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Resolution Of Grandi's Paradox And Investigations On Related Series*

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Abstract

Grandi's paradox is resolved by introducing a consistent truncation concept for series expansions of Grandi-type functions. In addition, a convergence improvement technique by successive averaging of truncated series of subsequent orders is presented. Resulting rectified series expansions satisfy certain critical points exactly and represent the corresponding generating functions considerably better compared to standard expansions. Finally, successive averages together with collocation technique are applied to series expansions of arbitrarily selected functions for demonstrating the improvements in series representations.

1 Introduction

Luigi Guido Grandi (1671–1742) was an Italian mathematician who specialized on the infinitesimal or differential calculus but studied quite a variety of problems as it was common during those times. Grandi collected his works in the book *Quadratura circoli et hyperbolae per infinitas hyperbolas et parabolas quadrabiles geometrice exhibita*[1] which is briefly referred to as *Quadratura*. The book does not contain much original material but does introduce some interesting aspects such as the first study of the curve which has become known as the Witch of Agnesi and the series expansion of $1/(1+x)$ which has led to the paradox named after Grandi [1].

2 Grandi's Paradox

In the *Quadratura*, Grandi considered the series expansion

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots \tag{1}$$

which converges for $x \le 1$ and diverges for $x > 1$. Herewith, for later purposes, $x = 1$ shall be called the threshold point as it marks the upper limit of convergence.

Before proceeding further, a method of rendering the series expansion in (1) convergent for $x > 1$ is introduced through a simple manipulation followed by a change of variable as given below.

$$\frac{1}{1+x} = \frac{1}{x(1/x+1)} = \frac{1/x}{1/x+1} = \frac{\tilde{x}}{1+\tilde{x}} = \tilde{x} - \tilde{x}^2 + \tilde{x}^3 - \tilde{x}^4 + \cdots \tag{2}$$

where $\tilde{x} = 1/x$ and obviously $\tilde{x} < 1$ for $x > 1$ hence unlike equation (1) its rearranged form (2) is convergent for $x > 1$.

Grandi indicated that for $x = 1$ the left-hand side of equation (1) would be $1/(1 + 1) = 1/2$ while the right-hand side could be collected as $(1 - 1) + (1 - 1) + (1 - 1) + \cdots = 0 + 0 + 0 + \cdots = 0$, thus revealing

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[1] Quadrature of the circle, and hyperbola for infinite hyperbolas and parabolas presented geometrically.

the paradox $1/2 = 0$. Naturally, quite a many mathematicians discussed Grandi's result; among them was Leibniz (1646-1716) who corresponded with most of the scholars in Europe. Leibniz's correspondence with Grandi, which had started in 1703, included this result [1].

When considered carefully Grandi's paradox may be posed in two different ways with two different results and this duality establishes a good starting point for the resolution of the paradox. If, as Grandi did, one includes in the series even number of terms or terms up to an odd power,

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + x^{2p} - x^{2p+1} \tag{3}$$

where p is an arbitrary integer, the resultant sum on the right, as Grandi argued, is zero when $x = 1$:

$$\frac{1}{1+1} = (1 - 1^1) + (1^2 - 1^3) + \cdots + (1^{2p} - 1^{2p+1}) = 0. \tag{4}$$

On the other hand, if one includes odd number of terms or terms up to an even power

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + x^{2p-2} - x^{2p-1} + x^{2p} \tag{5}$$

setting $x = 1$ results in a different value,

$$\frac{1}{1+1} = (1 - 1^1) + (1^2 - 1^3) + \cdots + (1^{2p-2} - 1^{2p-1}) + 1^{2p} = 1. \tag{6}$$

Thus, depending on the number of terms kept (even or odd) the result on the right is either 0 or 1. Indeed this is the heart of the problem, and if the paradox is to be resolved one must come up with an expression that would produce identical results irrespective of the number of terms kept.

The dual character of the series expansion (1) when different number of terms are retained is exposed quite clearly in Fig. 1 for four different truncation orders. For the region $x \geq 1$ equation (2) with corresponding terms is used. Note that depending on the number of terms (1) and (2) produce respectively 0 and 1 for the threshold point $x = 1$ but not the correct value $1/2$. For this reason, curves diverge rapidly from the actual function $1/(1 + x)$ in the vicinity of $x = 1$, making the approximation unacceptable.

3 Resolving Grandi's Paradox: A Consistent Truncation

If a definite numerical value is to be computed from a series expansion the number of terms retained cannot be indefinitely large or indeterminate; the series must be truncated at a finite number, albeit large. Returning now to equation (1) it has been observed that, depending on the number of terms retained, the result alternates between 0 and 1 for $x = 1$. Obviously then a reconciliatory approach is needed in order to devise a consistent truncation of the series so that the result would not alternate. To this end, equation (3) (even number of terms) and equation (5) (odd number of terms) must be combined. Without altering the left hand side the only possible operation on these equations would be linearly adding them side by side by multiplying each expression with an arbitrary constant. Thus, for achieving a combined equation free of inconsistency with respect to the number of terms kept, we first multiply equation (3) with an arbitrary constant α, equation (5) with β, add them side by side and impose a solvability condition that $\alpha + \beta = 1$ (left side remains the same after addition process) to get

