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Coincidence Points For Three Self Mappings Satisfying $F(\psi,\,\varphi)$-Contractions In m-Metric Spaces*

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Abstract

We discuss the existence and uniqueness of points of coincidence and common fixed points for three self mappings in m-metric spaces and apply our main result to derive fixed points for expansive type mappings and several new results in this setup. Finally, we give some examples to justify the validity of our result.

1 Introduction

Fixed point theory is an important branch of nonlinear analysis that can be applied to many areas of mathematics and applied sciences. The most celebrated result in this field is the Banach contraction principle [5]. Because of its simplicity and usefulness, it has become an important tool to solving existence and uniqueness problems in nonlinear functional analysis. After the appearance of Banach contraction principle, lots of generalizations have been made in different directions (see [10, 11, 12] and references therein). In 1994, Matthews [9] introduced the notion of partial metric spaces as a generalization of metric spaces and proved some important fixed point theorems including the well-known Banach contraction theorem in this new framework. Recently, Asadi et al. [3] extended the notion of partial metric spaces to m-metric spaces and studied some fixed point results in this setup. The main aim of this paper is to obtain a sufficient condition for the existence and uniqueness of points of coincidence and common fixed points for three self mappings satisfying a generalized contractive type condition in m-metric spaces. As some consequences of this study, we obtain several related results in the setting of m-metric spaces.

2 Some Basic Concepts

We recall some basic notations, definitions, and results in m-metric spaces.

Definition 1 ([3]) *Let X be a nonempty set. A function $\mu : X \times X \to \mathbb{R}^+$ is called an m-metric if the following conditions are satisfied:*

(m1) $\mu(x,x)=\mu(y,y)=\mu(x,y) \Longleftrightarrow x=y$,

(m2) $m_{xy} \le \mu(x,y)$,

(m3) $\mu(x,y)=\mu(y,x)$,

(m4) $(\mu(x,y)-m_{xy}) \le (\mu(x,z)-m_{xz}) + (\mu(z,y)-m_{zy})$,

where $m_{xy} := min\,\{\mu(x,x),\,\mu(y,y)\}$. Then the pair (X,μ) is called an m-metric space. The following notation is useful in the sequel $M_{xy} := max\,\{\mu(x,x),\,\mu(y,y)\}$.

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Example 1 ([3]) *Let $X = [0, \infty)$. Then $\mu(x, y) = \frac{x+y}{2}$ on X is an m-metric. It is valuable to note that μ is not a partial metric on X. In fact, if $x = 4,\ y = 2$ then $\mu(x, x) > \mu(x, y)$.*

Remark 1 ([3]) *For every $x,\ y \in X$,*

1. $0 \leq M_{xy} + m_{xy} = \mu(x, x) + \mu(y, y)$;
2. $0 \leq M_{xy} - m_{xy} = \mid \mu(x, x) - \mu(y, y) \mid$.

Example 2 ([3]) *Let μ be an m-metric. Put $\mu^w(x, y) = \mu(x, y) - 2m_{xy} + M_{xy}$. Then μ^w is an ordinary metric.*

Lemma 1 ([3]) *Every p-metric is an m-metric.*

It is clear that each m-metric μ on X generates a topology τ_μ on X. The set $\{ B_\mu(x, \epsilon) : x \in X, \epsilon > 0 \}$, where $B_\mu(x, \epsilon) = \{ y \in X : \mu(x, y) < m_{xy} + \epsilon \}$, for all $x \in X$ and $\epsilon > 0$, forms the base of τ_μ.

