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Quantum Hermite-Hadamard's Type Inequalities For Co-ordinated Convex Functions*

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Abstract

In this paper, we prove the correct q_1q_2-Hermite-Hadamard inequality, some new q_1q_2-Hermite-Hadamard inequalities, and generalized q_1q_2-Hermite-Hadamard inequality on q_1q_2-differentiable co-ordinated convex functions. Many results given in this paper provide extensions of others given in previous works.

1 Introduction

The study of calculus without limits is known as quantum calculus or q-calculus. The famous mathematician Euler initiated the study of q-calculus in the eighteenth century by introducing the parameter q in Newton's work of infinite series. In early twentieth century, Jackson [9] has started a symmetric study of q-calculus and introduced q-definite integrals. The subject of quantum calculus has numerous applications in various areas of mathematics and physics such as number theory, combinatorics, orthogonal polynomials, basic hyper-geometric functions, quantum theory, mechanics and in theory of relativity. This subject has received outstanding attention by many researchers and hence it is considered as an in-corporative subject between mathematics and physics. Interested readers are referred to [7, 8, 10] for some current advances in the theory of quantum calculus and theory of inequalities in quantum calculus.

In recent articles, Tariboon et al. [15] studied the concept of q-derivatives and q-integrals over the intervals of the form $[a,b] \subset \mathbb{R}$ and settled a number of quantum analogues of some well-known results such as Holder inequality, Hermite-Hadamard inequality and Ostrowski inequality, Cauchy-Bunyakovsky-Schwarz, Gruss, Gruss- Cebysev and other integral inequalities using classical convexity. Also, Noor et al. [12, 13], Sudsutad et al. [14] and Zhuang et al. [16] have contributed to the ongoing research and have developed some integral inequalities which provide quantum estimates for the right part of the quantum analogue of Hermite-Hadamard inequality through q-differentiable convex and q-differentiable quasi-convex functions.

In [1], Dragomir defined co-ordinated convex functions on the on a rectangle from the plane $\mathbb{R}^2$ as below:

$$f\left(\lambda x+(1-\lambda)z, \lambda y+(1-\lambda)w\right) \leq \lambda f(x,y)+(1-\lambda) f(z,w)$$

holds, for all $(x,y), (z,w) \in \mathbb{R}^2$ and $\lambda \in [0,1]$.

After that, Dragomir proved the following inequality of Hermite-Hadamard's type for co-ordinated convex functions:

Suppose that $f:[a,b]\times[c,d] \subset \mathbb{R}^2 \to \mathbb{R}$, where $-\infty < a < b < \infty$ and $-\infty < c < d < \infty$, is convex on the coordinates on $[a,b]\times[c,d]$. Then

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$$

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$$
\begin{aligned}
&\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\\
&\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\\
&\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right]\\
&\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \qquad (1)
\end{aligned}
$$

In [3], N. Alp proved the following q-Hermite-Hadamard inequalities for convex functions on quantum integral:

If $f:[a,b]\to\mathbb{R}$ be a convex differentiable function on $[a,b]$ and $0<q<1$. Then, q-Hermite-Hadamard inequalities

$$f\left(\frac{qa+b}{1+q}\right)\leq\frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx\ \leq\frac{qf(a)+f(b)}{1+q}, \qquad (2)$$

$$f\left(\frac{a+b}{2}\right)+\frac{(1-q)(b-a)}{2(1+q)}f'\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx\ \leq\frac{qf(a)+f(b)}{1+q}, \qquad (3)$$

$$f\left(\frac{a+qb}{1+q}\right)+\frac{(1-q)(b-a)}{1+q}f'\left(\frac{a+qb}{1+q}\right)\leq\frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx\ \leq\frac{qf(a)+f(b)}{1+q} \qquad (4)$$

hold.

In [2] M. A. Latif give the q_1q_2-Hermite-Hadamard inequality for co-ordinated convex functions. However, Latif's theorem and related proof are not correct, therefore we gave an example to the contrary in third section. After that, in the next section, we proved true q_1q_2-Hermite-Hadamard inequality and obtained q_1q_2-Hermite-Hadamard's type inequalities for co-ordinated convex functions on quantum integral by using (2), (3) and (4).

2 Notations and Preliminaries of q-Calculus

Throughout this paper, let $a<b$, $c<d$ and $0<q,q_1,q_2<1$ be constants. The following definitions are for q-derivative, q_1q_2-derivates, q-integral, q_1q_2-integral of a function f.

Definition 1 ([15]) *For a continuous function $f:[a,b]\to\mathbb{R}$, the q-derivative of f at $x\in[a,b]$ is characterized by the expression*

$${}_aD_qf(x)=\frac{f(x)-f(qx+(1-q)a)}{(1-q)(x-a)},\ \ x\neq a. \qquad (5)$$

Since $f:[a,b]\to\mathbb{R}$ is a continuous function, we have ${}_aD_qf(a)=\lim_{x\to a}\ {}_aD_qf(x)$. The function f is said to be q-differentiable on $[a,b]$ if ${}_aD_qf(t)$ exists for all $x\in[a,b]$. If $a=0$ in (5), then ${}_0D_qf(x)=D_qf(x)$, where $D_qf(x)$ is familiar q-derivative of f at $x\in[a,b]$ defined by the expression (see [10])

$$D_qf(x)=\frac{f(x)-f(qx)}{(1-q)x},\ \ x\neq 0. \qquad (6)$$

Definition 2 ([15]) *Let $f:[a,b]\rightarrow\mathbb{R}$ be a continuous function. Then, the q-definite integral on $[a,b]$ is delineated as*

$$\int_a^x f(t)\ {}_ad_qt \ = (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x+(1-q^n)a) \tag{7}$$

for $x\in[a,b]$.