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + x^{2p} - \alpha x^{2p+1} \tag{7}$$

where the number of terms on the right is even. Now, when the threshold point $x = 1$ is substituted into equation (7) we require $1/2 = 1 - \alpha$ so that $\alpha = 1/2$ and consequently, from $\alpha + \beta = 1$, $\beta = 1/2$. This β value corresponds to the case when the number of terms on the right is odd. Nevertheless, to see this explicitly the same procedure is applied once again by using constants α' and β' to odd number of terms

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + x^{2p-2} - x^{2p-1} + \alpha' x^{2p} \tag{8}$$

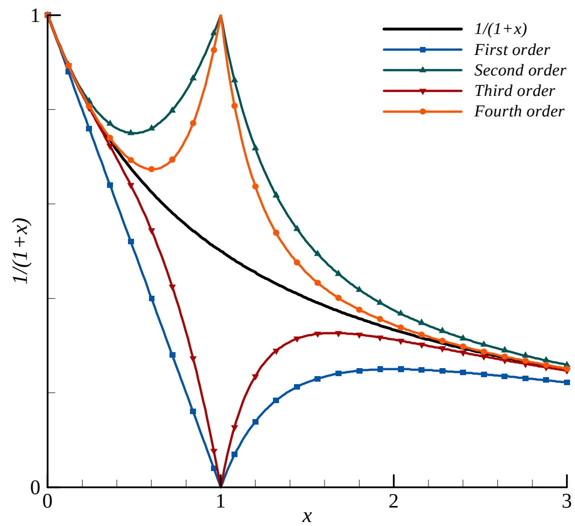

Figure 1: Function $1/(1+x)$ and its standard series expansions for different orders, demonstrating the duality of Grandi's paradox for $x = 1$.

which, as expected, the application of threshold condition requires $1/2 = \alpha'$. We are then in a position to suggest the following expansions, depending on the number of terms being even and odd respectively:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + x^{2p} - \frac{1}{2}x^{2p+1}, \tag{9}$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots - x^{2p-1} + \frac{1}{2}x^{2p}. \tag{10}$$

The rule is then simply to keep one-half of the last term in the truncated series for obtaining the same result for $x = 1$ irrespective of the number of terms. This may be checked easily by setting $x = 1$ in equations (9) and (10). Soundness of the resolution proposed for Grandi's paradox shall be more evident with further ramifications and different applications.

The above consistent arrangement fixes the threshold value of the series as $1/2$ for the threshold point $x = 1$. At the same time this corrective *collocation* improves the results computed from the series, as can be demonstrated with an example. Suppose we would like to compare the numerical values obtained from the function itself and the series for $x = 0.7$ when the highest power kept is four. We shall first employ the standard truncation (1)

$$\frac{1}{1+0.7} \simeq 1 - 0.7 + (0.7)^2 - (0.7)^3 + (0.7)^4.$$

It implies that

$$0.588 \simeq 0.687 \tag{11}$$

which shows that the series expansion value 0.687 is at variance with the correct value 0.588 by 16.9%. On the other hand, the corresponding consistent truncation (10) gives

$$\frac{1}{1+0.7} \simeq 1 - 0.7 + (0.7)^2 - (0.7)^3 + \frac{1}{2}(0.7)^4.$$

It implies that

$$0.588 \simeq 0.567 \tag{12}$$

which deviates from the exact result by 3.6%.

The consistent or rectified truncation is appreciably better compared to the standard one. More importantly, for the threshold point $x = 1$, the new arrangement produces the correct result $1/2$ regardless of the number of terms kept. Yet, much more can be done to improve the series and this is taken up now.

4 Convergence Improvement of Truncated Series

Reportedly, Gauss (1777–1855) was fond of saying that [2] "Nothing has been done if something remains to be done." Although quite a strict statement, it emphasizes Gauss' strong conviction that a subject cannot be considered finished unless treated thoroughly or exhausted completely. Adhering to this motto we shall extend the approach introduced in the previous part to enhance the convergence properties of the series expansion of $1/(1+x)$. Here, the convergence is meant to indicate the goodness of approximation to the actual function. The method is described for a special case but it is easily generalized and applied to different cases as demonstrated later.

Starting from the consistent truncation of the first-order, $1 - \frac{1}{2}x$, of the second-order $1 - x + \frac{1}{2}x^2$, and similarly of higher-orders, could we not combine these expressions successively to obtain truncated series with better convergence characteristics? This is indeed possible and we shall demonstrate this by taking expansions up to and including the fourth-order.

$1 - \frac{1}{2}x$	$1 - x + \frac{1}{2}x^2$	$1 - x + x^2 - \frac{1}{2}x^3$	$1 - x + x^2 - x^3 + \frac{1}{2}x^4$
$1 - \frac{3}{4}x + \frac{1}{4}x^2$	$1 - x + \frac{3}{4}x^2 - \frac{1}{4}x^3$	$1 - x + x^2 - \frac{3}{4}x^3 + \frac{1}{4}x^4$	
$1 - \frac{7}{8}x + \frac{4}{8}x^2 - \frac{1}{8}x^3$	$1 - x + \frac{7}{8}x^2 - \frac{4}{8}x^3 + \frac{1}{8}x^4$		
$1 - \frac{15}{16}x + \frac{11}{16}x^2 - \frac{5}{16}x^3 + \frac{1}{16}x^4$			

Table 1. Convergence improvement by successively averaging the consistently truncated series expansions of $1/(1+x)$ to the fourth-order.

