]\\
&+ \left[f^{(\alpha)}(\eta) + (f^{(\alpha)})'(\eta)(x-\eta) + o(x-\eta)\right]\\
&\times \left[\frac{1}{\frac{1}{\alpha}\eta^\alpha - \frac{1}{\alpha}a^\alpha} - \frac{\eta^{\alpha-1}}{(\frac{1}{\alpha}\eta^\alpha - \frac{1}{\alpha}a^\alpha)^2}(x-\eta) + o(x-\eta)\right]\\
&- \frac{f^{(\alpha)}(b)-f^{(\alpha)}(a)}{2(\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha)}.
\end{aligned}$$

Note $\Delta := \frac{1}{\alpha}\eta^\alpha - \frac{1}{\alpha}a^\alpha$. Notice that $(x-\eta)^2 = o(x-\eta), Ao(x-\eta) = o(x-\eta)$(A is a constant), $o(x-\eta)o(x-\eta) = o(x-\eta)$ (when $x \to \eta$), we can simplify the above formula and obtain

$$\begin{aligned}
G_f^{(\alpha)}(x) = &-\frac{f(\eta)-f(a)}{\Delta^2} + \frac{f^{(\alpha)}(\eta)}{\Delta} - \frac{f^{(\alpha)}(b)-f^{(\alpha)}(a)}{2(\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha)}\\
&+ (x-\eta)\left(\frac{(f^{(\alpha)})'(\eta)}{\Delta} - \frac{f'(\eta)}{\Delta^2}\right)\\
&- (x-\eta)\left[\frac{f'(\eta)}{\Delta^2} - \frac{2\eta^{\alpha-1}(f(\eta)-f(a))}{\Delta^3}\right]\\
&+ o(x-\eta).
\end{aligned} \tag{2}$$

Further more, the first line of the right-hand side of (2) is vanished, since η is an α-order Sahoo-Riedel's point of f, therefore

$$G_f^{(\alpha)}(x) = \frac{(x-\eta)}{\Delta^2}\left((f^{(\alpha)})'(\eta)\Delta - 2f'(\eta) + \frac{2\eta^{\alpha-1}(f(\eta)-f(a))}{\Delta}\right) + o(x-\eta).$$

We claim that

$$(f^{(\alpha)})'(\eta)\Delta - 2f'(\eta) + \frac{2\eta^{\alpha-1}(f(\eta)-f(a))}{\Delta} \neq 0.$$

Otherwise,

$$(f^{(\alpha)})'(\eta)\Delta - 2f'(\eta) + \frac{2\eta^{\alpha-1}(f(\eta)-f(a))}{\Delta} = 0,$$

then utilize

$$-\frac{f(\eta)-f(a)}{\Delta^2} + \frac{f^{(\alpha)}(\eta)}{\Delta} - \frac{f^{(\alpha)}(b)-f^{(\alpha)}(a)}{2(\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha)} = 0.$$

Note that $T_\alpha\left(T_\alpha f\right)(\eta) = (f^{(\alpha)})'(\eta)\eta^{1-\alpha}$, $f^{(\alpha)}(\eta) = f'(\eta)\eta^{1-\alpha}$. We get

$$T_\alpha\left(T_\alpha f\right)(\eta) = \frac{f^{(\alpha)}(b)-f^{(\alpha)}(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha},$$

which contradicts the conditions. Now we note $B := (f^{(\alpha)})'(\eta)\Delta - 2f'(\eta) + \frac{2\eta^{\alpha-1}(f(\eta)-f(a))}{\Delta}$. So

$$G_f^{(\alpha)}(x) = (x-\eta)\left(\frac{1}{\Delta^2}B + o(1)\right),$$

when $|x-\eta|$ is sufficiently small, the sign of $\frac{1}{\Delta^2}B + o(1)$ depends on B and B is nonzero, which means $G_f^{(\alpha)}(x)$ changes sign at point η in some neighborhood $N = (c,d) \subset (a,b)$.