If $a=0$ in (7), then $\int_0^x f(t)\ {}_0d_qt \ = \int_0^x f(t)\ d_qt$, where $\int_0^x f(t)\ d_qt$ is familiar q-definite integral on $[0,x]$ defined by the expression (see [10])

$$\int_0^x f(t)\ {}_0d_qt \ = \int_0^x f(t)\ d_qt \ = (1-q)\,x\sum_{n=0}^{\infty} q^n f(q^n x). \tag{8}$$

If $c\in(a,x)$, then the q-definite integral on $[c,x]$ is expressed as

$$\int_c^x f(t)\ {}_ad_qt \ = \int_a^x f(t)\ {}_ad_qt \ - \int_a^c f(t)\ {}_ad_qt \ . \tag{9}$$

In [2], M. A. Latif defined q_1q_2-derivatives, q_1q_2-integral and related properties for bi-variat functions as follows:

Definition 3 *Let $f:[a,b]\times[c,d]\subset\mathbb{R}^2\rightarrow\mathbb{R}$ be a continuous function of two variables and $0<q_1<1, 0<q_2<1$, the partial q_1-derivates, q_2-derivates and q_1q_2-derivates at $(x,y)\in[a,b]\times[c,d]$ can be defined as follows:*

$$\frac{{}_a\partial_{q_1} f(x,y)}{{}_a\partial_{q_1} x} = \frac{f(x,y)-f(q_1x+(1-q_1)a,y)}{(1-q_1)(x-a)},\ x\neq a,$$

$$\frac{{}_c\partial_{q_2} f(x,y)}{{}_c\partial_{q_2} y} = \frac{f(x,y)-f(x,q_2y+(1-q_2)c)}{(1-q_2)(y-c)},\ y\neq c,$$

and

$$\begin{aligned}\frac{{}_{a,c}\partial^2_{q_1q_2} f(x,y)}{{}_a\partial_{q_1} x\,{}_c\partial_{q_2} y} &= \frac{1}{(1-q_1)(1-q_2)(x-a)(y-c)}\\ &\quad\times[f(q_1x+(1-q_1)a,q_2y+(1-q_2)c)-f(q_1x+(1-q_1)a,y)\\ &\quad -f(x,q_2y+(1-q_2)c)+f(x,y),\ x\neq a, y\neq c\ .\end{aligned}$$

The function $f:[a,b]\times[c,d]\subset\mathbb{R}^2\rightarrow\mathbb{R}$ is said to be partially q_1-, q_2- and q_1q_2-differentiable on $[a,b]\times[c,d]$ if $\frac{{}_a\partial_{q_1} f(x,y)}{{}_a\partial_{q_1} x}$, $\frac{{}_c\partial_{q_2} f(x,y)}{{}_c\partial_{q_2} y}$ and $\frac{{}_{a,c}\partial^2_{q_1q_2} f(x,y)}{{}_a\partial_{q_1} x\,{}_c\partial_{q_2} y}$ exist for all $(x,y)\in[a,b]\times[c,d]$.

Definition 4 *Suppose that $f:[a,b]\times[c,d]\subset\mathbb{R}^2\rightarrow\mathbb{R}$ is continuous. Then, the definite q_1q_2-integral on $[a,b]\times[c,d]$ is defined by*

$$\begin{aligned}\int_c^y\int_a^x f(t,s)\ {}_ad_{q_1}t\ \ {}_cd_{q_2}s \quad &= \quad (1-q_1)(1-q_2)(x-a)(y-c)\\ &\quad \times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q_1^n q_2^m f(q_1^n x+(1-q_1^n)a, q_2^m y+(1-q_2^m)c)\end{aligned}$$

for $(x,y)\in[a,b]\times[c,d]$.

Moreover, in [2], M. A. Latif obtained the following q_1q_2-Hermite-Hadamard inequality for co-ordinated convex functions:

Theorem 1 *Let $f : [a,b] \times [c,d] \subset \mathbb{R}^2 \to \mathbb{R}$ be convex on co-ordinates on $[a,b] \times [c,d]$, the following inequalities hold*

$$\begin{aligned}
& f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
\leq\ & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\ {}_ad_{q_1}x\ +\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\ {}_cd_{q_2}y\ \right] \\
\leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\ {}_cd_{q_2}y\ \ {}_ad_{q_1}x \\
\leq\ & \frac{q_1}{2(1+q_1)(d-c)}\int_c^d f(a,y)\ {}_cd_{q_2}y\ +\frac{1}{2(1+q_1)(d-c)}\int_c^d f(b,y)\ {}_cd_{q_2}y \\
& +\frac{q_2}{2(1+q_2)(b-a)}\int_a^b f(x,c)\ {}_ad_{q_1}x\ +\frac{1}{2(1+q_2)(b-a)}\int_a^b f(x,d)\ {}_ad_{q_1}x \\
\leq\ & \frac{q_1q_2f(a,c)+q_1f(a,d)+q_2f(b,c)+f(b,d)}{(1+q_1)(1+q_2)}.
\end{aligned}\tag{10}$$

However, (10) is not correct. We give the following counter-example.

Example 1 *Let $f : [0,1] \times [0,1] \subset \mathbb{R}^2 \to \mathbb{R}$. Then, $f(x,y) = 1-x-y$ is convex continuous on co-ordinates on $[0,1]\times[0,1]$. Therefore, the function f satisfies (1). Then, from the inequality (10) the following inequality must hold for all $q_1, q_2 \in (0,1)$,*

$$\begin{aligned}
& f\left(\frac{0+1}{2},\frac{0+1}{2}\right) \\
\leq\ & \frac{1}{2}\left[\frac{1}{1-0}\int_0^1 f\left(x,\frac{0+1}{2}\right)\ {}_0d_{q_1}x\ +\frac{1}{1-0}\int_0^1 f\left(\frac{0+1}{2},y\right)\ {}_0d_{q_2}y\ \right] \\
\leq\ & \frac{1}{(1-0)(1-0)}\int_0^1\int_0^1 (1-x-y)\ {}_0d_{q_2}y\ \ {}_0d_{q_1}x \\
\leq\ & \frac{q_1}{2(1+q_1)(1-0)}\int_0^1 f(0,y)\ {}_cd_{q_2}y\ +\frac{1}{2(1+q_1)(1-0)}\int_0^1 f(1,y)\ {}_cd_{q_2}y \\
& +\frac{q_2}{2(1+q_2)(1-0)}\int_0^1 f(x,0)\ {}_0d_{q_1}x\ +\frac{1}{2(1+q_2)(1-0)}\int_0^1 f(x,1)\ {}_0d_{q_1}x \\
\leq\ & \frac{q_1q_2f(0,0)+q_1f(0,1)+q_2f(1,0)+f(1,1)}{(1+q_1)(1+q_2)}.
\end{aligned}$$