Note that the combination process is carried out by simply averaging the consistently truncated successive expansions. Specifically, adding $1 - \frac{1}{2}x$ to $1 - x + \frac{1}{2}x^2$ and averaging the sum gives $1 - \frac{3}{4}x + \frac{1}{4}x^2$. The entire process is carried out in this fashion and all the resulting expressions satisfy the threshold condition $1/2$ when $x = 1$ since each contribution individually does so.

In each row of Table 1 the first box on the left gives the highest possible combination for each order. Namely, we have for the first-order $1 - \frac{1}{2}x$, the second-order $1 - \frac{3}{4}x + \frac{1}{4}x^2$, the third-order $1 - \frac{7}{8}x + \frac{4}{8}x^2 - \frac{1}{8}x^3$, and finally the fourth-order $1 - \frac{15}{16}x + \frac{11}{16}x^2 - \frac{5}{16}x^3 + \frac{1}{16}x^4$. Let us also note that for $x = 0.7$ this fourth-order expression obtained from successive averages gives 0.588, which agrees perfectly with the exact value to the third decimal place. Compared to the unaveraged consistent truncation used in equation (12) this improvement is substantial and compared to the standard series in equation (11) the improvement is enormous. Finally, all the series satisfy the threshold point $x = 1$ exactly by yielding the threshold value $1/2$, which may also be viewed as a collocation point, the point satisfied exactly by both the function and the series.

Procedure of successive averages as applied to the consistently truncated series expansions of $1/(1+x)$ can be generalized to an arbitrary number of terms. For $x \leq 1$ the formulation for $p+1$ number of terms, where p is even or odd, is

$$\begin{aligned}\frac{1}{1+x} &= 1+\frac{(-1)^1}{2^p}\left[\binom{p}{1}+\binom{p}{2}+\cdots+\binom{p}{p-1}+\binom{p}{p}\right]x\\ &+\frac{(-1)^2}{2^p}\left[\binom{p}{2}+\binom{p}{3}+\cdots+\binom{p}{p-1}+\binom{p}{p}\right]x^2+\cdots\\ &+\frac{(-1)^{p-1}}{2^p}\left[\binom{p}{p-1}+\binom{p}{p}\right]x^{p-1}+\frac{(-1)^p}{2^p}\binom{p}{p}x^p. \qquad (13)\end{aligned}$$

As demonstrated in equation (2) the above formulation may be recast in a form which is convergent for $x \geq 1$

$$\begin{aligned}\frac{1}{1+x} &= \tilde{x}+\frac{(-1)^1}{2^p}\left[\binom{p}{1}+\binom{p}{2}+\cdots+\binom{p}{p-1}+\binom{p}{p}\right]\tilde{x}^2\\ &+\frac{(-1)^2}{2^p}\left[\binom{p}{2}+\binom{p}{3}+\cdots+\binom{p}{p-1}+\binom{p}{p}\right]\tilde{x}^3+\cdots\\ &+\frac{(-1)^{p-1}}{2^p}\left[\binom{p}{p-1}+\binom{p}{p}\right]\tilde{x}^p+\frac{(-1)^p}{2^p}\binom{p}{p}\tilde{x}^{p+1} \qquad (14)\end{aligned}$$

where $\tilde{x} = 1/x$ as defined before. The most remarkable aspect of the above expansions is the change of coefficients according to the number of terms retained. In this way the coefficients are adjusted such that a very accurate –if not the most accurate– representation of the expanded function for a given truncation is provided. This particular aspect is the fundamental novelty of the present *consistent truncation and successive averaging approach.*

Using equation (13) for $x \leq 1$ and (14) for $x \geq 1$ the graphs of above expressions for orders one to four are drawn in Fig. 2 within the range $0 \leq x \leq 3$. Naturally, the first-order truncation is quite crude while the second-order slightly stands out; the third-order is very good and the fourth-order expansion is virtually inseparable from the exact curve of $1/(1+x)$ for the entire range shown and obviously it is so for $x \to \infty$. Fig. 2 should be viewed as the counterpart of Fig. 1, which depicts precisely the same orders of expansions, only using the standard series (1). In this sense, Fig. 2 represents a visible demonstration of the resolution of Grandi's paradox.

5 Series Expansions of Associated and Similar Functions

In the previous section, the series expansions of $1/(1+x)$ with improved characteristics have been obtained. An obvious continuation of these would be integration of both sides of equations (13) and (14) to obtain series representations of natural logarithm.

