**Applied Mathematics E-Notes, 20(2020), 323-335 ©** ISSN 1607-2510
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# Some Approximation Properties Of King Type Generalization Of Modified Positive Linear Operators*

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Received 10 July 2019

**Abstract**

In manuscript, a variant of the operators defined in [1], which preserve the linear functions are investigated. The rate of convergence of these operators with the help of K-functional is discussed. Also, the modifications of the operators which preserve $x^2$ is defined. These modified operators yield better error estimates than the operator defined in [1]. Asymptotic formula, Quantitative Voronovskaya theorems and Grüss-type approximation results for these King's Type Generalized operators are established.

## 1 Introduction

In [1], the author constructed a class of linear and positive operators given by

$$S_n^{[v]}(f,x)=\sum_{k=0}^{\infty}\mathbf{p}_v(k,nx)f\left(\frac{k}{n}\right),\ n\in\mathbb{N}, \tag{1}$$

where $0\leq v<1$ and $\mathbf{p}_v(k,nx)=(1-v)\frac{1}{k!}(nx+vk)^ke^{-(nx+vk)}$. For $v=0$, the operators $S_n^{[v]}$ reduce to Szàsz-Mirakjan operators [2] as

$$S_n^{[0]}(f,x)=S_n(f,x)=e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right),\quad x\geq 0. \tag{2}$$

In [3], King presented an example of operators of Bernstein type which preserve the test functions $e_0(x)=1$ and $e_2(x)=x^2$ of the Bohman Korovkin theorem. Motivated by King's work [3], the Szàsz-Mirakjan operators were modified in [4] and [5]. Moreover, by letting $v_n(x):=x-\frac{1}{2n}$, $n\in\mathbb{N}$; it has been shown in [5] that the operators defined by $V_n(f;x):=S_n(f;v_n(x))$ do not preserve the test functions $e_1(x)=x$ and $e_2(x)=x^2$ but provide the best error estimation among all Szàsz-Mirakjan operators for all continuous bounded function $f$ on $[0,\infty)$ and for each $x\in[\frac{1}{2},\infty)$.

The aim of this manuscript is to investigate a variant of the operators (1) defined by Patel, which preserves the linear functions. The rate of convergence of these operators with the help of K-functional is disscussed. The modifications of these operators which preserve $x^2$ was introduced. These modified operators yield better error estimates than the operators defined in [1]. Also, an asymptotic formula, quantitative Voronovskaya theorems and Grüss-type approximation results for these King's type generalized operators are estabished.

## 2 Construction of The Operators

The following lemmas are needed for further discussion:

*Mathematics Subject Classifications: primary 41A25, 41A30, 41A36.
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**Lemma 1** ([1]) *Let* $0<\alpha<\infty$, $|\upsilon|<1$, $r\in\mathbb{N}$,

$$P(r,\alpha,\upsilon)=\sum_{k=0}^{\infty}\frac{1}{k!}(\alpha+\upsilon k)^{k+r}e^{-(\alpha+\upsilon k)}$$

*and*

$$(1-\upsilon)P(0,\alpha,\upsilon)=1.$$

*Then*

$$P(r,\alpha,\upsilon)=\sum_{k=0}^{\infty}\upsilon^k(\alpha+k\upsilon)P(r-1,\alpha+k\upsilon,\upsilon).$$

*When* $|\upsilon|<1$ , *one has*

$$\begin{aligned}
P(1,\alpha,\upsilon) &= \frac{\alpha}{(1-\upsilon)^2}+\frac{\upsilon^2}{(1-\upsilon)^3},\\
P(2,\alpha,\upsilon) &= \frac{\alpha^2}{(1-\upsilon)^3}+\frac{3\alpha\upsilon^2}{(1-\upsilon)^4}+\frac{\upsilon^3(1+2\upsilon)}{(1-\upsilon)^5},\\
P(3,\alpha,\upsilon) &= \frac{\alpha^3}{(1-\upsilon)^4}+\frac{6\alpha^2\upsilon^2}{(1-\upsilon)^5}+\frac{\alpha\upsilon^3\left(4+11\upsilon\right)}{(1-\upsilon)^6}+\frac{\upsilon^4+8\upsilon^5+6\upsilon^6}{(1-\upsilon)^7},\\
P(4,\alpha,\upsilon) &= \frac{\alpha^4}{(1-\upsilon)^5}+\frac{10\alpha^3\upsilon^2}{(1-\upsilon)^6}+\frac{5\alpha^2\left(2\upsilon^3+7\upsilon^4\right)}{(1-\upsilon)^7}\\
&\quad+\frac{5\alpha\left(\upsilon^4+10\upsilon^5+10\upsilon^6\right)}{(1-\upsilon)^8}+\frac{\upsilon^5+22\upsilon^6+58\upsilon^7+24\upsilon^8}{(1-\upsilon)^9},\\
P(5,\alpha,\upsilon) &= \frac{\alpha^5}{(1-\upsilon)^6}+\frac{15\alpha^4\upsilon^2}{(1-\upsilon)^7}+\frac{5\alpha^3\left(4\upsilon^3+17\upsilon^4\right)}{(1-\upsilon)^8}+\frac{15\alpha^2\left(\upsilon^4+12\upsilon^5+15\upsilon^6\right)}{(1-\upsilon)^9}\\
&\quad+\frac{\alpha\left(6\upsilon^5+157\upsilon^6+508\upsilon^7+274\upsilon^8\right)}{(1-\upsilon)^{10}}+\frac{\upsilon^6+52\upsilon^7+328\upsilon^8+444\upsilon^9+120\upsilon^{10}}{(1-\upsilon)^{11}},
\end{aligned}$$

$$\begin{aligned}
P(6,\alpha,\upsilon) &= \frac{\alpha^6}{(1-\upsilon)^7}+\frac{21\alpha^5\upsilon^2}{(1-\upsilon)^8}+\frac{35\alpha^4\left(\upsilon^3+5\upsilon^4\right)}{(1-\upsilon)^9}+\frac{35\alpha^3\left(\upsilon^4+14\upsilon^5+21\upsilon^6\right)}{(1-\upsilon)^{10}}\\
&\quad+\frac{7\alpha^2\left(3\upsilon^5+91\upsilon^6+349\upsilon^7+232\upsilon^8\right)}{(1-\upsilon)^{11}}+\frac{7\alpha\left(\upsilon^6+60\upsilon^7+444\upsilon^8+728\upsilon^9+252\upsilon^{10}\right)}{(1-\upsilon)^{12}}\\
&\quad+\frac{\upsilon^7+114\upsilon^8+1452\upsilon^9+4400\upsilon^{10}+3708\upsilon^{11}+720\upsilon^{12}}{(1-\upsilon)^{13}}.
\end{aligned}$$

## 3 Estimation of Moments

The following lemmas are required to prove main results.