By calculating above quantum integrals

$$I_1 = f\left(\frac{0+1}{2},\frac{0+1}{2}\right) = 1-\frac{1}{2}-\frac{1}{2} = 0,$$

$$I_2 = \frac{1}{2}\left[\frac{1}{1-0}\int_0^1 f\left(x, \frac{0+1}{2}\right) \ {}_0d_{q_1}x \ + \frac{1}{1-0}\int_0^1 f\left(\frac{0+1}{2}, y\right) \ {}_0d_{q_2}y \ \right]$$
$$= \frac{1}{2}\left[\int_0^1 \left(\frac{1}{2} - x\right) \ {}_0d_{q_1}x \ + \int_0^1 \left(\frac{1}{2} - y\right) \ {}_0d_{q_2}y \ \right]$$
$$= \frac{1}{2}\left[1 - \frac{1}{1+q_1} - \frac{1}{1+q_2}\right] = \frac{q_1q_2 - 1}{2(1+q_1)(1+q_2)},$$

$$I_3 = \int_0^1\int_0^1 (1-x-y) \ {}_0d_{q_2}y \ \ {}_0d_{q_1}x \ = 1 - \frac{1}{1+q_1} - \frac{1}{1+q_2} = \frac{q_1q_2-1}{(1+q_1)(1+q_2)},$$

$$I_{4a} = \frac{q_1}{2(1+q_1)(1-0)}\int_0^1 f(0,y) \ {}_cd_{q_2}y \ = \frac{q_1}{2(1+q_1)}\int_0^1 (1-y) \ {}_cd_{q_2}y \ = \frac{q_1q_2}{2(1+q_1)(1+q_2)},$$

$$I_{4b} = \frac{1}{2(1+q_1)(1-0)}\int_0^1 f(1,y) \ {}_cd_{q_2}y \ = \frac{1}{2(1+q_1)}\int_0^1 (-y) \ {}_cd_{q_2}y \ = -\frac{1}{2(1+q_1)(1+q_2)},$$

$$I_{4c} = \frac{q_2}{2(1+q_2)(1-0)}\int_0^1 f(x,0) \ {}_0d_{q_1}x \ = \frac{q_2}{2(1+q_2)}\int_0^1 (1-x) \ {}_0d_{q_1}x \ = \frac{q_1q_2}{2(1+q_1)(1+q_2)},$$

$$I_{4d} = \frac{1}{2(1+q_2)(1-0)}\int_0^1 f(x,1) \ {}_0d_{q_1}x \ = \frac{1}{2(1+q_2)}\int_0^1 (-x) \ {}_0d_{q_1}x \ = -\frac{1}{2(1+q_1)(1+q_2)},$$

$$I_4 = I_{4a} + I_{4b} + I_{4c} + I_{4d} = \frac{q_1q_2-1}{(1+q_1)(1+q_2)},$$

$$I_5 = \frac{q_1q_2-1}{(1+q_1)(1+q_2)}.$$

By using above equality, we have

$$I_1 \le I_2 \le I_3 \le I_4 \le I_5, \tag{11}$$

$$0 \le \frac{q_1q_2-1}{2(1+q_1)(1+q_2)} \le \frac{q_1q_2-1}{(1+q_1)(1+q_2)} \le \frac{q_1q_2-1}{(1+q_1)(1+q_2)} \le \frac{q_1q_2-1}{(1+q_1)(1+q_2)}.$$

If we choose $q_1 = q_2 = \frac{1}{2}$ in (11), then we have the following contradiction

$$0 \le -\frac{1}{6} \le -\frac{1}{3} \le -\frac{1}{3} \le -\frac{1}{3}.$$

It means that the left hand side of (10) is not correct. In the next section, we give the correct q_1q_2-Hermite–Hadamard inequality and some q_1q_2-Hermite–Hadamard inequalities for two variables the co-ordinated convex functions on quantum integral.

3 q_1q_2-Hermite-Hadamard Inequalities

In this section, we prove q_1q_2-Hermite-Hadamard inequality by using (2,3,4). After that we obtain varieties of q_1q_2-Hermite-Hadamard inequalities.

Theorem 2 (q_1q_2-Hermite-Hadamard Inequality) *Let $f : [a,b] \times [c,d] \subset \mathbb{R}^2 \to \mathbb{R}$ be convex on co-ordinates on $[a,b] \times [c,d]$, the following inequalities hold for all $q_1, q_2 \in (0,1)$*

$$f\left(\frac{aq_1+b}{1+q_1}, \frac{cq_2+d}{1+q_2}\right) \tag{12}$$

$$\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{cq_2+d}{1+q_2}\right)\ {}_ad_{q_1}x\ + \frac{1}{d-c}\int_c^d f\left(\frac{aq_1+b}{1+q_1}, y\right)\ {}_cd_{q_2}y\ \right] \tag{13}$$

$$\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\ {}_cd_{q_2}y\ \ {}_ad_{q_1}x \tag{14}$$

$$\leq \frac{q_1}{2(1+q_1)(d-c)}\int_c^d f(a,y)\ {}_cd_{q_2}y\ + \frac{1}{2(1+q_1)(d-c)}\int_c^d f(b,y)\ {}_cd_{q_2}y \tag{15}$$

$$+\frac{q_2}{2(1+q_2)(b-a)}\int_a^b f(x,c)\ {}_ad_{q_1}x\ + \frac{1}{2(1+q_2)(b-a)}\int_a^b f(x,d)\ {}_ad_{q_1}x$$

$$\leq \frac{q_1q_2f(a,c)+q_1f(a,d)+q_2f(b,c)+f(b,d)}{(1+q_1)(1+q_2)}. \tag{16}$$

Proof. Let $g_x : [c,d] \to \mathbb{R}$, $g_x(y) = f(x,y)$ is convex function. By using (2), q-Hermite-Hadamard inequality we have

$$g_x\left(\frac{cq_2+d}{1+q_2}\right) \leq \frac{1}{d-c}\int_c^d g_x(y)\ {}_cd_{q_2}y\ \leq \frac{q_2g_x(c)+g_x(d)}{1+q_2},$$

$$f\left(x, \frac{cq_2+d}{1+q_2}\right) \leq \frac{1}{d-c}\int_c^d f(x,y)\ {}_cd_{q_2}y\ \leq \frac{q_2f(x,c)+f(x,d)}{1+q_2}, \tag{17}$$

for all $x \in [a,b]$ and $q_1, q_2 \in (0,1)$. By q_1-integrating the inequality (17) on $[a,b]$, we have

$$\frac{1}{b-a}\int_a^b f\left(x, \frac{cq_2+d}{1+q_2}\right)\ {}_ad_{q_1}x\ \leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\ {}_cd_{q_2}y\ \ {}_ad_{q_1}x \tag{18}$$

$$\leq \frac{1}{b-a}\int_a^b \frac{q_2f(x,c)+f(x,d)}{1+q_2}\ {}_ad_{q_1}x\ .$$

By the same way, let $g_y : [a,b] \to \mathbb{R}$, $g_y(x) = f(x,y)$ be a convex function. By using (2), q-Hermite-Hadamard inequality we have