5.1 Series Representation of Natural Logarithm

We integrate equation (13) for $x \leq 1$ and (14) for $x \geq 1$ and obtain

$$\begin{aligned}\ln(1+x) &= x+\frac{(-1)^1}{2^p}\left[\binom{p}{1}+\binom{p}{2}+\cdots+\binom{p}{p-1}+\binom{p}{p}\right]\frac{x^2}{2}\\ &+\frac{(-1)^2}{2^p}\left[\binom{p}{2}+\binom{p}{3}+\cdots+\binom{p}{p-1}+\binom{p}{p}\right]\frac{x^3}{3}+\cdots\\ &+\frac{(-1)^{p-1}}{2^p}\left[\binom{p}{p-1}+\binom{p}{p}\right]\frac{x^p}{p}+\frac{(-1)^p}{2^p}\binom{p}{p}\frac{x^{p+1}}{p+1}, \qquad (15)\end{aligned}$$

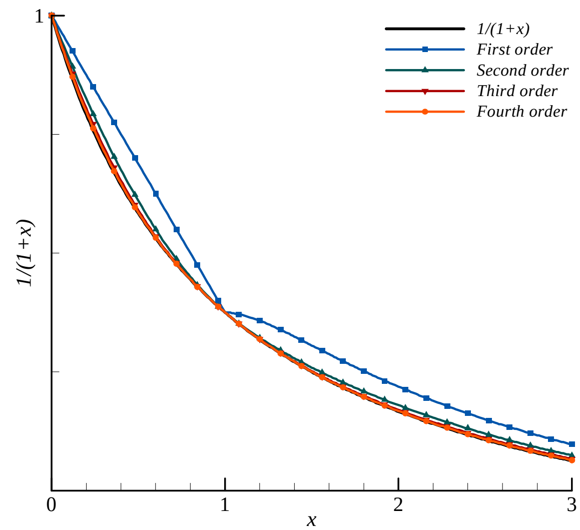

Figure 2: Function $1/(1+x)$ and its consistently truncated and successively averaged series expansions for different orders.

$$\begin{aligned} \ln(1+x) &= \ln x + \frac{(-1)^0}{2^p}\left[\binom{p}{1}+\binom{p}{2}+\cdots+\binom{p}{p-1}+\binom{p}{p}\right]\tilde{x} \\ &+\frac{(-1)^1}{2^p}\left[\binom{p}{2}+\binom{p}{3}+\cdots+\binom{p}{p-1}+\binom{p}{p}\right]\frac{\tilde{x}^2}{2}+\cdots \\ &+\frac{(-1)^{p-2}}{2^p}\left[\binom{p}{p-1}+\binom{p}{p}\right]\frac{\tilde{x}^{p-1}}{p-1}+\frac{(-1)^{p-1}}{2^p}\binom{p}{p}\frac{\tilde{x}^p}{p}. \end{aligned} \tag{16}$$

For $x \to \infty$ the terms involving $\tilde{x}$ on the right of equation (16) vanish while $\ln(1+x)$ on the left grows boundlessly. However, $\ln x$ arising from the integration of the first term on the right of equation (14) compensates this imbalance.

Consider now the truncated expansions in which the terms up to and including the fourth power are retained in the series.

$$\ln(1+x) = x - \frac{15}{16}\frac{x^2}{2} + \frac{11}{16}\frac{x^3}{3} - \frac{5}{16}\frac{x^4}{4} \quad \text{for } x \le 1, \tag{17}$$

$$\ln(1+x) = \ln x + \frac{15}{16}\frac{\tilde{x}}{1} - \frac{11}{16}\frac{\tilde{x}^2}{2} + \frac{5}{16}\frac{\tilde{x}^3}{3} - \frac{1}{16}\frac{\tilde{x}^4}{4} \quad \text{for } x \ge 1. \tag{18}$$

Integration of the fourth-order term for $x \le 1$ brings a fifth-order term but this fifth-order term is excluded to be consistent with the series representation for $x \ge 1$. The overall consistency of this ordering may be checked by setting $x = 1$ in (17) and (18), separately. The results obtained from the right-hand sides are the same for both expansions. Therefore, care should be observed in keeping the terms in equation (15); the last term proportional to the $p + 1^{\text{th}}$ power must be discharged for a correct truncation in practical use.

Numerical values corresponding to the threshold point are worth to recapitulate. The consistent truncation of function $1/(1+x)$ yields the exact threshold value $1/2$ for $x = 1$ but its integrated form $\ln(1+x)$ does not. The value obtained from equation (17) or (18) for $x = 1$ is $131/192 \simeq 0.682$, which deviates from the exact value $\ln 2 \simeq 0.693$ by around 1.6%. On the other hand, the standard series expansion of $\ln(1+x) = x - x^2/2 + x^3/3 - x^4/4$ for $x = 1$ gives $7/12 \simeq 0.583$, which is at variance with the exact value by 15.8%.

Fig. 3 depicts the standard series expansion and the convergence improved series, (17) and (18), against the function $\ln(1+x)$ for the range $0 \leq x \leq 3$. In evaluating the right-hand side of equation (18) within $1 \leq x \leq 2$ the $\ln x$ term is taken from (17) while the previous values of equation (18) is used within $2 \leq x \leq 3$ so that only the series expansions are used without resorting the function $\ln x$ itself. The same scheme is applied to construct the numerical values of the standard series as well. For this reason, as it is obvious from Fig. 3, the series representation errors of the standard expansion near $x = 1$ are carried forward repeatedly for every unit advancement. Although the same thing applies to the improved series, the overall approximation is very satisfactory for two reasons. First, the series representation (17) within $0 \leq x \leq 1$ is quite accurate. Second, at $x = 1$ both (17) and (18) have the same value hence at this point they connect seamlessly. Thus, the overall performance of the consistently truncated and successively averaged series is truly remarkable.