**Lemma 2** ([1]) *The operators* $S_n^{[\upsilon]}$, $n\geq 1$, *defined by* (1) *satisfies the following relations*

1. $S_n^{[\upsilon]}(1,x)=1$,
2. $S_n^{[\upsilon]}(t,x)=\frac{x}{(1-\upsilon)}+\frac{\upsilon}{n(1-\upsilon)^2}$,
3. $S_n^{[\upsilon]}(t^2,x)=\frac{x^2}{(1-\upsilon)^2}+\frac{x(1+2\upsilon)}{n(1-\upsilon)^3}+\frac{\upsilon(1+2\upsilon)}{n^2(1-\upsilon)^4}$,
4. $S_n^{[\upsilon]}(t^3,x)=\frac{x^3}{(1-\upsilon)^3}+\frac{3x^2(1+\upsilon)}{n(1-\upsilon)^4}+\frac{x\left(1+8\upsilon+6\upsilon^2\right)}{n^2(1-\upsilon)^5}+\frac{\upsilon\left(1+8\upsilon+6\upsilon^2\right)}{n^3(1-\upsilon)^6}$,

5. $^{[\upsilon]}_{n}(t^4,x)=\frac{x^4}{(1-\upsilon)^4}+\frac{2x^3(3+2\upsilon)}{n(1-\upsilon)^5}+\frac{x^2\left(7+26\upsilon+12\upsilon^2\right)}{n^2(1-\upsilon)^6}+\frac{x\left(1+22\upsilon+58\upsilon^2+24\upsilon^3\right)}{n^3(1-\upsilon)^7}+\frac{\upsilon\left(1+22\upsilon+58\upsilon^2+24\upsilon^3\right)}{n^4(1-\upsilon)^8}$.

By a straightforward calculation, one obtain

$$\begin{aligned} S_n^{[\upsilon]}(t^5,x) &= \frac{(1-\upsilon)}{n^5}\left(P(5,nx+5\upsilon,\upsilon)+10P(4,nx+4\upsilon,\upsilon)+25P(3,nx+3\upsilon,\upsilon)\right. \\ &\left. +15P(2,nx+2\upsilon,\upsilon)+P(1,nx+\upsilon,\upsilon)\right) \\ &= \frac{x^5}{(1-\upsilon)^5}+\frac{5x^4(2+\upsilon)}{n(1-\upsilon)^6}+\frac{5x^3\left(5+12\upsilon+4\upsilon^2\right)}{n^2(1-\upsilon)^7}+\frac{15x^2\left(1+9\upsilon+14\upsilon^2+4\upsilon^3\right)}{n^3(1-\upsilon)^8}. \\ &+\frac{x\left(1+52\upsilon+328\upsilon^2+444\upsilon^3+120\upsilon^4\right)}{n^4(1-\upsilon)^9}+\frac{\upsilon\left(1+52\upsilon+328\upsilon^2+444\upsilon^3+120\upsilon^4\right)}{n^5(1-\upsilon)^{10}}, \end{aligned}$$

$$\begin{aligned} S_n^{[\upsilon]}(t^6,x) &= \frac{(1-\upsilon)}{n^6}\left(P(6,nx+6\upsilon,\upsilon)+15P(5,nx+5\upsilon,\upsilon)\right. \\ &\left.+65P(4,nx+4\upsilon,\upsilon)+90P(3,nx+3\upsilon,\upsilon)+31P(2,nx+2\upsilon,\upsilon)+P(1,nx+\upsilon,\upsilon)\right). \end{aligned}$$

Hence, we have

$$\begin{aligned} S_n^{[\upsilon]}(t^6,x) &= \frac{x^6}{(1-\upsilon)^6}+\frac{3x^5(5+2\upsilon)}{n(1-\upsilon)^7}+\frac{5x^4\left(13+23\upsilon+6\upsilon^2\right)}{n^2(1-\upsilon)^8}+\frac{10x^3\left(9+50\upsilon+55\upsilon^2+12\upsilon^3\right)}{n^3(1-\upsilon)^9} \\ &+\frac{x^2\left(31+562\upsilon+1978\upsilon^2+1794\upsilon^3+360\upsilon^4\right)}{n^4(1-\upsilon)^{10}} \\ &+\frac{x\left(1+114\upsilon+1452\upsilon^2+4400\upsilon^3+3708\upsilon^4+720\upsilon^5\right)}{n^5(1-\upsilon)^{11}} \\ &+\frac{\upsilon\left(1+114\upsilon+1452\upsilon^2+4400\upsilon^3+3708\upsilon^4+720\upsilon^5\right)}{n^6(1-\upsilon)^{12}}. \end{aligned}$$

## 4 Approximation Properties

The modification of the classical Bernstein operators proposed by King [3]. These type of Bernstein operators preserve test function $e_0$ & $e_2$ and give faster convergence. This approach was applied to some well known operators, details are found in [6, 7, 8, 9, 10]. One can observe that, the operator $S_n^{[\upsilon]}$ introduced in (1) preserve only the constant so further modification of these operators is proposed to be made so that the modified operators preserve the constant as well as linear functions, for this purpose the modification of $S_n^{[\upsilon]}$ is defined as follows:

$$S_n^{*[\upsilon]}(f,x)=\sum_{k=0}^{\infty}\mathbf{p}_\upsilon(k,nr_n(x))f\left(\frac{k}{n}\right), \tag{3}$$

where

$$r_n(x)=\frac{nx(1-\upsilon)^2-\upsilon}{n(1-\upsilon)}=x(1-\upsilon)-\frac{\upsilon}{n(1-\upsilon)}$$

and $\mathbf{p}_\upsilon(k,nx)$ is as defined in (1).