$$g_y\left(\frac{aq_1+b}{1+q_1}\right) \leq \frac{1}{b-a}\int_a^b g_y(x)\ {}_ad_{q_1}x\ \leq \frac{q_2g_y(a)+g_y(b)}{1+q_1},$$

$$f\left(\frac{aq_1+b}{1+q_1}, y\right) \leq \frac{1}{b-a}\int_a^b f(x,y)\ {}_ad_{q_1}x\ \leq \frac{q_1f(a,y)+f(b,y)}{1+q_1}, \tag{19}$$

for all $y \in [c,d]$ and $q_1, q_2 \in (0,1)$. By q_2-integrating the inequality (19) on $[c,d]$, we have

$$\frac{1}{d-c}\int_c^d f\left(\frac{aq_1+b}{1+q_1}, y\right) \ {}_c d_{q_2} y \ \le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y) \ {}_c d_{q_2} y \ \ {}_a d_{q_1} x$$
$$\le \frac{1}{d-c}\int_c^d \frac{q_1 f(a,y)+f(b,y)}{1+q_1} \ {}_c d_{q_2} y \ . \tag{20}$$

By summing (18) and(20), we obtain following (14) and (15) inequalities in (12) as follows

$$\begin{aligned}
&\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{cq_2+d}{1+q_2}\right) \ {}_a d_{q_1} x \ + \frac{1}{d-c}\int_c^d f\left(\frac{aq_1+b}{1+q_1}, y\right) \ {}_c d_{q_2} y \ \right] \\
\le \ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y) \ {}_c d_{q_2} y \ \ {}_a d_{q_1} x \\
\le \ & \frac{q_2}{2(1+q_2)(b-a)}\int_a^b f(x,c) \ {}_a d_{q_1} x \ + \frac{1}{2(1+q_2)(b-a)}\int_a^b f(x,d) \ {}_a d_{q_1} x \\
& + \frac{q_1}{2(1+q_1)(d-c)}\int_c^d f(a,y) \ {}_c d_{q_2} y \ + \frac{1}{2(1+q_1)(d-c)}\int_c^d f(b,y) \ {}_c d_{q_2} y \ .
\end{aligned} \tag{21}$$

By choosing respectively $x = \frac{aq_1+b}{1+q_1}$ and $y = \frac{cq_2+d}{1+q_2}$ in (17) and (19) inequalities we have

$$f\left(\frac{aq_1+b}{1+q_1}, \frac{cq_2+d}{1+q_2}\right) \le \frac{1}{d-c}\int_c^d f\left(\frac{aq_1+b}{1+q_1}, y\right) \ {}_c d_{q_2} y \ \le \frac{q_2 f(\frac{aq_1+b}{1+q_1}, c) + f(\frac{aq_1+b}{1+q_1}, d)}{1+q_2}, \tag{22}$$

$$f\left(\frac{aq_1+b}{1+q_1}, \frac{cq_2+d}{1+q_2}\right) \le \frac{1}{b-a}\int_a^b f\left(x, \frac{cq_2+d}{1+q_2}\right) \ {}_a d_{q_1} x \ \le \frac{q_1 f(a, \frac{cq_2+d}{1+q_2}) + f(b, \frac{cq_2+d}{1+q_2})}{1+q_1}. \tag{23}$$

By summing (22) and(23), we obtain the following (13) inequality in (12) as follows

$$\begin{aligned}
& f\left(\frac{aq_1+b}{1+q_1}, \frac{cq_2+d}{1+q_2}\right) \\
\le \ & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{cq_2+d}{1+q_2}\right) \ {}_a d_{q_1} x \ + \frac{1}{d-c}\int_c^d f\left(\frac{aq_1+b}{1+q_1}, y\right) \ {}_c d_{q_2} y \ \right] \\
\le \ & \frac{1}{2}\left[\frac{q_2 f(\frac{aq_1+b}{1+q_1}, c) + f(\frac{aq_1+b}{1+q_1}, d)}{1+q_2} + \frac{q_1 f(a, \frac{cq_2+d}{1+q_2}) + f(b, \frac{cq_2+d}{1+q_2})}{1+q_1}\right].
\end{aligned} \tag{24}$$

Now finally, by using (2) on the right hand side of (21) we have

$$\frac{q_2}{2(1+q_2)(b-a)}\int_a^b f(x,c) \ {}_a d_{q_1} x \ \le \frac{q_2}{2(1+q_2)}\frac{q_1 f(a,c)+f(b,c)}{1+q_1}, \tag{25}$$

$$\frac{1}{2(1+q_2)(b-a)}\int_a^b f(x,d)\ {}_a d_{q_1}x \le \frac{1}{2(1+q_2)}\frac{q_1 f(a,d)+f(b,d)}{1+q_1}, \tag{26}$$

$$\frac{q_1}{2(1+q_1)(d-c)}\int_c^d f(a,y)\ {}_c d_{q_2}y \le \frac{q_1}{2(1+q_1)}\frac{q_2 f(a,c)+f(a,d)}{1+q_2}, \tag{27}$$

$$\frac{1}{2(1+q_1)(d-c)}\int_c^d f(b,y)\ {}_c d_{q_2}y \le \frac{1}{2(1+q_1)}\frac{q_2 f(b,c)+f(b,d)}{1+q_2}. \tag{28}$$

By summing (25), (26), (27) and (28) inequalities we obtain the (16) inequality in (12) and the proof is completed. ■

Remark 1 *In Theorem 2, if we take $q \to 1^-$, we recapture (1) inequality.*

Theorem 3 *Let $f:[a,b]\to\mathbb{R}$ be a convex differentiable function on $[a,b]$ and $0<q<1$. Then we have*

$$f\left(\frac{qa+b}{1+q}\right) \le \frac{1}{b-a}\int_a^b f(x)\ {}_a d_q x \le \frac{qf(a)+f(b)}{(1+q)^2}+\frac{q}{1+q}f(\frac{qa+b}{1+q}) \le \frac{qf(a)+f(b)}{1+q}. \tag{29}$$