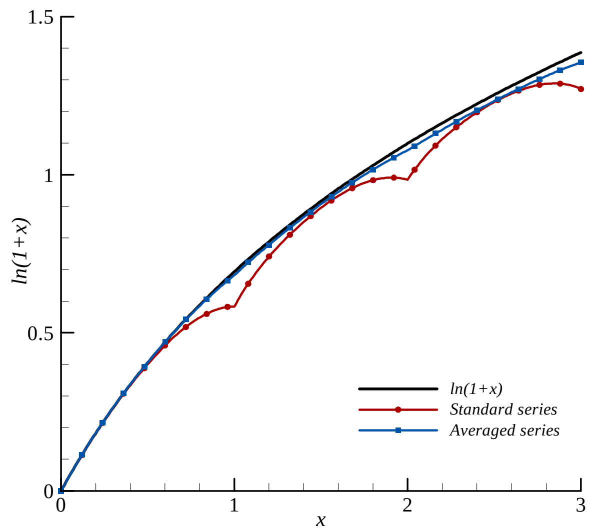

Figure 3: Function $\ln(1+x)$ compared with its standard and consistently truncated and averaged series expansions of the fourth-order.

5.2 Consistent Truncation and Convergence Improvement of $1/(1+x+x^2)$

Consistent truncation and successive averaging procedure is now applied to a different function, $1/(1+x+x^2)$ which has a standard series expansion

$$\frac{1}{1+x+x^2} = 1 - x + x^3 - x^4 + x^6 - x^7 + \cdots . \tag{19}$$

If the series is truncated with even number of terms

$$\frac{1}{1+x+x^2} = 1 - x + x^3 - x^4 + \cdots + x^{3p} - x^{3p+1} \tag{20}$$

and with odd number of terms

$$\frac{1}{1+x+x^2} = 1 - x + x^3 - x^4 + \cdots + x^{3p-3} - x^{3p-2} + x^{3p}. \tag{21}$$

Setting $x = 1$ in (20) and (21) give respectively $1/3 = 0$ and $1/3 = 1$, quite similar to Grandi's paradox.

To obtain a consistent truncation we apply the procedure proclaimed in §2. Thus, we obtain the following equation for the even number of terms

$$\frac{1}{1+x+x^2} = 1 - x + x^3 - x^4 + \cdots + x^{3p} - \alpha x^{3p+1}. \tag{22}$$

When the threshold point $x = 1$ is set in (22) we obtain $1/3 = 1 - \alpha$ so that $\alpha = 2/3$ and since $\alpha + \beta = 1$, $\beta = 1/3$. At this point it is also obvious that the last term of truncation is multiplied by $1/3$ when the number of terms kept is odd. Nevertheless, let us apply the same procedure with the constants α' and β' to the case of odd number of terms

$$\frac{1}{1+x+x^2} = 1 - x + x^3 - x^4 + \cdots + x^{3p-3} - x^{3p-2} + \alpha'^{3p}. \tag{23}$$

The application of threshold condition requires $1/3 = \alpha'$ so that $\beta' = 2/3$. We then establish the following expansions depending on the number of terms kept being even and odd, respectively:

$$\frac{1}{1+x+x^2} = 1 - x + x^3 - x^4 + \cdots + x^{3p} - \frac{2}{3}x^{3p+1}, \tag{24}$$

$$\frac{1}{1+x+x^2} = 1 - x + x^3 - x^4 + \cdots - x^{3p-2} + \frac{1}{3}x^{3p}. \tag{25}$$

The rule is then to take two-third of the highest-order term when the number of retained terms is even and one-third of the highest-order term when the number of retained terms is odd.

Next step is to improve the convergence of the above consistent truncation approach. As before, convergence improvement is done by use of simple averaging. The method is successively applied to higher levels until all the terms but first are modified through averaging.

It is not strictly necessary to apply simple averaging, the weighted averaging such as using the above $\alpha = 2/3$ and $\beta = 1/3$ values to respectively even and odd expansions is also possible. Since each individual truncated expansion (odd or even number) satisfies the threshold value of the function, any weighting may be used. However, we shall employ the simple averaging as done in §4. The reason for this particular choice is that the ultimate formulation is more satisfactory compared to any other weighted averaging.

$1 - \frac{2}{3}x$	$1 - x + \frac{1}{3}x^3$	$1 - x + x^3 - \frac{2}{3}x^4$	$1 - x + x^3 - x^4 + \frac{1}{3}x^6$
$1 - \frac{5}{6}x + \frac{1}{6}x^3$	$1 - x + \frac{2}{3}x^3 - \frac{1}{3}x^4$	$1 - x + x^3 - \frac{5}{6}x^4 + \frac{1}{6}x^6$	
$1 - \frac{11}{12}x + \frac{5}{12}x^3 - \frac{1}{6}x^4$	$1 - x + \frac{5}{6}x^3 - \frac{7}{12}x^4 + \frac{1}{12}x^6$		
$1 - \frac{23}{24}x + \frac{15}{24}x^3 - \frac{9}{24}x^4 + \frac{1}{24}x^6$			

Table 2. Convergence improvement by successively averaging the consistently truncated series expansions of $1/(1+x+x^2)$ to the sixth-order.

