Applied Mathematics E-Notes, 20(2020), 236-247 © ISSN 1607-2510
http://www.math.nthu.edu.tw/∼amen/

Inequalities For Generalized Preinvex Functions I*

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Received: April 24, 2019

Abstract

In this paper, we introduce and consider a new class of generalized preinvex functions involving two non-negative functions. These generalized preinvex functions include several new and known classes of convex and preinvex functions as special cases. We derive several new integral inequalities for these preinvex functions. Some special cases are also discussed. The idea and techniques of this paper may stimulate further research in this field.

1 Introduction

Recently, convex sets and convex functions have been generalized in several directions using novel and innovative techniques. These new generalizations have been proved to be very useful in considering very complicated and difficult problem, which arise in engineering, physics and mathematical sciences. Hanson [4] introduced the concept of invexity in mathematical programming, which proved to be useful concept in optimization theory. Ben-Israel and Mond [1] defined invex sets and preinvex functions. They proved that the differentiable preinvex are invex functions. Mohan and Neogy [7] proved that the invex functions are also preinvex functions under suitable conditions. It have been shown that the invex sets and preinvex functions include convex sets and convex functions as special cases, but the converse is not true. Weir and Mond [23] discussed the applications in multiple objective optimization. Noor [9] proved that the optimality conditions of a differentiable preinvex function can be characterized by a class of variational inequalities, which are called the variational-like inequalities. For the applications, numerical methods and other applications of variational inequalities, see [8, 9, 10, 12, 11, 20] and the references therein. Noor [14] proved that a function is a preinvex function, if and only if, it satisfies the Hermite-Hadamard type inequality. Noor [13, 14, 15, 16] derived several Hermite-Hadamard type inequalities for preinvex functions, which proved to be the starting point for establishing integral inequalities for various type of preinvex functions. For the applications, properties and other aspects of the preinvex functions, see [13, 14, 15, 16, 17, 18, 20] and the references therein. Noor et al. [19] also introduced the concept of beta-preinvex functions and established some error estimates involving the Euler beta function for the class of functions whose certain powers of the absolute value are beta-preinvex function.

In this paper, we introduce and consider a new class of preinvex functions involving two arbitrary non-negative functions. We established some new integral inequalities. For suitable and appropriate choice of the arbitrary functions, one can obtain several new and known results for convex functions and preinvex functions as special cases. Results proved in this paper continue to hold for these special cases.

2 Preliminaries

Let K_η be a nonempty closed set in $\mathbb{R}$. Let $f : K_\eta = [x, x + \eta(y, x)] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function and $\eta(\cdot, \cdot) : K_\eta \times K_\eta \to \mathbb{R}$ be a continuous bifunction. In this section, we discuss some new and known results.

*Mathematics Subject Classifications: 26D15, 26D10, 90C23.

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Definition 1 ([20]) *A set $K_\eta \subset \mathbb{R}$ is said to be invex with respect to a bifunction $\eta(\cdot,\cdot) : K_\eta \times K_\eta \to \mathbb{R}$, if*

$$x + t\eta(y,x) \in K_\eta, \qquad \forall x, y \in K_\eta, t \in [0,1].$$

We now introduce some new classes of preinvex functions involving two arbitrary non-negative functions.

Definition 2 *Let K_η be an invex set and $h_1, h_2 : J \subset \mathbb{R} \to \mathbb{R}$ be non-negative functions, where J is an interval such that $(0,1) \subset J$ and $h_1, h_2 \neq 0$. A function $f : K_\eta \subset \mathbb{R} \to \mathbb{R}$ is said to be (h_1, h_2)-preinvex function, if*

$$f(x + t\eta(y,x)) \leq h_1(t) f(x) + h_2(t) f(y), \quad \forall x, y \in K_\eta, t \in [0,1].$$

For $t = \frac{1}{2}$, we have

$$f\left(\frac{2x + \eta(y,x)}{2}\right) \leq h_1\left(\frac{1}{2}\right) f(x) + h_2\left(\frac{1}{2}\right) f(y), \quad \forall x, y \in K_\eta. \tag{1}$$

The function f is said to be (h_1, h_2) Jensen preinvex function with respect to $\eta(y,x)$.

The class of (h_1, h_2)-preinvex function contains several new and known classes.

I. If $h_1(t) = (1-t)$ and $h_2(t) = t$, then (h_1, h_2)-preinvex function reduces to classical preinvex function, see [23].

II. If $h_1(t) = (1-t)^p t^q$ and $h_2(t) = t^p(1-t)^q$, then (h_1, h_2)-preinvex function is a beta-preinvex function, where $p, q \geq -1$, see [19].