Proof. The left hand side of (29) inequality was proved in (2). Here we will prove right hand side of (29) for convex functions on quantum integral. Since f is a convex function on $[a,b]$, the f function is under the $k_1(x)$ and $k_2(x)$ lines which connecting the points $(a,f(a))$, $\left(\frac{qa+b}{1+q}, f\left(\frac{qa+b}{1+q}\right)\right)$ and $(b,f(b))$. This lines can be expressed as

$$k_1(x)=\frac{f(\frac{qa+b}{1+q})-f(a)}{\frac{qa+b}{1+q}-a}(x-a)+f(a) \quad\text{and}\quad k_2(x)=\frac{f(\frac{qa+b}{1+q})-f(b)}{\frac{qa+b}{1+q}-b}(x-b)+f(b).$$

Then, we have the following inequalities

$$f(x)\le k_1(x)=(1+q)\frac{f(\frac{qa+b}{1+q})-f(a)}{b-a}(x-a)+f(a),\ \text{on}\ \left[a,\frac{qa+b}{1+q}\right], \tag{30}$$

$$f(x)\le k_2(x)=(1+q)\frac{f(\frac{qa+b}{1+q})-f(b)}{q(a-b)}(x-b)+f(b),\ \text{on}\ \left[\frac{qa+b}{1+q},b\right], \tag{31}$$

$$h(x)=\frac{f(b)-f(a)}{b-a}(x-a)+f(a).$$

For all $x\in[a,b]$, q-integrating the inequalities (30) on $\left[a,\frac{qa+b}{1+q}\right]$and (31) on $\left[\frac{qa+b}{1+q},b\right]$, we have

$$\int_a^b f(x)\ {}_a d_q x \le \int_a^{\frac{qa+b}{1+q}} k_1(x)\ {}_a d_q x + \int_{\frac{qa+b}{1+q}}^b k_2(x)\ {}_a d_q x\ . \tag{32}$$

Now calculate q-integrals in (32) we have

$$\int_a^{\frac{qa+b}{1+q}} k_1(x)\ {}_a d_q x = \frac{b-a}{(1+q)^2}\left(f(\frac{qa+b}{1+q})+qf(a)\right), \tag{33}$$

$$\int_{\frac{qa+b}{1+q}}^{b} k_2(x)\ {}_a d_q x = \frac{b-a}{(1+q)^2}\left[f(b)+\left(q^2+q-1\right)f\left(\frac{qa+b}{1+q}\right)\right]. \tag{34}$$

By summing (33), (34), we have

$$\int_a^b f(x)\ {}_a d_q x \le \int_a^{\frac{qa+b}{1+q}} k_1(x)\ {}_a d_q x + \int_{\frac{qa+b}{1+q}}^{b} k_2(x)\ {}_a d_q x = (b-a)\left(\frac{qf(a)+f(b)}{(1+q)^2}+\frac{q}{1+q}f\left(\frac{qa+b}{1+q}\right)\right). \tag{35}$$

Finally, in figure 3 $k_1(x) \le h(x)$ on $\left[a, \frac{qa+b}{1+q}\right]$ and $k_2(x) \le h(x)$ on $\left[\frac{qa+b}{1+q}, b\right]$. By taking q-integrals, we have

$$\int_a^{\frac{qa+b}{1+q}} k_1(x)\ {}_a d_q x + \int_{\frac{qa+b}{1+q}}^{b} k_2(x)\ {}_a d_q x \le \int_a^b h(x)\ {}_a d_q x$$

$$\frac{qf(a)+f(b)}{(1+q)^2}+\frac{q}{1+q}f\left(\frac{qa+b}{1+q}\right) \le \frac{qf(a)+f(b)}{1+q}$$

and we obtain right hand side of (29) so the proof is accomplished. ■

Remark 2 *In Theorem 3, if we take $q \to 1^-$, we recapture the following classical Hermite-Hadmard type inequality:*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{4}+\frac{1}{2}f\left(\frac{a+b}{2}\right) \le \frac{f(a)+f(b)}{2}.$$

Theorem 4 *Let $f : [a,b]\times[c,d] \subset \mathbb{R}^2 \to \mathbb{R}$ be convex on co-ordinates on $[a,b]\times[c,d]$, the following inequalities hold for all $q_1, q_2 \in (0,1)$*

$$f\left(\frac{aq_1+b}{1+q_1}, \frac{cq_2+d}{1+q_2}\right)$$

$$\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{cq_2+d}{1+q_2}\right)\ {}_a d_{q_1} x + \frac{1}{d-c}\int_c^d f\left(\frac{aq_1+b}{1+q_1}, y\right)\ {}_c d_{q_2} y\right] \tag{36}$$

$$\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\ {}_c d_{q_2} y\ {}_a d_{q_1} x \tag{37}$$

$$\le \frac{q_2}{2(b-a)(1+q_2)^2}\int_a^b f(x,c)\ {}_a d_{q_1} x \tag{38}$$

$$+\frac{1}{2(b-a)(1+q_2)^2}\int_a^b f(x,d)\ {}_a d_{q_1} x$$

$$+\frac{q_2}{2(b-a)(1+q_2)}\int_a^b f\left(x, \frac{cq_2+d}{1+q_2}\right)\ {}_a d_{q_1} x$$

$$+\frac{q_1}{2(d-c)(1+q_1)^2}\int_c^d f(a,y)\ {}_c d_{q_2} y$$

$$+\frac{1}{2\left(d-c\right)\left(1+q_1\right)^2}\int_c^d f(b,y)\ {}_cd_{q_2}y$$

$$+\frac{q_1}{2\left(d-c\right)\left(1+q_1\right)}\int_c^d f\left(\frac{aq_1+b}{1+q_1},y\right)\ {}_cd_{q_2}y$$

$$\leq\frac{q_1q_2f(a,c)+q_1f(a,d)+q_2f(b,c)+f(b,d)}{\left(1+q_1\right)^2\left(1+q_2\right)^2} \tag{39}$$

$$+\frac{q_1q_2}{\left(1+q_1\right)\left(1+q_2\right)}f\left(\frac{aq_1+b}{1+q_1},\frac{cq_2+d}{1+q_2}\right)$$

$$+\frac{q_1q_2}{\left(1+q_1\right)\left(1+q_2\right)^2}f(\frac{aq_1+b}{1+q_1},c)+\frac{q_1}{\left(1+q_1\right)\left(1+q_2\right)^2}f(\frac{aq_1+b}{1+q_1},d)$$

$$+\frac{q_1q_2}{\left(1+q_1\right)^2\left(1+q_2\right)}f\left(a,\frac{cq_2+d}{1+q_2}\right)+\frac{q_2}{\left(1+q_1\right)^2\left(1+q_2\right)}f\left(b,\frac{cq_2+d}{1+q_2}\right)$$