**Lemma 3** *The operators defined in (3) satisfies for each $x\geq 0$, the following identities*

$$S_n^{*[\upsilon]}(1,x)=1,$$

$$S_n^{*[\upsilon]}(t,x)=x,$$

$$S_n^{*[\upsilon]}(t^2,x)=x^2+\frac{x}{n(1-\upsilon)^2}+\frac{\upsilon^2}{n^2(1-\upsilon)^4},$$

$$S_n^{*[\upsilon]}(t^3,x)=x^3+\frac{3x^2}{n(1-\upsilon)^2}+\frac{x\left(1+2\upsilon+3\upsilon^2\right)}{n^2(1-\upsilon)^4}+\frac{\upsilon^2(3+2\upsilon)}{n^3(1-\upsilon)^6},$$

$$S_n^{*[\upsilon]}(t^4,x)=x^4+\frac{6x^3}{n(1-\upsilon)^2}+\frac{x^2\left(7+8\upsilon+6\upsilon^2\right)}{n^2(1-\upsilon)^4}+\frac{x\left(1+8\upsilon+24\upsilon^2+8\upsilon^3\right)}{n^3(1-\upsilon)^6}+\frac{\upsilon^2\left(7+20\upsilon+9\upsilon^2\right)}{n^4(1-\upsilon)^8},$$

$$\begin{aligned}S_n^{*[\upsilon]}(t^5,x) &= x^5+\frac{10x^4}{n(1-\upsilon)^2}+\frac{5x^3\left(5+4\upsilon+2\upsilon^2\right)}{n^2(1-\upsilon)^4}+\frac{5x^2\left(3+12\upsilon+18\upsilon^2+4\upsilon^3\right)}{n^3(1-\upsilon)^6}\\ &\quad+\frac{x\left(1+22\upsilon+133\upsilon^2+164\upsilon^3+45\upsilon^4\right)}{n^4(1-\upsilon)^8}+\frac{\upsilon^2\left(15+110\upsilon+160\upsilon^2+44\upsilon^3\right)}{n^5(1-\upsilon)^{10}},\end{aligned}$$

$$\begin{aligned}S_n^{*[\upsilon]}(t^6,x) &= x^6+\frac{15x^5}{n(-1+\upsilon)^2}+\frac{5x^4\left(13+8\upsilon+3\upsilon^2\right)}{n^2(-1+\upsilon)^4}+\frac{10x^3\left(9+24\upsilon+24\upsilon^2+4\upsilon^3\right)}{n^3(-1+\upsilon)^6}\\ &\quad+\frac{x^2\left(31+292\upsilon+868\upsilon^2+684\upsilon^3+135\upsilon^4\right)}{n^4(-1+\upsilon)^8}\\ &\quad+\frac{x\left(1+52\upsilon+598\upsilon^2+1684\upsilon^3+1385\upsilon^4+264\upsilon^5\right)}{n^5(-1+\upsilon)^{10}}\\ &\quad+\frac{31\upsilon^2+472\upsilon^3+1543\upsilon^4+1344\upsilon^5+265\upsilon^6}{n^6(-1+\upsilon)^{12}}.\end{aligned}$$

By direct computation, the proof of the above identities can be obtained. By linear properties of the operators $S_n^{*[\upsilon]}$, one has

$$S_n^{*[\upsilon]}(t-x,x)=0,$$

$$S_n^{*[\upsilon]}((t-x)^2,x)=\frac{x}{n(1-\upsilon)^2}+\frac{\upsilon^2}{n^2(1-\upsilon)^4},$$

$$S_n^{*[\upsilon]}((t-x)^3,x)=\frac{x(1+2\upsilon)}{n^2(1-\upsilon)^4}+\frac{\upsilon^2(3+2\upsilon)}{n^3(1-\upsilon)^6},$$

$$S_n^{*[\upsilon]}((t-x)^4,x)=\frac{3x^2}{n^2(1-\upsilon)^4}+\frac{x\left(1+8\upsilon+12\upsilon^2\right)}{n^3(1-\upsilon)^6}+\frac{\upsilon^2\left(7+20\upsilon+9\upsilon^2\right)}{n^4(1-\upsilon)^8},$$

$$S_n^{*[\upsilon]}((t-x)^5,x)=\frac{10x^2(1+2\upsilon)}{n^3(1-\upsilon)^6}+\frac{x\left(1+22\upsilon+98\upsilon^2+64\upsilon^3\right)}{n^4(1-\upsilon)^8}+\frac{\upsilon^2\left(15+110\upsilon+160\upsilon^2+44\upsilon^3\right)}{n^5(1-\upsilon)^{10}},$$

$$\begin{aligned}S_n^{*[\upsilon]}((t-x)^6,x) &= \frac{15x^3}{n^3(1-\upsilon)^6}+\frac{5x^2\left(5+32\upsilon+35\upsilon^2\right)}{n^4(1-\upsilon)^8}+\frac{x\left(1+52\upsilon+508\upsilon^2+1024\upsilon^3+425\upsilon^4\right)}{n^5(1-\upsilon)^{10}}\\ &\quad+\frac{\upsilon^2\left(31+472\upsilon+1543\upsilon^2+1344\upsilon^3+265\upsilon^4\right)}{n^6(1-\upsilon)^{12}}.\end{aligned}$$

**Remark 1** *Let $n>1$ be a given number. For every $0<\upsilon<1$, one has*

$$S_n^{*[\upsilon]}((t-x)^2,x)\leq\frac{1}{n(1-\upsilon)^2}\left(x+\frac{1}{n(1-\upsilon)^2}\right).$$