III. If $h_1(t) = (1-t^s)$ and $h_2(t) = t^s$, then (h_1, h_2)-preinvex function is a s-preinvex function of first kind.

Definition 3 *Let K_η be an invex set and $h_1, h_2 : J \subset \mathbb{R} \to \mathbb{R}$ be non-negative functions, where J is an interval such that $(0,1) \subset J$ and $h_1, h_2 \neq 0$. A function $f : K_\eta \subset \mathbb{R} \to \mathbb{R}$ is said to be (h_1, h_2)-*log*-preinvex function, if*

$$f(x + t\eta(y,x)) \leq [f(x)]^{h_1(t)} [f(y)]^{h_2(t)}, \quad \forall x, y \in K_\eta, t \in [0,1]. \tag{2}$$

From (2) it follows that

$$\log f(x + t\eta(y,x)) \leq h_1(t) \log f(x) + h_2(t) \log f(y), \qquad \forall x, y \in K_\eta,\ t \in [0,1].$$

We recall the following special function which is known as Beta function

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x, y > 0,$$

where $\Gamma(\cdot)$ is the gamma function.

We need the well-known Condition C introduced by Mohan and Neogy in [7].

Condition *C*: Let $K_\eta \subseteq \mathbb{R}$ be an invex set with respect to bifunction $\eta(\cdot,\cdot) : K_\eta \times K_\eta \to \mathbb{R}$. For any $x, y \in K_\eta$ and any $t \in [0,1]$, we have

$$\eta(y, y + t\eta(x,y)) = -t\eta(x,y),$$

$$\eta(x, y + t\eta(x,y)) = (1-t)\eta(x,y).$$

3 Main Results

In this section, we obtain some new integral inequalities related to (h_1, h_2)-preinvex function.

Theorem 1 *Let $h_1, h_2 : J \subset \mathbb{R} \to \mathbb{R}$ be integrable and non-negative functions, where J is an interval such that $(0,1) \subset J$ and $h_1, h_2 \neq 0$. Let $f : K_\eta = [a, a+\eta(b,a)] \subset \mathbb{R} \to \mathbb{R}$ be a non-negative (h_1, h_2)-preinvex function. Then*

$$\begin{aligned} \frac{1}{h_1(\frac{1}{2}) + h_2(\frac{1}{2})} f\left(\frac{2a+\eta(b,a)}{2}\right) &\leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\mathrm{d}x \\ &\leq f(a)\int_0^1 h_1(t)\mathrm{d}t + f(b)\int_0^1 h_2(t)\mathrm{d}t. \end{aligned} \tag{3}$$

Proof. Let f be (h_1, h_2)-preinvex function. Then taking $x = a + t\eta(b,a)$ and $y = a + (1-t)\eta(b,a)$ in (1), and using condition C, we have

$$\begin{aligned} f\left(\frac{2a+\eta(b,a)}{2}\right) &\leq h_1\left(\frac{1}{2}\right) f(a+t\eta(b,a)) + h_2\left(\frac{1}{2}\right) f(a+(1-t)\eta(b,a)) \\ &= \int_0^1 h_1\left(\frac{1}{2}\right) f(a+t\eta(b,a))\mathrm{d}t + \int_0^1 h_2\left(\frac{1}{2}\right) f(a+(1-t)\eta(b,a))\mathrm{d}t. \end{aligned}$$

This implies

$$\frac{1}{h_1(\frac{1}{2}) + h_2(\frac{1}{2})} f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\mathrm{d}x.$$

Now,

$$\begin{aligned} \frac{1}{h_1(\frac{1}{2}) + h_2(\frac{1}{2})} f\left(\frac{2a+\eta(b,a)}{2}\right) &\leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\mathrm{d}x \\ &= \int_0^1 f\big(a+t\eta(b,a)\big)\mathrm{d}t \\ &\leq f(a)\int_0^1 h_1(t)\mathrm{d}t + f(b)\int_0^1 h_2(t)\mathrm{d}t, \end{aligned}$$

which is the required result. ■

Now we discuss some special cases of Theorem 1.

I. If $h_1(t) = 1-t$ and $h_2(t) = t$, then we have result for preinvex functions.

Corollary 2 *Let $f : K_\eta = [a, a+\eta(b,a)] \subset \mathbb{R} \to \mathbb{R}$ be a preinvex function. Then*

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}.$$

II. If $h_1(t) = (1-t)^p t^q$ and $h_2(t) = t^p(1-t)^q$, then we have a result for beta-preinvex functions.

Corollary 3 *Let $f : K_\eta = [a, a+\eta(b,a)] \subset \mathbb{R} \to \mathbb{R}$ be beta preinvex function, where $p, q \geq -1$. Then*

$$\begin{aligned} 2^{p+q-1} f\left(\frac{2a+\eta(b,a)}{2}\right) &\leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\mathrm{d}x \\ &\leq \beta(p+1, q+1)[f(a)+f(b)]. \end{aligned}$$

III. If $h_1(t) = 1-t^s$ and $h_2(t) = t^s$, then we have a result for s-preinvex functions of first kind.

Corollary 4 *Let $f : K_\eta = [a, a + \eta(b, a)] \subset \mathbb{R} \to \mathbb{R}$ be s-preinvex function of first kind, where $s \in (0, 1)$. Then*

$$f\left(\frac{2a + \eta(b, a)}{2}\right) \leq \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)\mathrm{d}x \leq \frac{sf(a)}{s + 1} + \frac{f(b)}{s + 1}.$$