Proof. (36) and (37) is proved in Theorem 2. We will prove (38) and (39). Let $g_x:[c,d]\to\mathbb{R}$, $g_x(y)=f\left(x,y\right)$ be convex function. By using (29), q-Hermite-Hadamard type inequality we have

$$\frac{1}{d-c}\int_c^d g_x(y)\ {}_cd_{q_2}y\ \leq\frac{q_2g_x(c)+g_x(d)}{\left(1+q_2\right)^2}+\frac{q_2}{1+q_2}g_x\left(\frac{cq_2+d}{1+q_2}\right),$$

$$\frac{1}{d-c}\int_c^d f\left(x,y\right)\ \ {}_cd_{q_2}y\ \leq\frac{q_2f(x,c)+f(x,d)}{\left(1+q_2\right)^2}+\frac{q_2}{1+q_2}f\left(x,\frac{cq_2+d}{1+q_2}\right), \tag{40}$$

for all $x\in[a,b]$ and $q_1,q_2\in(0,1)$. By q_1-integrating the inequality (40) on $[a,b]$, we have

$$\frac{1}{(d-c)}\int_a^b\int_c^d f\left(x,y\right)\ \ {}_cd_{q_2}y\ \ \ {}_ad_{q_1}x$$

$$\leq\ \frac{q_2}{\left(1+q_2\right)^2}\int_a^b f(x,c)\ \ {}_ad_{q_1}x\ +\frac{1}{\left(1+q_2\right)^2}\int_a^b f(x,d)\ \ {}_ad_{q_1}x\ +\frac{q_2}{1+q_2}\int_a^b f(x,\frac{cq_2+d}{1+q_2})\ \ {}_ad_{q_1}x\ . \tag{41}$$

By he same way, let $g_y:[a,b]\to\mathbb{R}$, $g_y(x)=f\left(x,y\right)$ is convex function. By using (29) q-Hermite-Hadamard inequality, we have

$$\frac{1}{b-a}\int_a^b g_y(x)\ {}_ad_{q_1}x\ \leq\frac{q_1g_y(a)+g_y(b)}{\left(1+q_1\right)^2}+\frac{q_1}{1+q_1}g_y\left(\frac{aq_1+b}{1+q_1}\right),$$

$$\frac{1}{b-a}\int_a^b f\left(x,y\right)\ \ {}_ad_{q_1}x\ \leq\frac{q_1f(a,y)+f(b,y)}{\left(1+q_1\right)^2}+\frac{q_1}{1+q_1}f\left(\frac{aq_1+b}{1+q_1},y\right), \tag{42}$$

for all $y\in[c,d]$ and $q_1,q_2\in(0,1)$. By q_2-integrating the inequality (42) on $[c,d]$, we have

$$\frac{1}{(b-a)}\int_a^b\int_c^d f\left(x,y\right)\ \ {}_cd_{q_2}y\ \ \ {}_ad_{q_1}x$$

$$\leq \frac{q_1}{(1+q_1)^2}\int_c^d f(a,y)\ {}_cd_{q_2}y + \frac{1}{(1+q_1)^2}\int_c^d f(b,y)\ {}_cd_{q_2}y$$
$$+\frac{q_1}{1+q_1}\int_c^d f\left(\frac{aq_1+b}{1+q_1},y\right)\ {}_cd_{q_2}y\ . \tag{43}$$

If (41) is divided by $(b-a)$ and (43) is divided by $(d-c)$ and (41 with (43) are summed, then (38) and (39) inequalities are obtained as follow:

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\ {}_cd_{q_2}y\ {}_ad_{q_1}x$$
$$\leq \frac{1}{2(b-a)}\frac{q_2}{(1+q_2)^2}\int_a^b f(x,c)\ {}_ad_{q_1}x + \frac{1}{2(b-a)}\frac{1}{(1+q_2)^2}\int_a^b f(x,d)\ {}_ad_{q_1}x$$
$$+\frac{1}{2(b-a)}\frac{q_2}{1+q_2}\int_a^b f(x,\frac{cq_2+d}{1+q_2})\ {}_ad_{q_1}x + \frac{1}{2(d-c)}\frac{q_1}{(1+q_1)^2}\int_c^d f(a,y)\ {}_cd_{q_2}y$$
$$+\frac{1}{2(d-c)}\frac{1}{(1+q_1)^2}\int_c^d f(b,y)\ {}_cd_{q_2}y + \frac{1}{2(d-c)}\frac{q_1}{1+q_1}\int_c^d f\left(\frac{aq_1+b}{1+q_1},y\right)\ {}_cd_{q_2}y\ . \tag{44}$$

Finally, by using (2) on the right hand side of (44) we have

$$\frac{q_2}{2(1+q_2)^2}\frac{1}{b-a}\int_a^b f(x,c)\ {}_ad_{q_1}x \leq \frac{q_2}{2(1+q_2)^2}\left[\frac{q_1f(a,c)+f(b,c)}{(1+q_1)^2}+\frac{q_1}{1+q_1}f(\frac{aq_1+b}{1+q_1},c)\right], \tag{45}$$

$$\frac{1}{2(1+q_2)^2}\frac{1}{b-a}\int_a^b f(x,d)\ {}_ad_{q_1}x \leq \frac{1}{2(1+q_2)^2}\left[\frac{q_1f(a,d)+f(b,d)}{(1+q_1)^2}+\frac{q_1}{1+q_1}f(\frac{aq_1+b}{1+q_1},d)\right], \tag{46}$$

$$\frac{q_1}{2(1+q_1)^2}\frac{1}{d-c}\int_c^d f(a,y)\ {}_cd_{q_2}y \leq \frac{q_1}{2(1+q_1)^2}\left[\frac{q_2f(a,c)+f(a,d)}{(1+q_1)^2}+\frac{q_2}{1+q_2}f\left(a,\frac{cq_2+d}{1+q_2}\right)\right], \tag{47}$$

$$\frac{1}{2(1+q_1)^2}\frac{1}{d-c}\int_c^d f(b,y)\ {}_cd_{q_2}y \leq \frac{1}{2(1+q_1)^2}\left[\frac{q_2f(b,c)+f(b,d)}{(1+q_1)^2}+\frac{q_2}{1+q_2}f\left(b,\frac{cq_2+d}{1+q_2}\right)\right], \tag{48}$$

$$\frac{q_1}{2(1+q_1)}\frac{1}{d-c}\int_c^d f\left(\frac{aq_1+b}{1+q_1},y\right)\ {}_cd_{q_2}y$$
$$\leq \frac{q_1}{2(1+q_1)}\left[\frac{q_2f(\frac{aq_1+b}{1+q_1},c)+f(\frac{aq_1+b}{1+q_1},d)}{(1+q_2)^2}+\frac{q_2}{1+q_2}f\left(\frac{aq_1+b}{1+q_1},\frac{cq_2+d}{1+q_2}\right)\right], \tag{49}$$

$$\frac{q_2}{2(1+q_2)}\frac{1}{b-a}\int_a^b f\left(x,\frac{cq_2+d}{1+q_2}\right)\ {}_ad_{q_1}x$$

$$\leq \frac{q_2}{2(1+q_2)}\left[\frac{q_1 f(a,\frac{cq_2+d}{1+q_2})+f(b,\frac{cq_2+d}{1+q_2})}{(1+q_1)^2}+\frac{q_1}{1+q_1}f\left(\frac{aq_1+b}{1+q_1},\frac{cq_2+d}{1+q_2}\right)\right]. \tag{50}$$