Definition 2 ([3]) *Let (X, μ) be an m-metric space. Then:*

1. *A sequence (x_n) in an m-metric space (X, μ) converges to a point $x \in X$ if $\lim_{n\to\infty}(\mu(x_n, x) - m_{x_n x}) = 0$.*
2. *A sequence (x_n) in an m-metric space (X, μ) is called an m-Cauchy sequence if $\lim_{n,m\to\infty}(\mu(x_n, x_m) - m_{x_n x_m})$ and $\lim_{n,m\to\infty}(M_{x_n x_m} - m_{x_n x_m})$ exist(and are finite).*
3. *An m-metric space (X, μ) is said to be complete if every m-Cauchy sequence (x_n) in X converges, with respect to τ_μ, to a point $x \in X$ such that $\lim_{n\to\infty}(\mu(x_n, x) - m_{x_n x}) = 0$ and $\lim_{n\to\infty}(M_{x_n x} - m_{x_n x}) = 0$.*

Lemma 2 ([3]) *Let (X, μ) be an m-metric space. Then:*

1. *(x_n) is an m-Cauchy sequence in (X, μ) if and only if it is a Cauchy sequence in the metric space (X, μ^w).*
2. *An m-metric space (X, μ) is complete if and only if the metric space (X, μ^w) is complete.*

Definition 3 ([7]) *A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if it satisfies the following properties:*

(i) *ψ is strictly increasing and continuous;*

(ii) *$\psi(t) = 0$ if and only if $t = 0$.*

The class of all altering distance functions is denoted by Ψ.

Definition 4 ([2]) *A function $\varphi : [0, \infty) \to [0, \infty)$ is called an ultra altering distance function if φ is continuous, and $\varphi(0) \geq 0$, $\varphi(t) > 0$, $t \neq 0$. The class of all ultra altering distance functions is denoted by Φ.*

Definition 5 ([2]) *A mapping $F : [0, \infty)^2 \to \mathbb{R}$ is called a C-class function if it is continuous and satisfies the following axioms:*

1. *$F(s, t) \leq s$;*
2. *$F(s, t) = s$ implies that either $s = 0$ or $t = 0$.*

We denote the C-class functions by $\mathcal{C}$.

Example 3 (**[2]**) *The following functions are elements of $\mathcal{C}$.*

1. $F(s,t) = s - t$.
2. $F(s,t) = ms,\ 0 < m < 1$.
3. $F(s,t) = s\beta(s),\ \beta : [0,\infty) \to [0,1)$ *and is continuous.*

Definition 6 (**[1]**) *Let $T,\, S : X \to X$ be two self mappings on a set X. If $y = Tx = Sx$ for some x in X, then x is called a coincidence point of T and S and y is called a point of coincidence of T and S.*

Definition 7 (**[6]**) *The mappings $T,\, S : X \to X$ are called weakly compatible if they commute at their coincidence points, i.e., if $T(Sx) = S(Tx)$ whenever $Sx = Tx$.*

Lemma 3 (**[4]**) *Let X be a nonempty set and the mappings $S,\, T,\, f : X \to X$ be such that $(S,\, f)$ and $(T,\, f)$ are weakly compatible. If $S,\, T$ and f have a unique point of coincidence y in X, then y is the unique common fixed point of $S,\, T$ and f in X.*

Definition 8 *Let (X,μ) be an m-metric space. A mapping $f : X \to X$ is called expansive if there exists a positive number $k > 1$ such that*

$$\mu(fx, fy) \geq k\,\mu(x,y),\ \forall x,\, y \in X.$$

3 Main Results

Theorem 4 *Let (X,μ) be an m-metric space and let the mappings $S,\, T,\, f : X \to X$ satisfy the following condition:*

$$max\,\{\psi(\mu(Sx,Ty)), \psi(\mu(Tx,Sy)), \psi(\mu(Tx,Ty)), \psi(\mu(Sx,Sy))\} \leq F(\psi(\mu(fx,fy)), \varphi(\mu(fx,fy))) \tag{1}$$

for all $x,\, y \in X$, where $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathcal{C}$. Suppose that $S(X) \cup T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X. Then S, T and f have a unique point of coincidence u(say) in $f(X)$ with $\mu(u,u) = 0$. Moreover, if (S,f) and (T,f) are weakly compatible, then S, T and f have a unique common fixed point in $f(X)$.

Proof. Let $x_0 \in X$ be arbitrary and choose a point $x_1 \in X$ such that $fx_1 = Sx_0$ which is possible since $S(X) \subseteq f(X)$. Similarly, there is a point $x_2 \in X$ such that $fx_2 = Tx_1$. Continuing this process, we can construct a sequence (fx_n) in $f(X)$ by $fx_n = Sx_{n-1}$, if n is odd and $fx_n = Tx_{n-1}$, if n is even.