**Remark 2** *Let $n>1$ be a given number. For every $0<\upsilon<1$, one has*

$$\begin{aligned}S_n^{*[\upsilon]}((t-x)^4,x) &= \frac{3n^2x^2(1-\upsilon)^4+\upsilon^2\left(7+20\upsilon+9\upsilon^2\right)+nx(1-\upsilon)^2\left(1+8\upsilon+12\upsilon^2\right)}{n^4(1-s\upsilon)^8}\\ &\leq \frac{3n^2x^2+36+21nx}{n^4(1-\upsilon)^8}\leq\frac{36(x^2+x+1)}{n^2(1-\upsilon)^8}.\end{aligned}$$

**Remark 3** *Let $n > 1$ be a given number. For every $0 < \upsilon < 1$ and $x \in (0, \infty)$, one has*

$$\begin{aligned}\frac{S_n^{*[\upsilon]}((t-x)^6,x)}{S_n^{*[\upsilon]}((t-x)^2,x)} &= \frac{15n^3x^3(1-\upsilon)^6+5n^2x^2(1-\upsilon)^4(1+5\upsilon)(5+7\upsilon)+}{n^4(1-\upsilon)^8\left(nx(1-\upsilon)^2+n^2x^2(1-\upsilon)^4+\upsilon^2\right)}\\ &\quad+\frac{nx(1-\upsilon)^2(1+\upsilon(52+\upsilon(508+\upsilon(1024+425\upsilon))))}{n^4(1-\upsilon)^8\left(nx(1-\upsilon)^2+n^2x^2(1-\upsilon)^4+\upsilon^2\right)}\\ &\quad+\frac{\upsilon^2(31+\upsilon(472+\upsilon(1543+\upsilon(1344+265\upsilon))))}{n^4(1-\upsilon)^8\left(nx(1-\upsilon)^2+n^2x^2(1-\upsilon)^4+\upsilon^2\right)}\\ &\le \frac{15n^3x^3+360n^2x^2+2010nx+3655}{n^4(1-\upsilon)^8\left(nx(1-\upsilon)^2+n^2x^2(1-\upsilon)^4+\upsilon^2\right)}. \qquad (4)\end{aligned}$$

*Using $(1-\upsilon)^4 \le (1-\upsilon)^2$ and $1 < n < n^2$, one get*

$$nx(1-\upsilon)^2+n^2x^2(1-\upsilon)^4+\upsilon^2 \ge nx(1-\upsilon)^4+nx^2(1-\upsilon)^4+\upsilon^2 \ge n(1-\upsilon)^4(x+x^2).$$

*Therefore,*

$$\frac{1}{nx(1-\upsilon)^2+n^2x^2(1-\upsilon)^4+\upsilon^2} \le \frac{1}{n(1-\upsilon)^4(x+x^2)}. \qquad (5)$$

*Also,*

$$\begin{aligned}15n^3x^3+360n^2x^2+2010nx+3655 &\le 360n^3x^2(x+1)+3655n(x+1)\\ &\le 3655(x+1)n^3(x^2+1). \qquad (6)\end{aligned}$$

*Using (5) and (6) in (4), one has*

$$\frac{S_n^{*[\upsilon]}((t-x)^6,x)}{S_n^{*[\upsilon]}((t-x)^2,x)} \le \frac{3655(x+1)n^3(x^2+1)}{n^5(1-\upsilon)^{12}(x+x^2)} = \frac{3655(x^2+1)}{n^2(1-\upsilon)^{12}x}.$$

*The general technique to construct sequences of operators of discrete type with the same property as given in [3] was obtained by Agratini [11]. With the help of this technique, the function is defined as*

$$\widetilde{r}_n(x) = \frac{\sqrt{1-4\upsilon^2+4x^2n^2(1-\upsilon)^4}-1-2\upsilon}{2n(1-\upsilon)}.$$

*Using this function, the following linear and positive operators are defined*

$$\widetilde{S}_n^{*[\upsilon]}(f,x) = \sum_{k=0}^{\infty} \mathbf{p}_\upsilon(k, n\widetilde{r}_n(x)) f\left(\frac{k}{n}\right). \qquad (7)$$

*Direct computation gives*

$$\widetilde{S}_n^{*[\upsilon]}(1,x) = 1,$$

$$\widetilde{S}_n^{*[\upsilon]}(t,x) = \frac{\sqrt{1-4\upsilon^2+4n^2x^2(1-\upsilon)^4}-1}{2n(1-\upsilon)^2},$$

$$\widetilde{S}_n^{*[\upsilon]}(t^2,x) = x^2.$$

*Also,*

$$\widetilde{S}_n^{*[\upsilon]}(t-x,x) = \frac{\sqrt{1-4\upsilon^2+4n^2x^2(1-\upsilon)^4}-1}{2n(1-\upsilon)^2} - x,$$

$$\widetilde{S}_n^{*[\upsilon]}((t-x)^2,x) = 2x^2 + x\left(\frac{1-\sqrt{1-4\upsilon^2+4n^2x^2(-1+\upsilon)^4}}{n(1-\upsilon)^2}\right).$$

*Now, by defining a function similar to the one used in [5],*

$$\widehat{r}_n(x) = (1-\upsilon)\left(x - \frac{1}{2n}\right) - \frac{\upsilon}{n(1-\upsilon)}.$$

*The linear and positive operators are define, for $n \in \mathbb{N}$ and $f \in C([0,\infty))$,*

$$\widehat{S}_n^{*[\upsilon]}(f,x) = \sum_{k=0}^{\infty} \mathbf{p}_\upsilon(k, n\widehat{r}_n(x)) f\left(\frac{k}{n}\right). \tag{8}$$