By summing (45)–(50) inequalities the proof is completed. ■

Remark 3 *In Theorem 4, if we take $q \to 1$, we recapture [5, Theorem 2.2].*

Theorem 5 *Let $f : [a,b] \times [c,d] \subset \mathbb{R}^2 \to \mathbb{R}$ be convex on co-ordinates on $[a,b] \times [c,d]$, the following inequalities hold for all $q_1, q_2 \in (0,1)$,*

$$\begin{aligned}
& f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{(1-q_1)(b-a)}{4(1+q_1)}\left[\frac{\partial f}{\partial x}\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{1}{d-c}\int_c^d \frac{\partial}{\partial x}f\left(\frac{a+b}{2},y\right)\ {}_c d_{q_2}y\ \right]\\
& +\frac{(1-q_2)(d-c)}{4(1+q_2)}\left[\frac{\partial f}{\partial y}\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{1}{b-a}\int_a^b \frac{\partial}{\partial y}f\left(x,\frac{c+d}{2}\right)\ {}_a d_{q_1}x\ \right]\\
\leq\ & \frac{1}{2(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)\ {}_a d_{q_1}x\ +\frac{1}{2(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)\ {}_c d_{q_2}y\\
& +\frac{(1-q_2)(d-c)}{4(1+q_2)(b-a)}\int_a^b \frac{\partial}{\partial y}f\left(x,\frac{c+d}{2}\right)\ {}_a d_{q_1}x\ +\frac{(1-q_1)(b-a)}{4(1+q_1)(d-c)}\int_c^d \frac{\partial}{\partial x}f\left(\frac{a+b}{2},y\right)\ {}_c d_{q_2}y\\
\leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\ {}_c d_{q_2}y\ \ {}_a d_{q_1}x\\
\leq\ & \frac{q_1}{2(1+q_1)(d-c)}\int_c^d f(a,y)\ {}_c d_{q_2}y\ +\frac{1}{2(1+q_1)(d-c)}\int_c^d f(b,y)\ {}_c d_{q_2}y\\
& +\frac{q_2}{2(1+q_2)(b-a)}\int_a^b f(x,c)\ {}_a d_{q_1}x\ +\frac{1}{2(1+q_2)(b-a)}\int_a^b f(x,d)\ {}_a d_{q_1}x\\
\leq\ & \frac{q_1q_2 f(a,c)+q_1 f(a,d)+q_2 f(b,c)+f(b,d)}{(1+q_1)(1+q_2)}.
\end{aligned}$$

Proof. The right hand side of Theorem 5 was proved in Theorem 2. We will prove the left hand side of Theorem 5.

Let $g_x : [c,d] \to \mathbb{R}$, $g_x(y) = f(x,y)$ is convex function. By using (3), q-Hermite-Hadamard inequality we have

$$g_x\left(\frac{c+d}{2}\right)+\frac{(1-q_2)(d-c)}{2(1+q_2)}\frac{\partial g_x}{\partial y}\left(\frac{c+d}{2}\right)\leq\frac{1}{d-c}\int_c^d g_x(y)\ {}_c d_{q_2}y\ ,$$

$$f\left(x,\frac{c+d}{2}\right)+\frac{(1-q_2)(d-c)}{2(1+q_2)}\frac{\partial f}{\partial y}\left(x,\frac{c+d}{2}\right)\leq\frac{1}{d-c}\int_c^d f(x,y)\ {}_c d_{q_2}y\ . \tag{51}$$

By the same way, let $g_y : [a,b] \to \mathbb{R}$, $g_y(x) = f(x,y)$ is convex function. By using (2) q-Hermite-Hadamard inequality we have

$$g_y\left(\frac{a+b}{2}\right)+\frac{(1-q_1)(b-a)}{2(1+q_1)}\frac{\partial g_y}{\partial x}\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b g_y(x)\ {}_a d_{q_1}x\ ,$$

$$f\left(\frac{a+b}{2},y\right)+\frac{(1-q_1)(b-a)}{2(1+q_1)}\frac{\partial f}{\partial x}\left(\frac{a+b}{2},y\right)\le\frac{1}{b-a}\int_a^b f(x,y)\ {}_a d_{q_1}x\ . \tag{52}$$

By choosing respectively $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$ in (51) and (52) inequalities we have, we have

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{(1-q_2)(d-c)}{2(1+q_2)}\frac{\partial f}{\partial y}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\le\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\ {}_c d_{q_2}y\ , \tag{53}$$

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{(1-q_1)(b-a)}{2(1+q_1)}\frac{\partial f}{\partial x}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\le\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\ {}_a d_{q_1}x\ . \tag{54}$$

By summing (51) and(52), we obtain the following the left hand side of Theorem 5

$$\begin{aligned}&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{(1-q_1)(b-a)}{4(1+q_1)}\frac{\partial f}{\partial x}\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{(1-q_2)(d-c)}{4(1+q_2)}\frac{\partial f}{\partial y}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\ \le\ &\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\ {}_a d_{q_1}x\ +\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\ {}_c d_{q_2}y\ \right]\end{aligned} \tag{55}$$

for all $x\in[a,b]$ and $q_1,q_2\in(0,1)$. By q_1-integrating the inequality (51) on $[a,b]$ and divide by $(b-a)$, we have

$$\begin{aligned}&\frac{1}{b-a}\int_a^b\left[f\left(x,\frac{c+d}{2}\right)+\frac{(1-q_2)(d-c)}{2(1+q_2)}\frac{\partial f}{\partial y}\left(x,\frac{c+d}{2}\right)\right]\ {}_a d_{q_1}x\\ \le\ &\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\ {}_c d_{q_2}y\ \ {}_a d_{q_1}x\ ,\end{aligned}$$