If $n \in \mathbb{N}$ is odd, then by using condition (1) we have

$$\begin{aligned} & \psi(\mu(fx_n, fx_{n+1})) \\ = \ & \psi(\mu(Sx_{n-1}, Tx_n)) \\ \leq \ & \max\{\psi(\mu(Sx_{n-1}, Tx_n)), \psi(\mu(Tx_{n-1}, Sx_n)), \psi(\mu(Tx_{n-1}, Tx_n)), \psi(\mu(Sx_{n-1}, Sx_n))\} \\ \leq \ & F(\psi(\mu(fx_{n-1}, fx_n)), \varphi(\mu(fx_{n-1}, fx_n))). \end{aligned}$$

If $n \in \mathbb{N}$ is even, then similarly we get

$$\psi(\mu(fx_n, fx_{n+1})) \leq F(\psi(\mu(fx_{n-1}, fx_n)), \varphi(\mu(fx_{n-1}, fx_n))).$$

Thus for all $n \in \mathbb{N}$, we must have

$$\psi(\mu(fx_n, fx_{n+1})) \leq F(\psi(\mu(fx_{n-1}, fx_n)), \varphi(\mu(fx_{n-1}, fx_n))) \leq \psi(\mu(fx_{n-1}, fx_n)). \tag{2}$$

We shall show that $\lim_{n\to\infty} \mu(fx_n, fx_{n+1}) = 0$. If $\mu(fx_{n_0}, fx_{n_0+1}) = 0$ for some $n_0 \in \mathbb{N}$, then by using condition (2), we have

$$0 \leq \psi(\mu(fx_{n_0+1}, fx_{n_0+2})) \leq \psi(\mu(fx_{n_0}, fx_{n_0+1}))$$

which implies that $\psi(\mu(fx_{n_0+1}, fx_{n_0+2})) = 0$ and hence $\mu(fx_{n_0+1}, fx_{n_0+2}) = 0$. This means that

$$\mu(fx_n, fx_{n+1}) = 0 \text{ } for \text{ } all \text{ } n \geq n_0$$

and so $\lim_{n\to\infty} \mu(fx_n, fx_{n+1}) = 0$.

We now suppose that $\mu(fx_n, fx_{n+1}) > 0$ for all $n \in \mathbb{N}$. Since ψ is strictly increasing, it follows from condition (2) that $\mu(fx_n, fx_{n+1}) \leq \mu(fx_{n-1}, fx_n)$. Therefore, $(\mu(fx_n, fx_{n+1}))$ is a decreasing sequence of nonnegative real numbers. So, there exists $r \geq 0$ such that $\lim_{n\to\infty} \mu(fx_n, fx_{n+1}) = r$. We shall show that $r = 0$. From condition (2), we get

$$\begin{aligned}
\limsup_{n\to\infty} \psi(\mu(fx_n, fx_{n+1})) &\leq \limsup_{n\to\infty} F(\psi(\mu(fx_{n-1}, fx_n)), \varphi(\mu(fx_{n-1}, fx_n))) \\
&\leq \limsup_{n\to\infty} \psi(\mu(fx_{n-1}, fx_n)).
\end{aligned}$$

Therefore, $\psi(r) \leq F(\psi(r), \varphi(r)) \leq \psi(r)$. So, it must be the case that $F(\psi(r), \varphi(r)) = \psi(r)$ which ensures that either $\psi(r) = 0$ or $\varphi(r) = 0$. By using the properties of ψ and φ, it follows that in each case, $r = 0$. Therefore,

$$\lim_{n\to\infty} \mu(fx_n, fx_{n+1}) = 0. \tag{3}$$

Now we prove that (fx_n) is an m-Cauchy sequence in $(f(X), \mu)$.