Note that each one of the expressions in Table 2 satisfies the threshold condition of rendering $1/3$ when $x = 1$. We now have the improved truncation $1-\frac{23}{24}x+\frac{15}{24}x^3-\frac{9}{24}x^4+\frac{1}{24}x^6$ for $x \leq 1$. For $x \geq 1$ following the approach given in equation (2) it is straightforward to obtain the improved truncation $\tilde{x}^2 - \frac{23}{24}\tilde{x}^3 + \frac{15}{24}\tilde{x}^5 - \frac{9}{24}\tilde{x}^6 + \frac{1}{24}\tilde{x}^8$ with $\tilde{x} = 1/x$ as defined before.

Using these truncated forms separately for $x \leq 1$ and $x \geq 1$ their performance in representing the actual function $1/(1+x+x^2)$ is shown in Fig. 4 for the range $0 \leq x \leq 3$. The consistently truncated and successively averaged expansion is inseparable from the exact expression for the entire range shown and continues in the same manner for $x \to \infty$. The standard expansion, $1 - x + x^3 - x^4 + x^6$ for $x \leq 1$ and $\tilde{x}^2 - \tilde{x}^3 + \tilde{x}^5 - \tilde{x}^6 + \tilde{x}^8$ for $x \geq 1$, on the other hand exhibits considerably poor performance.

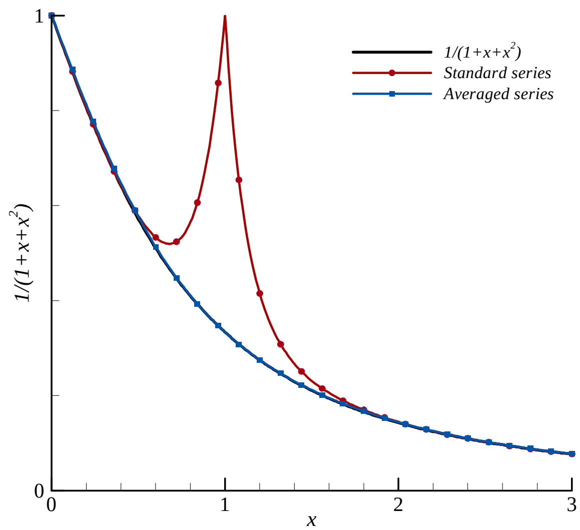

Figure 4: Function $1/(1+x+x^2)$ compared with its standard, and consistently truncated and averaged series expansions of the sixth-order.

6 Generalized Forms

The particular function $1/(1+x)$ considered by Grandi may be generalized quite easily as

$$\frac{a \cdot b}{a+x} = \frac{b}{1+x/a} = \frac{b}{1+\bar{x}} = b\left(1 - \bar{x} + \bar{x}^2 - \bar{x}^3 + \cdots\right) \tag{26}$$

where a and b are arbitrary real constants and $\bar{x} = x/a$. The threshold point is now $x = a$ instead of $x = 1$ and the threshold value is $b/2$. All the approaches introduced for $1/(1+x)$ can identically be applied to $b/(1+x/a)$ as well.

A simple extension of the above generalization is

$$\frac{a^n \cdot b}{a^n + x^n} = \frac{b}{1+(x/a)^n} = \frac{b}{1+\bar{x}^n} = b\left(1 - \bar{x}^n + \bar{x}^{2n} - \bar{x}^{3n} + \cdots\right) \tag{27}$$

where $n \geq 1$ is an integer. The threshold point $x = a$ and the corresponding value $b/2$ are the same as (26).

Finally, a function that includes the previously treated functions as special cases is

$$\begin{aligned} & \frac{a^n \cdot b}{a^n + a^{n-1}x + a^{n-2}x^2 + \cdots + ax^{n-1} + x^n} \\ = & \frac{b}{1 + (x/a) + (x/a)^2 + \cdots + (x/a)^{n-1} + (x/a)^n} \\ = & \frac{b}{1 + \bar{x} + \bar{x}^2 + \cdots + \bar{x}^{n-1} + \bar{x}^n} \\ = & \; b\left(1 - \bar{x} + \bar{x}^{n+1} - \bar{x}^{n+2} + \bar{x}^{2n+2} - \bar{x}^{2n+3} + \cdots\right). \end{aligned} \tag{28}$$

The above function has the threshold point $x = a$ and the threshold value $b/(1+n)$. Grandi's function is a special case with $a = 1$, $b = 1$, and $n = 1$ so that the threshold point is $x = a = 1$ and the threshold value is $1/2$. The function used in §5.2 corresponds to $a = 1$, $b = 1$, and $n = 2$ therefore the threshold point is $x = a = 1$ and the threshold value is $b/(1+n) = 1/(1+2) = 1/3$.