*The following identities, for the operators $\widehat{S}_n^{*[\upsilon]}$, $x \geq \frac{1}{2}$, can be obtained*

$$\widetilde{S}_n^{*[\upsilon]}(1,x) = 1,$$

$$\widetilde{S}_n^{*[\upsilon]}(t,x) = x - \frac{1}{2n},$$

$$\widetilde{S}_n^{*[\upsilon]}(t^2,x) = x^2 + \frac{x(2-\upsilon)\upsilon}{n(1-\upsilon)^2} + \frac{-1+8\upsilon^2-4\upsilon^3+\upsilon^4}{4n^2(1-\upsilon)^4}.$$

*and*

$$\widetilde{S}_n^{*[\upsilon]}(t-x,x) = \frac{1}{2n},$$

$$\widetilde{S}_n^{*[\upsilon]}((t-x)^2,x) = \frac{x}{n(1-\upsilon)^2} + \frac{-1+8\upsilon^2-4\upsilon^3+\upsilon^4}{4n^2(1-\upsilon)^4}.$$

*The modification of Jain operators [12], the author consider the restriction that $\upsilon_n \to 0$ as $n \to \infty$, in order to get convergence. Here, in (3), (7) and (8) need not to be take any restriction on $\upsilon$. Because of this interesting property its worth to study these operators.*

## 4.1 Asymptotic Formula

**Theorem 4** *Let $f$ be a continuous function on $[0,\infty)$ and $0 \leq a < b < \infty$. Then the sequence $\{S_n^{*[\upsilon]}(f,\cdot)\}$ converges uniformly to $f$ as $n \to \infty$ in $[a,b]$.*

**Proof.** Using Lemma 3, observe that $S_n^{*[\upsilon]}(1,x)$, $S_n^{*[\upsilon]}(t,x)$ and $S_n^{*[\upsilon]}(t^2,x)$ converges uniformly to 1, $x$ and $x^2$ as $n \to \infty$ respectively on every compact subset of $[0,\infty)$. Thus, the required result follows from Bohman-Korovkin theorem. ∎

Similarly, we have

**Theorem 5** *Let $f$ be a continuous function on $[0,\infty)$ and $0 \leq a < b < \infty$. Then the sequence $\{\widetilde{S}_n^{*[\upsilon]}(f,\cdot)\}$ converges uniformly to $f$ as $n \to \infty$ in $[a,b]$.*

**Theorem 6** *Let $f$ be a continuous function on $[0,\infty)$ and $0 \leq a < b < \infty$. Then the sequence $\{\widehat{S}_n^{*[\upsilon]}(f,x)\}$ converges uniformly to $f(x)$ as $n \to \infty$ in $[a,b]$.*

**Theorem 7** *Let $f$ be a bounded integrable function on $[0,\infty)$ which has second derivative at a point $x \in [0,\infty)$. Then*

$$\lim_{n\to\infty} n\left[S_n^{*[\upsilon]}(f,x) - f(x)\right] = \frac{x}{2(1-\upsilon)^2} f''(x).$$

**Proof.** By the Taylor's expansion of $f$, one has

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''^2 + r(t,x)(t-x)^2, \tag{9}$$

where $r(t,x)$ is the remainder term and $\lim_{t\to x} r(t,x) = 0$. Operating $S_n^*$ to the equation (9), one obtains

$$S_n^{*[\upsilon]}(f,x) - f(x) = f'(x)S_n^{*[\upsilon]}((t-x),x) + \frac{f''(x)}{2}S_n^{*[\upsilon]}((t-x)^2,x) + S_n^{*[\upsilon]}(r(t,x)(t-x)^2,x). \tag{10}$$

Using the Cauchy-Schwarz inequality, one has

$$S_n^{*[\upsilon]}(r(t,x)(t-x)^2,x) \le \sqrt{S_n^{*[\upsilon]}((r(t,x))^2,x)}\sqrt{S_n^{*[\upsilon]}((t-x)^4,x)}. \tag{11}$$

As $r(x,x) = 0$ and $r^2(t,x) \in C_2^*[0,\infty)$, one has

$$\lim_{n\to\infty} S_n^{*[\upsilon]}((r(t,x))^2,x) = (r(x,x))^2 = 0 \tag{12}$$

uniformly with respect to $x \in [0,A]$. From (11) and (12), one has

$$\begin{aligned}
\lim_{n\to\infty} n\left[S_n^{*[\upsilon]}(f,x) - f(x)\right] &= \lim_{n\to\infty} n\left[f'(x)S_n^{*[\upsilon]}((t-x),x) + \frac{f''(x)}{2}S_n^{*[\upsilon]}((t-x)^2,x)\right. \\
&\quad \left. + S_n^{*[\upsilon]}(r(t,x)(t-x)^2,x)\right]. \\
&= f'(x)\lim_{n\to\infty} nS_n^{*[\upsilon]}((t-x),x) + \frac{f''(x)}{2}\lim_{n\to\infty} nS_n^{*[\upsilon]}((t-x)^2,x) \\
&= \frac{f''(x)}{2}\lim_{n\to\infty}\left[\frac{x}{(1-\upsilon)^2} + \frac{\upsilon^2}{n(1-\upsilon)^4}\right] = \frac{x}{2(1-\upsilon)^2}f''(x).
\end{aligned}$$

■

**Theorem 8** *Let $f$ be a bounded integrable function on $[0,\infty)$ which has second derivative at a point $x \in [0,\infty)$. Then*

$$\lim_{n\to\infty} n\left[\widehat{S}_n^{*[\upsilon]}(f,x) - f(x)\right] = \frac{x}{2(1-\upsilon)^2}f''(x).$$

The proof of the above theorem follows along the lines of Theorem 7, thus the details are omitted.