For all $n \in \mathbb{N}$, we obtain by using condition (1) that

$$\begin{aligned}
\psi(\mu(fx_{n+1}, fx_{n+1})) &\leq \max\{\psi(\mu(Sx_n, Tx_n)), \psi(\mu(Tx_n, Tx_n)), \psi(\mu(Sx_n, Sx_n))\} \\
&\leq F(\psi(\mu(fx_n, fx_n)), \varphi(\mu(fx_n, fx_n))) \\
&\leq \psi(\mu(fx_n, fx_n)).
\end{aligned}$$

Since ψ is strictly increasing, we have $\mu(fx_{n+1}, fx_{n+1}) \leq \mu(fx_n, fx_n)$. This assures that the sequence $(\mu(fx_n, fx_n))$ is decreasing. So, $\lim\limits_{n\to\infty} \mu(fx_n, fx_n)$ exists. By using $(m2)$, we get

$$0 \leq m_{fx_n fx_{n+1}} = \mu(fx_{n+1}, fx_{n+1}) \leq \mu(fx_n, fx_{n+1}).$$

This implies that $\lim_{n\to\infty} \mu(fx_n, fx_n) = 0$. On the other hand, $m_{fx_n fx_m} = \min\{\mu(fx_n, fx_n),\ \mu(fx_m, fx_m)\}$ implies that

$$\lim_{n,m\to\infty} m_{fx_n fx_m} = 0. \tag{4}$$

As $0 \leq M_{fx_n fx_m} - m_{fx_n fx_m} = |\ \mu(fx_n, fx_n) - \mu(fx_m, fx_m)\ |$, it follows that

$$\lim_{n,m\to\infty} (M_{fx_n fx_m} - m_{fx_n fx_m}) = 0.$$

We now show that $\lim_{n,m\to\infty}(\mu(fx_n, fx_m) - m_{fx_n fx_m}) = 0$. Let $\mu^*(x, y) := \mu(x, y) - m_{xy}$, for all x, $y \in X$. Therefore,

$$\lim_{n\to\infty} \mu^*(fx_n, fx_{n+1}) = 0. \tag{5}$$

Suppose that $\lim_{n,m\to\infty} \mu^*(fx_n, fx_m) \neq 0$. Then there exist $\epsilon > 0$ and two subsequences (fx_{n_k}) and (fx_{m_k}) of (fx_n) with $k \leq n_k < m_k$ and

$$\mu^*(fx_{m_k-1}, fx_{n_k}) < \epsilon \leq \mu^*(fx_{m_k}, fx_{n_k})\ \forall k \in \mathbb{N}.$$

By $(m4)$, we have

$$\begin{aligned}
\epsilon \leq \mu^*(fx_{m_k}, fx_{n_k}) &\leq \mu^*(fx_{m_k}, fx_{m_k-1}) + \mu^*(fx_{m_k-1}, fx_{n_k}) \\
&< \mu^*(fx_{m_k}, fx_{m_k-1}) + \epsilon.
\end{aligned}$$

Taking limit as $k \to \infty$ and using condition (5), we get

$$\lim_{k\to\infty} \mu^*(fx_{m_k}, fx_{n_k}) = \epsilon.$$

i.e.,

$$\lim_{k\to\infty} (\mu(fx_{m_k}, fx_{n_k}) - m_{fx_{m_k}fx_{n_k}}) = \epsilon.$$

As $\lim_{k\to\infty} m_{fx_{m_k}fx_{n_k}} = 0$, it follows that

$$\lim_{k\to\infty} \mu(fx_{m_k}, fx_{n_k}) = \epsilon. \tag{6}$$

By repeated use of $(m4)$, we get

$$\mu^*(fx_{m_k}, fx_{n_k}) \leq \mu^*(fx_{m_k}, fx_{m_k+1}) + \mu^*(fx_{m_k+1}, fx_{n_k+1}) + \mu^*(fx_{n_k+1}, fx_{n_k})$$

and

$$\mu^*(fx_{m_k+1}, fx_{n_k+1}) \leq \mu^*(fx_{m_k+1}, fx_{m_k}) + \mu^*(fx_{m_k}, fx_{n_k}) + \mu^*(fx_{n_k}, fx_{n_k+1}).$$