3) Next, we show that if $|h(x)-h(a)-(f(x)-f(a))| < \delta$ for all x in N, then $|G_h(x) - G_f(x)| < \delta_0$ for all x in N,where $\delta_0 = \frac{\delta}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}$ and $G_h(x)$ is a function defined on $[a,b]$:

$$G_h(x) = \begin{cases} \frac{h(x)-h(a)}{\frac{1}{\alpha}x^\alpha - \frac{1}{\alpha}a^\alpha} - \frac{1}{2}\frac{h^{(\alpha)}(b)-h^{(\alpha)}(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}(\frac{1}{\alpha}x^\alpha - \frac{1}{\alpha}a^\alpha), & a < x \le b, \\ h^{(\alpha)}(a), & x = a. \end{cases}$$

Since

$$\begin{aligned} &\left(\frac{1}{\alpha}x^\alpha - \frac{1}{\alpha}a^\alpha\right)G_h(x) - \left(\frac{1}{\alpha}x^\alpha - \frac{1}{\alpha}a^\alpha\right)G_f(x) \\ &= [h(x)-h(a)-f(x)+f(a)] \\ &- \frac{1}{2}\frac{(\frac{1}{\alpha}x^\alpha - \frac{1}{\alpha}a^\alpha)^2}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}(h^{(\alpha)}(b) - h^{(\alpha)}(a) - (f^{(\alpha)}(b) - f^{(\alpha)}(a))) \\ &= h(x) - h(a) - f(x) + f(a) \end{aligned}$$

and $|h(x)-h(a)-(f(x)-f(a))| < \delta$ for all x in N, set $\delta = \delta_0(\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha)$, we get $|G_h(x)-G_f(x)| < \delta_0$.

4) Last, from 1)–3), by Lemma 2, there exists a point $\xi \in N$ with $G_h^{(\alpha)}(\xi) = 0$ and $|\xi - \eta| < \varepsilon$. Obviously ξ is an α-order Sahoo-Riedel's point of h.

■

Corollary 3 *Suppose $0 < a < b < \infty, \alpha \in (0,1)$, $f(x)$ is α-differentiable in $[a,b]$, $f^{(\alpha)}(a) = f^{(\alpha)}(b)$ and η be an α-order Flett's point of f in (a,b). If f has 2nd α-order derivative at η and*

$$T_\alpha\left(T_\alpha f\right)(\eta) \neq 0,$$

then corresponding to any $\varepsilon > 0$ and any neighborhood $N \subset (a,b)$ of η, there exists a $\delta > 0$, such that for every h satisfying $|h(x)-h(a)-(f(x)-f(a))| < \delta$ for all x in N and $h^{(\alpha)}(b) = h^{(\alpha)}(a)$ and there exists a point $\xi \in N$ such that ξ is an α-order Flett's point of h and $|\xi - \eta| < \varepsilon$.

Proof. It follows from the above theorem. Since we add the hypothesis $f^{(\alpha)}(a) = f^{(\alpha)}(b)$ and $h^{(\alpha)}(b) = h^{(\alpha)}(a)$, the α-order Sahoo-Riedel's points α-order degenerate into Flett's points. ■

At the end of this paper, we give another form of stability of Flett's points for conformable functions.

Theorem 11 *Suppose $0 < a < b < \infty, \alpha \in (0,1)$ and $I = (a', b')$ with $I \subset (a, b)$. Here we use the notation: $a' \triangleq \frac{1}{\alpha}a^\alpha$, $b' \triangleq \frac{1}{\alpha}b^\alpha$, $x' \triangleq \frac{1}{\alpha}x^\alpha$. Let $f : I \to \mathbb{C}$ be a function which is continuous on I and continuously α-differentiable on I. Assume that $\varphi : I \to [0, \infty)$ with*

$$\int_{a'}^{b'} \frac{\varphi(\tau)}{\tau - a'} d\tau < \infty.$$

Then if f satisfies

$$\left| f^{(\alpha)}(x) - \frac{f(x) - f(a)}{x' - a'} \right| \leq \varphi(x) \quad \text{and} \quad |f'(x)| \leq \frac{\varphi(x)(x - x')}{M}, \quad \forall x \in (a', b')$$

where $M = a' + a'(b')^{1-\alpha} + \frac{1-\alpha}{\alpha}b'$. Then there exists a function $y : [a, b] \to \mathbb{C}$, which is continuously α-differentiable on (a, b) with

$$y^{(\alpha)}(x) = \frac{y(x) - y(a)}{x' - a'} \quad \text{and} \quad |f(x') - y(x)| \leq (x' - a') \int_{a'}^{x'} \frac{\varphi(\tau)}{\tau - a'} d\tau.$$

Proof. In the Theorem 6, we put

$$g(x) \triangleq \frac{-1}{x - a'}, \quad h(x) \triangleq \frac{f(a)}{x - a'} \quad \text{and} \quad p(x) \triangleq f(x).$$

1) Easy to see that the function $g(x) = \frac{-1}{x - a'}$ is integrable on (c, b') for any $a' < c < b'$.