All the above more general forms may be manipulated to be convergent for $x \geq a$ and treated by the methods described in preceding sections; however, we shall not attempt to pursue any such works for the sake of briefness.

7 Convergence Improvement by Successive Averages and Collocation

Convergence improvement introduced in §4 is applied to series with a threshold point and a corresponding threshold value. Nevertheless, it is possible to improve the convergence of series without a definite threshold point by successive averages alone or by collocation and successive averages.

As observed for Grandi type series expansions a threshold point is a particular value which renders different threshold values depending on the number of terms kept in the series. Threshold values for odd and even number of terms differ discontinuously or sharply as 0 and 1. Besides, the threshold point sets a demarcation line between convergence and divergence of a series expansion. Ordinarily, series expansions do not have threshold points as defined in this sense. Therefore, when applying the consistent truncation and then convergence improvement we must introduce a slightly different approach.

The convergence improvement of a series without a definite threshold point may be accomplished by simple successive averages as in §4. Alternatively, a collocation point may be selected so that the corresponding value of the generating function is satisfied exactly by every successive truncation of the series and then the collocated series are successively averaged. These slightly different approaches are demonstrated below for two arbitrarily selected functions without definite threshold points.

7.1 Convergence Improvement of $(1+x)^{1/2}$

The first function is $(1+x)^{1/2}$, which has the corresponding Maclaurin series

$$(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \cdots \tag{29}$$

which is convergent for $|x| \leq 1$. Since the above series expansion does not have a given threshold point in the sense attributed to Grandi-type expansions, an arbitrary collocation point x^* is selected. For this collocation point, the sum of any two successive truncations multiplied by arbitrary constants is required to satisfy exactly the corresponding value of the generating function, $(1+x^*)^{1/2}$. The sum of the arbitrary constants is set to unity as a solvability condition. Overall, the procedure is exactly like the consistent truncation approach of §3, the only difference is the use of a collocation point chosen at will instead of a definite threshold point. To make the treatment more general we shall not assign a numerical value to the collocation point yet and keep the arbitrary constants undetermined. The application below should make it all clear.

$1+\frac{\alpha_1}{2}x$	$1+\frac{1}{2}x-\frac{\alpha_2}{8}x^2$	$1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{\alpha_3}{16}x^3$
$1+\frac{(\alpha_1+1)}{4}x-\frac{\alpha_2}{16}x^2$	$1+\frac{1}{2}x-\frac{(\alpha_2+1)}{16}x^2+\frac{\alpha_3}{32}x^3$	
$1+\frac{(\alpha_1+3)}{8}x-\frac{(2\alpha_2+1)}{32}x^2+\frac{\alpha_3}{64}x^3$		

Table 3. Convergence improvement by collocation and successive averages for the series expansion of $(1+x)^{1/2}$ to the third-order.

First, to obtain the improved series corresponding to simple averaging without collocation we set $\alpha_1 = \alpha_2 = \alpha_3 = 1$ so that

$$(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{3}{32}x^2 + \frac{1}{64}x^3. \tag{30}$$

On the other hand, if a collocation point, say $x^* = 3$, is selected then to satisfy the exact value $(1+3)^{1/2} = 2$ for each truncation order separately; that is, $2 = 1 + \frac{\alpha_1}{2}3$, $2 = 1 + \frac{1}{2}3 - \frac{\alpha_2}{8}3^2$, and $2 = 1 + \frac{1}{2}3 - \frac{1}{8}3^2 + \frac{\alpha_3}{16}3^3$, we must set $\alpha_1 = 2/3$, $\alpha_2 = 4/9$, and $\alpha_3 = 10/27$ in the last expression of Table 3. The improved series is then

$$(1+x)^{1/2} = 1 + \frac{11}{24}x - \frac{17}{288}x^2 + \frac{5}{864}x^3. \tag{31}$$

The above series exactly satisfies the value of function $(1+3)^{1/2} = 2$ for the collocation point $x^* = 3$.

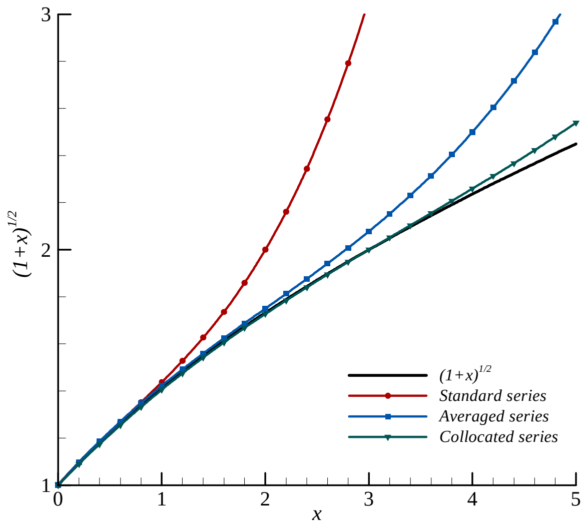

Figure 5: Function $(1+x)^{1/2}$ compared with its standard, successively averaged, and collocated and successively averaged series expansions of the third-order.

Fig. 5 depicts the exact function $(1+x)^{1/2}$ against the truncated series (29), (30), and (31) for the range $0 \le x \le 5$. The formal validity range of the expansion (29) is $-1 \le x \le 1$.