Taking limit as $k \to \infty$ and using conditions (3) and (6), we have

$$\lim_{k\to\infty} \mu^*(fx_{m_k+1}, fx_{n_k+1}) = \epsilon.$$

This together with condition (4) imply that

$$\lim_{k\to\infty} \mu(fx_{m_k+1}, fx_{n_k+1}) = \epsilon. \tag{7}$$

By using condition (1), we obtain

$$\begin{aligned} & \psi(\mu(fx_{m_k+1}, fx_{n_k+1})) \\ \leq\ & \max\{\psi(\mu(Sx_{m_k}, Tx_{n_k})), \psi(\mu(Tx_{m_k}, Sx_{n_k})), \psi(\mu(Tx_{m_k}, Tx_{n_k})), \psi(\mu(Sx_{m_k}, Sx_{n_k}))\} \\ \leq\ & F(\psi(\mu(fx_{m_k}, fx_{n_k})), \varphi(\mu(fx_{m_k}, fx_{n_k}))) \\ \leq\ & \psi(\mu(fx_{m_k}, fx_{n_k})). \end{aligned}$$

Taking limit as $k \to \infty$ and using conditions (6) and (7), we get

$$\psi(\epsilon) \leq F(\psi(\epsilon), \varphi(\epsilon)) \leq \psi(\epsilon).$$

So $F(\psi(\epsilon), \varphi(\epsilon)) = \psi(\epsilon)$. The definition of F ensures that either $\psi(\epsilon) = 0$ or $\varphi(\epsilon) = 0$. In each case, we have $\epsilon = 0$, which is a contradiction. Therefore, $\lim_{n,m\to\infty}(\mu(fx_n, fx_m) - m_{fx_nfx_m}) = 0$. Thus, (fx_n) is an m-Cauchy sequence in $f(X)$. Since $f(X)$ is complete, there exists $u \in f(X)$ such that $fx_n \to u = ft$ for some $t \in X$. So it must be the case that

$$\lim_{n\to\infty} (\mu(fx_n, u) - m_{fx_nu}) = 0 \text{ and } \lim_{n\to\infty} (M_{fx_nu} - m_{fx_nu}) = 0.$$

As $\lim_{n\to\infty} m_{fx_nu} = 0$, it follows that $\lim_{n\to\infty} \mu(fx_n, u) = 0$. Moreover, $0 \leq M_{fx_nu} - m_{fx_nu} =| \mu(fx_n, fx_n) - \mu(u, u) |$ implies that $\mu(u, u) = 0$.By using condition (1), we get

$$\begin{aligned} \psi(\mu(fx_{2n+1}, Tt)) &= \psi(\mu(Sx_{2n}, Tt)) \\ &\leq \max\{\psi(\mu(Sx_{2n}, Tt)), \psi(\mu(Tx_{2n}, St)), \psi(\mu(Tx_{2n}, Tt)), \psi(\mu(Sx_{2n}, St))\} \\ &\leq F(\psi(\mu(fx_{2n}, ft)), \varphi(\mu(fx_{2n}, ft))) \\ &\leq \psi(\mu(fx_{2n}, ft)). \end{aligned}$$

Since ψ is strictly increasing, we have

$$\mu(fx_{2n+1}, Tt) \leq \mu(fx_{2n}, ft). \tag{8}$$

By an argument similar to that used above, we obtain

$$\mu(fx_{2n+2}, St) \leq \mu(fx_{2n+1}, ft).$$

By using condition (8), we have

$$\begin{aligned} 0 \leq \mu(ft, Tt) &= \mu(ft, Tt) - m_{ftTt} \\ &\leq \mu(ft, fx_{2n+1}) + \mu(fx_{2n+1}, Tt) \\ &\leq \mu(ft, fx_{2n+1}) + \mu(fx_{2n}, ft). \end{aligned}$$