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\ {}_a d_{q_1}x\ +\frac{(1-q_2)(d-c)}{2(1+q_2)(b-a)}\int_a^b\frac{\partial}{\partial y}f\left(x,\frac{c+d}{2}\right)\ {}_a d_{q_1}x\\ \le\ &\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\ {}_c d_{q_2}y\ \ {}_a d_{q_1}x\ ,\end{aligned} \tag{56}$$

for all $y\in[c,d]$ and $q_1,q_2\in(0,1)$. By q_2-integrating the inequality (52) on $[c,d]$ and divide by $(d-c)$, we have

$$f\left(\frac{a+b}{2},y\right)+\frac{(1-q_1)(b-a)}{2(1+q_1)}\frac{\partial f}{\partial x}\left(\frac{a+b}{2},y\right)\le\frac{1}{b-a}\int_a^b f(x,y)\ {}_a d_{q_1}x\ ,$$

$$\begin{aligned}&\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\ {}_c d_{q_2}y\ +\frac{(1-q_1)(b-a)}{2(1+q_1)(d-c)}\int_c^d\frac{\partial}{\partial x}f\left(\frac{a+b}{2},y\right)\ {}_c d_{q_2}y\\ \le\ &\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\ {}_c d_{q_2}y\ \ {}_a d_{q_1}x\ .\end{aligned} \tag{57}$$

By summing (56) and(57), we obtain the following (14) and (15) inequalities in Theorem 5

$$\begin{aligned}
&\frac{1}{2(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)\ {}_ad_{q_1}x\ +\frac{1}{2(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)\ {}_cd_{q_2}y\\
&+\frac{(1-q_2)(d-c)}{4(1+q_2)(b-a)}\int_a^b \frac{\partial}{\partial y}f\left(x,\frac{c+d}{2}\right)\ {}_ad_{q_1}x\\
&+\frac{(1-q_1)(b-a)}{4(1+q_1)(d-c)}\int_c^d \frac{\partial}{\partial x}f\left(\frac{a+b}{2},y\right)\ {}_cd_{q_2}y\\
\leq\ &\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\ {}_cd_{q_2}y\ \ {}_ad_{q_1}x\ .
\end{aligned}\tag{58}$$

By using 55) and (58), the proof is completed. ■

Theorem 6 *Let $f:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ be convex on co-ordinates on $[a,b]\times[c,d]$, the following inequalities hold for all $q_1,q_2\in(0,1)$*

$$\begin{aligned}
&f\left(\frac{a+q_1b}{1+q_1},\frac{c+q_2d}{1+q_2}\right)+\frac{(1-q_1)(b-a)}{2(1+q_1)}\times\\
&\left[\frac{\partial f}{\partial x}\left(\frac{a+q_1b}{1+q_1},\frac{c+q_2d}{1+q_2}\right)+\frac{1}{d-c}\int_c^d \frac{\partial}{\partial x}f\left(\frac{a+q_1b}{1+q_1},y\right)\ {}_cd_{q_2}y\ \right]\\
&+\frac{(1-q_2)(d-c)}{2(1+q_2)}\left[\frac{\partial f}{\partial y}\left(\frac{a+q_1b}{1+q_1},\frac{c+q_2d}{1+q_2}\right)+\frac{1}{b-a}\int_a^b \frac{\partial}{\partial y}f\left(x,\frac{c+q_2d}{1+q_2}\right)\ {}_ad_{q_1}x\ \right]\\
\leq\ &\frac{1}{2(b-a)}\int_a^b f\left(x,\frac{c+q_2d}{1+q_2}\right)\ {}_ad_{q_1}x\ +\frac{1}{2(d-c)}\int_c^d f\left(\frac{a+q_1b}{1+q_1},y\right)\ {}_cd_{q_2}y\\
&+\frac{(1-q_2)(d-c)}{2(1+q_2)(b-a)}\int_a^b \frac{\partial}{\partial y}f\left(x,\frac{c+q_2d}{1+q_2}\right)\ {}_ad_{q_1}x\\
&+\frac{(1-q_1)(b-a)}{2(1+q_1)(d-c)}\int_c^d \frac{\partial}{\partial x}f\left(\frac{a+q_1b}{1+q_1},y\right)\ {}_cd_{q_2}y\\
\leq\ &\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\ {}_cd_{q_2}y\ \ {}_ad_{q_1}x\\
\leq\ &\frac{q_1}{2(1+q_1)(d-c)}\int_c^d f(a,y)\ {}_cd_{q_2}y\ +\frac{1}{2(1+q_1)(d-c)}\int_c^d f(b,y)\ {}_cd_{q_2}y\\
&+\frac{q_2}{2(1+q_2)(b-a)}\int_a^b f(x,c)\ {}_ad_{q_1}x\ +\frac{1}{2(1+q_2)(b-a)}\int_a^b f(x,d)\ {}_ad_{q_1}x\\
\leq\ &\frac{q_1q_2f(a,c)+q_1f(a,d)+q_2f(b,c)+f(b,d)}{(1+q_1)(1+q_2)}.
\end{aligned}$$

Proof. The right hand side of Theorem 6 was proved in Theorem 2. We will prove the left hand side of Theorem 6.

By using (4) inequality The proof is done with same method in Theorem 5. ■

4 Conclusion

With the help of these results, it will be possible to find a range for co-ordianetd convex functions whose q-integral can not be calculated.

Example 2 *The quantum integral of* $f(x,y) = e^{x^2+y^2}$ *can not be calculated on* $I = [0,1] \times [0,1]$. *But, we can obtain an interval for q-integral for this function on* I *by* q_1q_2*-Hermite-Hadamard Inequality as follows:*

$$\begin{aligned} e^{\frac{1}{(1+q_1)^2}+\frac{1}{(1+q_2)^2}} &\leq \frac{1}{2}\left(e^{\frac{1}{(1+q_2)^2}}+e^{\frac{1}{(1+q_1)^2}}\right)\int_0^1 e^{x^2} d_qx \\ &\leq \int_0^1\int_0^1 e^{x^2+y^2} d_{q_2}y d_{q_1}x \leq \frac{q+e}{(1+q)}\int_0^1 e^{x^2} d_qx \\ &\leq \frac{q_1q_2+q_1e+q_2e+e^2}{(1+q_1)(1+q_2)}. \end{aligned}$$

where $0 < q, q_1, q_2 < 1$.

As can be seen, the inequalities obtained are very useful in quantum integral estimates calculations.

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