The standard expansion (29) is highly accurate in the neighborhood of $x = 0$ but diverges rapidly for $x > 1$. The simple averaged series (30) performs definitely better while the collocation at the point of $x^* = 3$, equation (31), widens the range of approximation considerably. It must be pointed out that when viewed closer the accuracy in the shorter range lessens with increased range of collocation. Nevertheless, overall improvement in representing the generating function is quite impressive for both (30) and (31).

We have only considered the positive region $x \ge 1$ as it has a larger domain of application compared to the negative region $-1 \le x \le 0$. Since the collocation is applied in the positive region the performance in the negative region is not appreciable; however, collocating the series at point $x = -1$ for instance would improve it considerably in the negative region and moderately in the positive region.

7.2 Convergence Improvement of $\tanh x$

The second function considered is $\tanh x$, which arises in water wave mechanics problems and has a Maclaurin series

$$\tanh x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \cdots . \tag{32}$$

As in the previous example this function does not have a specific threshold point. Therefore, we shall proceed by averaging and then selecting an arbitrary collocation point.

$x - \frac{\alpha_1}{3}x^3$	$x - \frac{1}{3}x^3 + \frac{2\alpha_2}{15}x^5$	$x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17\alpha_3}{315}x^7$
$x - \frac{(\alpha_1+1)}{6}x^3 + \frac{\alpha_2}{15}x^5$	$x - \frac{1}{3}x^3 + \frac{(\alpha_2+1)}{15}x^5 - \frac{17\alpha_3}{630}x^7$	
$x - \frac{(\alpha_1+3)}{12}x^3 + \frac{(2\alpha_2+1)}{30}x^5 - \frac{17\alpha_3}{1260}x^7$		

Table 4. Convergence improvement by collocation and successive averages for the series expansion of $\tanh x$ to the seventh-order.

Setting $\alpha_1 = \alpha_2 = \alpha_3 = 1$ gives the improved series corresponding to simple averaging

$$\tanh x = x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{17}{1260}x^7. \tag{33}$$

We also do the improvement by selecting an arbitrary collocation point, say $x^* = 2$. The value of function for $x^* = 2$ is approximately $\tanh 2 \simeq 0.964$. Satisfying this value for each truncation order separately gives $\alpha_1 = 0.3885$, $\alpha_2 = 0.3822$ and $\alpha_3 = 0.3816$. Substituting these values into the final averaged form results in the following improved series

$$\tanh x = x - 0.2824\,x^3 + 0.0588\,x^5 - 0.00515\,x^7 \tag{34}$$

which gives quite nearly 0.964 for $x^* = 2$.

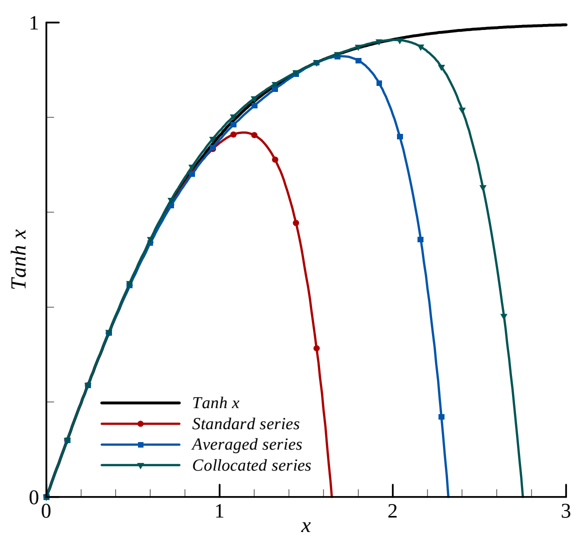

Figure 6: Function $\tanh x$ compared with its standard, successively averaged, and collocated and successively averaged series expansions of the seventh-order.

Fig. 6 compares the standard series (32), the averaged series (33), and the collocated and averaged series (34) with the exact function $\tanh x$ for the range $0 \leq x \leq 3$. Note however that the formal validity range of the expansion (32) is $|x| \leq 1$.

The formal expansion (32) is highly accurate in the neighborhood of $x = 0$ but diverges rapidly for $x > 1$. The successively averaged series (33) does considerably better by extending the accuracy range to approximately $x = 2$, which is twice the range of the standard series. Finally, the collocated and then averaged series (34) widens the range of approximation farther to nearly $x = 2.5$ and supersedes the others but the performances of both (33) and (34) are impressive.

8 Concluding Remarks

Resolution of Grandi's paradox has opened up quite wide vistas far beyond the expectations in treatment of series. Results force us to change our usual philosophy concerning the series by making a distinction between finite and infinite series. Finite series should not be considered as merely truncated forms of infinite series anymore. Depending on the number of terms retained the coefficients of a truncated series may be redefined for improving various aspects of the series. When the number of terms tends to become indefinitely large, only then the coefficients approach to the usual coefficients given by usual infinite series expansions. This renders the standard series expansions special cases rather than the general ones. For functions that have series expansions with coefficients of alternating signs, there is always the possibility of obtaining better representing truncated series whose coefficients are different from those of the standard infinite series expansions.

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