Taking limit as $n \to \infty$, we get $\mu(ft, Tt) = 0$. Similarly, by using the inequality

$$0 \leq \mu(ft, St) \leq \mu(ft, fx_{2n+2}) + \mu(fx_{2n+2}, St),$$

we obtain, $\mu(ft, St) = 0$. Again,

$$\begin{aligned} 0 \leq \psi(\mu(Tt, Tt)) &\leq max\{\psi(\mu(St, Tt)),\, \psi(\mu(Tt, Tt)),\, \psi(\mu(St, St))\} \\ &\leq F(\psi(\mu(ft, ft)), \varphi(\mu(ft, ft))) \\ &\leq \psi(\mu(ft, ft)) = 0. \end{aligned}$$

This gives that $\mu(Tt, Tt) = 0$. Similarly, $\mu(St, St) = 0$. Therefore, $\mu(ft, ft) = \mu(ft, Tt) = \mu(ft, St) = \mu(Tt, Tt) = \mu(St, St)$ and hence $ft = Tt = St = u$. This shows that u is a point of coincidence of S, T and f.

For uniqueness, assume that there is another point $v \in f(X)$ such that $fx = Tx = Sx = v$ for some $x \in X$ and $\mu(v, v) = 0$. Then,

$$\begin{aligned} \psi(\mu(u, v)) &= \psi(\mu(St, Tx)) \\ &\leq \max\{\psi(\mu(St, Tx)), \psi(\mu(Tt, Sx)), \psi(\mu(Tt, Tx)), \psi(\mu(St, Sx))\} \\ &\leq F(\psi(\mu(ft, fx)), \varphi(\mu(ft, fx))) \leq \psi(\mu(u, v)), \end{aligned}$$

which implies that $F(\psi(\mu(u, v)), \varphi(\mu(u, v))) = \psi(\mu(u, v))$. Using properties of F, we have $\psi(\mu(u, v)) = 0$ or $\varphi(\mu(u, v)) = 0$. In each case, $\mu(u, v) = 0$. Thus, $\mu(u, u) = \mu(v, v) = \mu(u, v)$ which ensures that $u = v$. Therefore, S, T and f have a unique point of coincidence in $f(X)$. If $(S,\, f)$ and $(T,\, f)$ are weakly compatible, then by Lemma 3, S, T and f have a unique common fixed point in $f(X)$. ■

Remark 2 *It is worth mentioning that we can replace the continuity of F in Theorem 4 by its upper semi-continuity.*

Corollary 5 *Let (X, μ) be an m-metric space and let the mappings $T,\, f : X \to X$ satisfy the following condition:*

$$\psi(\mu(Tx, Ty)) \leq F(\psi(\mu(fx, fy)),\, \varphi(\mu(fx, fy)))$$

for all $x,\, y \in X$, where $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathcal{C}$. Suppose that $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X. Then T and f have a unique point of coincidence u(say) in $f(X)$ with $\mu(u, u) = 0$. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point in $f(X)$.

Proof. The proof can be obtained from Theorem 4 by taking $S = T$. ■

Corollary 6 *Let (X, μ) be an m-metric space and let the mappings $T,\, f : X \to X$ satisfy the following condition:*

$$\psi(\mu(Tx, Ty)) \leq \psi(\mu(fx, fy)) - \varphi(\mu(fx, fy))$$

for all $x,\, y \in X$, where $\psi \in \Psi$, $\varphi \in \Phi$. Suppose that $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X. Then T and f have a unique point of coincidence u(say) in $f(X)$ with $\mu(u, u) = 0$. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point in $f(X)$.

Proof. The proof follows from Theorem 4 by taking $S = T$ and $F(s,t) = s - t$. ■

Corollary 7 *Let (X,μ) be an m-metric space and let the mappings T, $f : X \to X$ satisfy the following condition:*

$$\mu(Tx, Ty) \leq \mu(fx, fy)\beta(\mu(fx, fy))$$

for all x, $y \in X$, where $\beta : [0,\infty) \to [0,1)$ is a continuous function. Suppose that $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X. Then T and f have a unique point of coincidence u(say) in $f(X)$ with $\mu(u,u) = 0$. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point in $f(X)$.