2) By calculation, we obtain the following formula

$$\int_c^{b'} \exp\left\{ -\int_{b'}^{\tau} \frac{du}{u - a'} \right\} \frac{f(a)}{\tau - a'} d\tau = (b' - a') \left\{ \frac{f(a)}{c - a'} - \frac{f(a)}{b' - a'} \right\} < \infty$$

for every $c \in (a', b')$.

3) We claim that

$$|p'(x) + g(x)p(x) + h(x)| \leq \varphi(x), \forall x \in (a', b'),$$

i.e.

$$|f'(x)(x - a') - f(x) + f(a)| \leq \varphi(x)(x - a'), \forall x \in (a', b').$$

In fact, we have

$$\begin{aligned}
& |f'(x)(x - a') - f(x) + f(a)| \\
\leq\; & \left| f'(x)(x - a') - f^{(\alpha)}(x)(x' - a') \right| + \left| f^{(\alpha)}(x)(x' - a') - f(x) + f(a) \right| \\
\leq\; & \left| f'(x) \left[a'x^{1-\alpha} + (1 - \frac{1}{\alpha})x - a' \right] \right| + \varphi(x)(x' - a') \\
\leq\; & |f'(x)M| + \varphi(x)(x' - a') \\
\leq\; & \varphi(x)(x - a').
\end{aligned}$$

4) According to the condition

$$\int_{a'}^{b'} \frac{\varphi(\tau)}{\tau - a'} d\tau < \infty,$$

we know

$$\varphi(x) \exp\left\{ \Re \left(\int_{b'}^{x} g(u) \mathrm{d}u \right) \right\} = \varphi(x) \frac{b' - a'}{x - a'}$$

is integrable on I.

5) From the 1)–4), we check the functions p, h, g, φ satisfying the conditions of Theorem 6 and note that $x \in (a', b') \Rightarrow x' \in (a', b')$, since $(a', b') \subset (a, b)$. So there exists a unique complex number z such that

$$\left| f(x') - \exp\left\{\int_{b'}^{x'} \frac{du}{u-a'}\right\}\left(z - \int_{b'}^{x'} \exp\left\{-\int_{b'}^{\tau} \frac{du}{u-a'}\right\} \frac{f(a)}{\tau - a'} d\tau\right)\right| \\ \leq \exp\left\{\int_{b'}^{x'} \frac{du}{u-a'}\right\} \int_{a'}^{x'} \varphi(\tau) \exp\left\{-\int_{b'}^{\tau} \frac{du}{u-a'}\right\} d\tau, \quad \forall\, x' \in (a', b').$$

6) Set $y(x) = \frac{z-f(a)}{b'-a'}x' + \frac{b'f(a)-za'}{b'-a'}$, then y is continuously α-differentiable in (a', b') and $y(a) = f(a)$. Then the above implies

$$|f(x') - y(x)| \leq (x' - a') \int_{a'}^{x'} \frac{\varphi(\tau)}{\tau - a'} d\tau.$$

Further more,

$$\begin{aligned} y^{(\alpha)}(x) &= \frac{z - f(a)}{b' - a'} \\ &= \frac{1}{x' - a'}\left(\frac{z - y(a)}{b' - a'}x' - \frac{z - y(a)}{b' - a'}a'\right) \\ &= \frac{1}{x' - a'}\left(\frac{z - y(a)}{b' - a'}x' + \frac{b'y(a) - za'}{b' - a'} - y(a)\right) \\ &= \frac{y(x) - y(a)}{x' - a'} \end{aligned}$$

for all $x \in (a, b)$.

■

Remark 4 *(1) The condition $(a', b') \subset (a, b)$ can be satisfied when $0 < a \leq \sqrt[\alpha-1]{\alpha} \leq b < \infty$.*
(2) In fact the condition $|f'(x)| \leq \frac{\varphi(x)(x-x')}{M}, \forall x \in (a', b')$ implies $x' \leq x$, so $x \geq \sqrt[\alpha-1]{\alpha}$. So this theorem needs $a = a' = \sqrt[\alpha-1]{\alpha}$ to make all assumptions satisfied.
(3) In fact, from $y^{(\alpha)}(x) = \frac{y(x)-y(a)}{x'-a'}$ we attain a differential equation $x^{1-\alpha}y'(x) = \alpha\frac{y(x)-y(a)}{x^{\alpha}-a^{\alpha}}$. The solution of the equation is

$$y(x) = \frac{A}{\alpha}x^{\alpha} + B,$$

where A and B are undetermined constants.

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