## 4.2 The Rate of Convergence

The set $C_B([0,\infty))$ is the class of real valued continuous bounded functions $f$ on $x \in [0,\infty)$ with the norm $\|f\| = \sup_{x\in[0,\infty)}|f(x)|$. For $f \in C_B([0,\infty))$ and $\delta > 0$ the $m$-th order modulus of continuity is defined as

$$\omega_m(f,\delta) = \sup_{0\le h\le\delta}\sup_{x\in[0,\infty)}|\triangle_h^m f(x)|,$$

where $\triangle$ is the forward difference. For $m = 1$, $\omega_m(f,\delta)$ is the usual modulus of continuity. The Peetre's $K$-functional is defined as

$$K_2(f,\delta) = \inf_{g\in C_B^2([0,\infty))}\{\|f-g\| + \delta\|g''\| : g \in C_B^2([0,\infty))\},$$

where

$$C_B^2([0,\infty)) = \{g \in C_B([0,\infty)) : g', g'' \in C_B([0,\infty))\}.$$

The following direct results are established in this section:

**Theorem 9** *Let $f \in C_B([0,\infty))$ and $0 \le \upsilon < 1$. Then*

$$|S_n^{*[\upsilon]}(f,x) - f(x)| \le 2K\left(f, \frac{1}{2n(1-\upsilon)^2}\left(x + \frac{1}{n(1-\upsilon)^2}\right)\right). \tag{13}$$

**Proof.** Consider any $g \in C_B^2([0,\infty))$ and an arbitrarily fixed $x \in [0,\infty)$. Using the Taylor formula, one obtain

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)du, \quad t \ge 0.$$

Since $S_n^{*[\upsilon]}$ is a linear and positive operator, one obtains

$$\begin{aligned}
|S_n^{*[\upsilon]}(g,x) - g(x)| &\le |g'(x)S_n^{*[\upsilon]}((t-x),x)| + \left|S_n^{*[\upsilon]}\left(\int_x^t (t-u)g''(u)du, x\right)\right| \\
&\le S_n^{*[\upsilon]}\left(\left|\int_x^t (t-u)g''(u)du\right|, x\right) \\
&\le \|g''\| S_n^{*[\upsilon]}\left(\left|\int_x^t (t-u)du\right|, x\right) \\
&\le \|g''\| S_n^{*[\upsilon]}\left((t-x)^2, x\right) \le \frac{\|g''\|}{n(1-\upsilon)^2}\left(x + \frac{1}{n(1-\upsilon)^2}\right).
\end{aligned}$$

For $f \in C_B([0,\infty))$, one obtain

$$\begin{aligned}
|S_n^{*[\upsilon]}(f,x) - f(x)| &\le |S_n^{*[\upsilon]}(f-g,x)| + |S_n^{*[\upsilon]}(g,x) - g(x)| + |g(x) - f(x)| \\
&\le 2\|f-g\| + \frac{\|g''\|}{n(1-\upsilon)^2}\left(x + \frac{1}{n(1-\upsilon)^2}\right).
\end{aligned}$$

By using the $K$-functional described by

$$K(f,\delta) = \inf_{g \in C_B([0,\infty))} \left(\|f-g\| + \delta\|g\|\right),$$

for $x \ge 0$,

$$|S_n^{*[\upsilon]}(f,x) - f(x)| \le 2K\left(f, \frac{1}{2n(1-\upsilon)^2}\left(x + \frac{1}{n(1-\upsilon)^2}\right)\right).$$

■

**Theorem 10** *Let $f \in C_B([0,\infty))$ and $0 \le \upsilon < 1$. Then*

$$|\widetilde{S}_n^{*[\upsilon]}(f,x) - f(x)| \le 2\omega(f, \widetilde{\delta}(x)),$$

*where*

$$\widetilde{\delta}(x) = \sqrt{2x^2 + x\left(\frac{1 - \sqrt{1 - 4\upsilon^2 + 4n^2x^2(-1+\upsilon)^4}}{n(1-\upsilon)^2}\right)}.$$

**Proof.** Let $f \in C_B([0,\infty))$ and $x \ge 0$. Using linearity and monotonicity of $\widetilde{S}_n$, one gets

$$|\widetilde{S}_n^{*[\upsilon]}(f,x) - f(x)| \le \omega(f,\delta)\left(1 + \frac{1}{\delta}\sqrt{\widetilde{S}_n^{*[\upsilon]}((t-x)^2, x)}\right).$$

Choosing $\delta = \widetilde{\delta}(x)$, the proof follows. ■

**Theorem 11** *Let $f \in C_B([0,\infty))$, $0 \le \upsilon < 1$ and $x \ge \frac{1}{2}$. Then*

$$|\widehat{S}_n^{*[\upsilon]}(f,x) - f(x)| \le 2\omega\left(f, \widehat{\delta}(x)\right),$$

*where*

$$\widehat{\delta}(x) = \sqrt{\frac{x}{n(1-\upsilon)^2} + \frac{-1+8\upsilon^2-4\upsilon^3+\upsilon^4}{4n^2(1-\upsilon)^4}}.$$

## 4.3 Quantitative Voronovskaya Theorem

The weighted modulus of continuity is denoted by $\Omega(f,\delta)$ and given by

$$\Omega(f,\delta) = \sup_{0\le h<\delta, x\in[0,\infty)} \frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)}$$

for $f \in C_{x^2}([0,\infty))$. For every $f \in C^*_{x^2}([0,\infty))$,

$$\lim_{\delta\to 0} \Omega(f,\delta) = 0$$

and

$$\Omega(f,\lambda\delta) = 2(1+\lambda)(1+\delta^2)\Omega(f,\delta), \qquad \lambda > 0, \tag{14}$$

which was proved in [13, p. 359–360]. Using the definition of the weighted modulus of continuity, one write

$$|f(y)-f(x)| \le (1+x^2)(1+(y-x)^2)\Omega(f;|x-y|).$$

Also, using the property (14), one has

$$\begin{aligned} |f(y)-f(x)| &\le (1+(y-x)^2)(1+x^2)\Omega(f;|x-y|) \\ &\le 2\left(1+\frac{|y-x|}{\delta}\right)\left(1+\delta^2\right)\Omega(f,\delta)\left(1+(y-x)^2\right)(1+x^2). \end{aligned} \tag{15}$$

Now, we give the quantitative Voronovskaya-type theorem in weighted spaces, as follows:

**Theorem 12** *Let $S_n^{*[\upsilon]}$ be defined as in (3), where $0 \le \upsilon < 1$. If $f \in C([0,\infty))$ and $f'' \in C^*_{x^2}([0,\infty))$, then one has for $x \in (0,\infty)$ that*

$$\begin{aligned} & |S_n^{*[\upsilon]}(f,x) - f(x) - \frac{f''(x)}{2} S_n^{*[\upsilon]}((t-x)^2,x)| \\ \le\ & \frac{16(1+x^2)}{n(1-\upsilon)^2}\Omega\left(f^{(2)}, \sqrt[4]{\frac{3655(x^2+1)}{n^2(1-\upsilon)^{12}x}}\right)\left(x+\frac{1}{n(1-\upsilon)^2}\right). \end{aligned}$$

**Proof.** Using the Taylor's expansion of $f$, one obtain

$$f(t) = f(x) + f'(x)(t-x) + \frac{f''(x)}{2}(t-x)^2 + R_2(f;t,x), \tag{16}$$

where

$$R_2(f;t,x) = \frac{(t-x)^2}{2!}\left(f^{(2)}(\xi) - f^{(2)}(x)\right) \quad \text{for } \xi \in (t,x).$$

Applying the operators $S_n^{*[\upsilon]}$ to both sides of (16), we obtain

$$S_n^{*[\upsilon]}(f,x) - f(x) - f'(x)S_n^{*[\upsilon]}(t-x,x) - \frac{f''(x)}{2}S_n^{*[\upsilon]}((t-x)^2,x) = S_n^{*[\upsilon]}(R_2(f;t,x),x). \tag{17}$$

Hence, using the inequality $|\xi - x| \leq |t-x|$,

$$\begin{aligned}
|R_2^*(f;t,x)| := & \quad \frac{1}{2}|f^{(2)}(\xi) - f^{(2)}(x)| \\
\leq & \quad \frac{1}{2}\Omega(f^{(2)}; |\xi - x|)\left(1+(\xi-x)^2\right)(1+x^2) \\
\leq & \quad \frac{1}{2}\Omega(f^{(2)}; |t - x|)\left(1+(t-x)^2\right)(1+x^2) \\
\leq & \quad \left(1+\frac{|t-x|}{\delta}\right)(1+\delta^2)\,\Omega(f^{(2)},\delta)\left(1+(t-x)^2\right)(1+x^2).
\end{aligned}$$

If $|t-x| < \delta$, one has

$$|R_2^*(f;t,x)| \leq 2\left(1+\delta^2\right)^2\Omega(f^{(2)},\delta)(1+x^2)$$

and if $|t-x| \geq \delta$, one has

$$|R_2^*(f;t,x)| \leq 2\left(1+\delta^2\right)^2\frac{(t-x)^4}{\delta^4}\Omega(f^{(2)},\delta)(1+x^2).$$

Therefore any for $t \in [0,\infty)$ and choosing $\delta < 1$, we get

$$\begin{aligned}
|R_2^*(f;t,x)| \quad \leq & \quad 2\left(1+\delta^2\right)^2\left(1+\frac{(t-x)^4}{\delta^4}\right)\Omega(f^{(2)},\delta)(1+x^2) \\
\leq & \quad 8\left(1+\frac{(t-x)^4}{\delta^4}\right)(1+x^2)\Omega(f^{(2)},\delta).
\end{aligned}$$

Using the above inequality, we deduce

$$\begin{aligned}
S_n^{*[\upsilon]}(|R_2(f;t,x)|,x) \quad = & \quad S_n^{*[\upsilon]}((t-x)^2|R_2^*(f;t,x)|,x) \\
\leq & \quad 8(1+x^2)\Omega(f^{(2)},\delta)\left(S_n^{*[\upsilon]}((t-x)^2,x)+\frac{1}{\delta^4}S_n^{*[\upsilon]}((t-x)^6,x)\right).
\end{aligned}$$

Using above inequality, Remarks 1, 3 and the equation (17), one has

$$\begin{aligned}
|S_n^{*[\upsilon]}(f,x) \quad - & \quad f(x) - \frac{f''(x)}{2}S_n^{*[\upsilon]}((t-x)^2,x)| \\
\leq & \quad 8(1+x^2)\Omega(f^{(2)},\delta)\left(S_n^{*[\upsilon]}((t-x)^2,x)+\frac{1}{\delta^4}S_n^{*[\upsilon]}((t-x)^6,x)\right) \\
\leq & \quad 8(1+x^2)\Omega(f^{(2)},\delta)S_n^{*[\upsilon]}((t-x)^2,x)\left(1+\frac{1}{\delta^4}\left(\frac{3655(x^2+1)}{n^2(1-\upsilon)^{12}x}\right)\right).
\end{aligned}$$

Choosing $\delta = \left(\frac{3655(x^2+1)}{n^2(1-\upsilon)^{12}x}\right)^{1/4}$, one has

$$|S_n^{*[\upsilon]}(f,x) - f(x) - \frac{f''(x)}{2}S_n^{*[\upsilon]}((t-x)^2,x)| \leq \frac{16(1+x^2)}{n(1-\upsilon)^2}\Omega(f^{(2)},\delta)\left(x+\frac{1}{n(1-\upsilon)^2}\right),$$

as desired. ■

## 4.4 Grüss-Type Approximation Results

**Theorem 13** *Let $S_n^{*[\upsilon]}$ be defined as in (3), where $0 \leq \upsilon < 1$. Let $E$ be subspace of $C([0,\infty))$. $f,g \in E \cup C_{x^2}^*([0,\infty))$ be such that $f^2, g^2 \in E \cup C_{x^2}^*([0,\infty))$ and $fg \in E$. Then for fixed $x \in [0,\infty)$ the inequality*

$$|S_n^{*[\upsilon]}(fg,x) - S_n^{*[\upsilon]}(f,x)S_n^{*[\upsilon]}(g,x)| \leq A_f(x)A_g(x),$$

*where*

$$A_f(x) = \sqrt{32(1+x^2)\left(\Omega(f^2,\delta) + (1+M_\upsilon)(1+x^2)\|f\|_{x^2}\Omega(f,\delta)\right)}$$