Proof. The proof can be obtained from Theorem 4 by taking $S = T$, $\psi(t) = t$, $\forall\, t \geq 0$ and $F(s,t) = s\beta(s)$. ■

Corollary 8 *Let (X,μ) be an m-metric space and let the mappings S, T, $f : X \to X$ satisfy the following condition:*

$$max\, \{\mu(Sx, Ty),\, \mu(Tx, Sy), \mu(Tx, Ty), \mu(Sx, Sy)\} \leq k\; \mu(fx, fy)$$

for all x, $y \in X$, where $0 < k < 1$ is a constant. If $S(X) \cup T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X, then S, T and f have a unique point of coincidence u(say) in $f(X)$ with $\mu(u,u) = 0$. Moreover, if $(S,\, f)$ and $(T,\, f)$ are weakly compatible, then S, T and f have a unique common fixed point in $f(X)$.

Proof. The proof follows from Theorem 4 by taking $\psi(t) = t$, $\forall\, t \geq 0$ and $F(s,t) = ks$, $0 < k < 1$. ■

The following corollary is a generalization of [3, Theorem 3.1].

Corollary 9 *Let (X,μ) be an m-metric space and let the mappings T, $f : X \to X$ satisfy the following condition:*

$$\mu(Tx, Ty) \leq k\; \mu(fx, fy)$$

for all x, $y \in X$, where $0 < k < 1$ is a constant. If $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X, then T and f have a unique point of coincidence u(say) in $f(X)$ with $\mu(u,u) = 0$. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point in $f(X)$.

Proof. The proof follows from Theorem 4 by taking $S = T$, $\psi(t) = t$ for all $t \in [0,\infty)$ and $F(s,t) = ks$, where $0 < k < 1$. ■

Taking $T = I$ in above corollary, we obtain the following result.

Corollary 10 *Let (X,μ) be a complete m-metric space and let $f : X \to X$ be an onto expansive mapping. Then f has a unique fixed point in X.*

We now give some examples to support our main result.

Example 4 *Let $X = [0,\infty)$ and $\mu : X \times X \to \mathbb{R}^+$ be defined by $\mu(x,y) = \frac{x+y}{2}$ for all $x, y \in X$. Then (X,μ) is a complete m-metric space. Let S, T, $f : X \to X$ be defined by $Sx = \frac{x}{3}$, if $0 \leq x < 1$ and $Sx = 0$, if $1 \leq x < \infty$; $Tx = 0$, if $0 \leq x < 1$ and $Tx = \frac{x}{3}$, if $1 \leq x < \infty$; $fx = 2x$ for all $x \in X$. Clearly, $S(X) \cup T(X) \subseteq f(X) = X$. Take $\psi(t) = 3t$ for all $t \geq 0$ and $F(s,t) = \frac{1}{3}s$ for all s, $t \geq 0$. It is easy to verify that all the hypotheses of Theorem 4 hold true and 0 is the unique common fixed point of S, T and f in $f(X)$ with $\mu(0,0) = 0$.*

Example 5 *Let $X = [0,\infty)$ and $\mu : X \times X \to \mathbb{R}^+$ be defined by $\mu(x,y) = min\{x,y\} + \frac{x+y}{2}$ for all x, $y \in X$. Then (X,μ) is a complete m-metric space. Let S, T, $f : X \to X$ be defined by $Sx = \frac{(x-1)^2}{3}$, if $0 \leq x < 1$ and $Sx = 0$, if $1 \leq x < \infty$; $Tx = 0$, if $0 \leq x < 1$ and $Tx = \frac{(x-1)^2}{3}$, if $1 \leq x < \infty$; $fx = (x-1)^2$ for all $x \in X$. Clearly, $S(X) \cup T(X) \subseteq f(X) = X$. Take $\psi(t) = 2t$ for all $t \geq 0$ and $F(s,t) = \frac{1}{3}s$ for all s, $t \geq 0$. It is observed that $S(1) = T(1) = f(1) = 0$ but $f(S(1)) \neq S(f(1))$. Therefore, S and f are not weakly compatible. However, all the other conditions of Theorem 4 are satisfied. We find that 0 is the unique point of coincidence of S, T and f in $f(X)$ with $\mu(0,0) = 0$. It should be noticed that Theorem 4 can not assure the existence of a common fixed point of S, T and f.*

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