*and*

$$A_g(x) = \sqrt{32(1+x^2)\left(\Omega(g^2,\delta) + (1+M_\upsilon)(1+x^2)\|g\|_{x^2}\Omega(g,\delta)\right)}.$$

**Proof.** Denote

$$D_n(f,g;x) = S_n^{*[\upsilon]}(fg,x) - S_n^{*[\upsilon]}(f,x)S_n^{*[\upsilon]}(g,x). \tag{18}$$

Applyig the Cauchy-Schwarz inequality, one has

$$|D_n(f,g;x)| \leq \sqrt{D_n(f,f;x)D_n(g,g;x)}.$$

On the other hand, using (15),

$$|S_n^{*[\upsilon]}(f,x) - f(x)| \leq 2(1+x^2)(1+\delta^2)\Omega(f,\delta)S_n^{*[\upsilon]}\left(\left(1+\frac{|y-x|}{\delta}\right)\left(1+(y-x)^2\right),x\right). \tag{19}$$

Let $A(x,y) := \left(1+\frac{|y-x|}{\delta}\right)\left(1+(y-x)^2\right)$ . Since

$$A(x,y) \leq \begin{cases} 2(1+\delta^2), & |y-x| \leq \delta, \\ 2(1+\delta^2)\frac{(y-x)^4}{\delta^4}, & |y-x| \geq \delta, \end{cases}$$

we obtain for all $x,t \in [0,\infty)$ that

$$A(x,y) \leq 2(1+\delta^2)\left(1+\frac{(y-x)^4}{\delta^4}\right). \tag{20}$$

Using the inequalities (19) and (20), one gets

$$\begin{aligned} |S_n^{*[\upsilon]}(f,x) - f(x)| &\leq 4(1+x^2)(1+\delta^2)^2\Omega(f,\delta)S_n^{*[\upsilon]}\left(\left(1+\frac{(y-x)^4}{\delta^4}\right),x\right) \\ &\leq 16(1+x^2)\Omega(f,\delta)\left(1+\delta^4 S_n^{*[\upsilon]}\left((y-x)^4,x\right)\right). \end{aligned} \tag{21}$$

By definition (18),

$$\begin{aligned} D_n(f,f;x) &= S_n^{*[\upsilon]}(f^2,x) - f^2(x) + f^2(x) - \left(S_n^{*[\upsilon]}(f,x)\right)^2 \\ &= S_n^{*[\upsilon]}(f^2,x) - f^2(x) + \left(f(x) - S_n^{*[\upsilon]}(f,x)\right)\left(f(x) + S_n^{*[\upsilon]}(f,x)\right). \end{aligned}$$

Since $f \in C^*_{x^2}[0,\infty)$,

$$\frac{S_n^{*[\upsilon]}(f,x)}{1+x^2} \leq \frac{\|f\|_{x^2}S_n^{*[\upsilon]}((1+t^2),x)}{1+x^2}.$$

Now, using $0<\upsilon<1$ and $n \geq 1$, one gets

$$\begin{aligned} S_n^{*[\upsilon]}((1+t^2),x) &= 1+x^2+\frac{x}{n(1-\upsilon)^2}+\frac{\upsilon^2}{n^2(1-\upsilon)^4} \\ &\leq 1+x^2+\frac{\upsilon^2+nx(1-\upsilon)^2}{n^2(-1+\upsilon)^4} \\ &\leq 1+x^2+\frac{1}{n^2(1-\upsilon)^4}+\frac{x}{n(1-\upsilon)^4} \\ &\leq \frac{1}{(1-\upsilon)^4}\left(2+x^2+x\right). \end{aligned}$$

If $x \in [0,1]$, then $x \leq 1$ and if $x \in (1,\infty)$, then $x \leq x^2$. Hence, one has

$$S_n^{*[\upsilon]}((1+t^2),x) \leq \frac{2\left(1+x^2\right)}{(1-\upsilon)^4}.$$

Finally,

$$\frac{S_n^{*[\upsilon]}(f,x)}{1+x^2} \leq \frac{\|f\|_{x^2} M_\upsilon (1+x^2)}{1+x^2} = \|f\|_{x^2} M_\upsilon.$$

Hence,

$$|D_n(f,f;x)| \leq |S_n^{*[\upsilon]}(f^2,x) - f^2(x)| + |f(x) - S_n^{*[\upsilon]}(f,x)| \left(\|f\|_{x^2} + M_\upsilon \|f\|_{x^2}\right)(1+x^2).$$

Using the inequality (21), one deduce

$$\begin{aligned} |D_n(f,f;x)| &\leq 16(1+x^2)\Omega(f^2,\delta)\left(1+\delta^4 S_n^{*[\upsilon]}\left((y-x)^4,x\right)\right) \\ &\quad +16(1+x^2)^2(1+M_\upsilon)\|f\|_{x^2}\Omega(f,\delta)\left(1+\delta^4 S_n^{*[\upsilon]}\left((y-x)^4,x\right)\right) \\ &\leq 16(1+x^2)\Omega(f^2,\delta)\left(1+\delta^4\frac{36(x^2+x+1)}{n^2(1-\upsilon)^8}\right) \\ &\quad +16(1+x^2)^2(1+M_\upsilon)\|f\|_{x^2}\Omega(f,\delta)\left(1+\delta^4\frac{36(x^2+x+1)}{n^2(1-\upsilon)^8}\right) \end{aligned}$$

and choosing $\delta = \sqrt[4]{\frac{36(x^2+x+1)}{n^2(1-\upsilon)^8}}$, one has

$$|D_n(f,f;x)| \leq 32(1+x^2)\left(\Omega(f^2,\delta) + (1+M_\upsilon)(1+x^2)\|f\|_{x^2}\Omega(f,\delta)\right).$$

We may obtain similar estimate for $|D_n(g,g;x)|$, which completes the proof. ■

**Acknowledgments**. The author would like to thank the referee for his/her valuable suggestions which improved the paper considerably.

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