Theorem 5 *Let $h_1, h_2 : J \subset \mathbb{R} \to \mathbb{R}$ be integrable and non-negative functions, where J is an interval such that $(0, 1) \subset J$ and $h_1, h_2 \neq 0$. Let $f : K_\eta = [a, a + \eta(b, a)] \subset \mathbb{R} \to \mathbb{R}$ be a non-negative (h_1, h_2)-preinvex function. Then*

$$\begin{aligned} & \frac{2f(a)}{\eta(b, a)} \int_a^{a+\eta(b,a)} h_1\left(\frac{x - a}{\eta(b, a)}\right) f(x)\mathrm{d}x + \frac{2f(b)}{\eta(b, a)} \int_a^{a+\eta(b,a)} h_2\left(\frac{x - b}{\eta(b, a)}\right) f(x)\mathrm{d}x \\ \leq \; & \frac{1}{\eta(b, a)} \int_a^b f^2(x)\mathrm{d}x + f^2(a) \int_0^1 h_1^2(t)\mathrm{d}t + f^2(b) \int_0^1 h_2^2(t)\mathrm{d}t \\ & + 2f(a)f(b) \int_0^1 h_1(t)h_2(t)\mathrm{d}t. \end{aligned}$$

Proof. Since f is (h_1, h_2)-preinvex function on $[a, a + \eta(b, a)]$, we see that we have

$$f(a + t\eta(b, a)) \leq h_1(t)f(a) + h_2(t)f(b), \quad \forall a, b \in K_\eta,\ t \in [0, 1].$$

Using the inequality $G(x, y) \leq A(x, y)$ we can write

$$\sqrt{f(a + t\eta(b, a))[h_1(t)f(a) + h_2(t)f(b)]} \leq \frac{f(a + t\eta(b, a)) + [h_1(t)f(a) + h_2(t)f(b)]}{2}.$$

Thus

$$\begin{aligned} 2f(a + t\eta(b, a))[h_1(t)f(a) + h_2(t)f(b)] \leq \; & f^2(a + t\eta(b, a)) + h_1^2(t)f^2(a) \\ & + h_2^2(t)f^2(b) + 2h_1(t)h_2(t)f(a)f(b). \end{aligned}$$

Since f and h_1, h_2 are integral functions, we can integrate over the interval $[0, 1]$ to obtain

$$\begin{aligned} & 2f(a) \int_0^1 h_1(t)f(a + t\eta(b, a))\mathrm{d}t + 2f(b) \int_0^1 h_2(t)f(a + t\eta(b, a))\mathrm{d}t \\ \leq \; & \int_0^1 f^2(a + t\eta(b, a))\mathrm{d}t + f^2(a) \int_0^1 h_1^2(t)\mathrm{d}t + f^2(b) \int_0^1 h_2^2(t)\mathrm{d}t \\ & + 2f(a)f(b) \int_0^1 h_1(t)h_2(t)\mathrm{d}t. \end{aligned} \tag{4}$$

If we make change of variable $a + t\eta(b, a) = x$, then we can write

$$\int_0^1 h_1(t)f(a + t\eta(b, a))\mathrm{d}t = \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} h_1\left(\frac{x - a}{\eta(b, a)}\right) f(x)\mathrm{d}x,$$

and similarly

$$\int_0^1 h_2(t)f(a + t\eta(b, a))\mathrm{d}t = \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} h_2\left(\frac{x - b}{\eta(b, a)}\right) f(x)\mathrm{d}x.$$

Therefore the inequality (4) can be rewritten as

$$\begin{aligned} & \frac{2f(a)}{\eta(b, a)} \int_a^{a+\eta(b,a)} h_1\left(\frac{x - a}{\eta(b, a)}\right) f(x)\mathrm{d}x + \frac{2f(b)}{\eta(b, a)} \int_a^{a+\eta(b,a)} h_2\left(\frac{x - b}{\eta(b, a)}\right) f(x)\mathrm{d}x \\ \leq \; & \frac{1}{\eta(b, a)} \int_a^b f^2(x)\mathrm{d}x + f^2(a) \int_0^1 h_1^2(t)\mathrm{d}t + f^2(b) \int_0^1 h_2^2(t)\mathrm{d}t \\ & + 2f(a)f(b) \int_0^1 h_1(t)h_2(t)\mathrm{d}t, \end{aligned}$$

which is the required result. ■

Now we discuss some special cases of Theorem 5.

I. If $h_1(t) = 1 - t$ and $h_2(t) = t$, then we have a new result for preinvex functions:

Corollary 6 *Let $f : K_\eta = [a, a + \eta(b, a)] \subset \mathbb{R} \to \mathbb{R}$ be a preinvex function. Then*

$$\begin{aligned}&\frac{2f(a)}{\eta(b,a)}\int_a^{a+\eta(b,a)}\left(1-\frac{x-a}{\eta(b,a)}\right)f(x)\mathrm{d}x+\frac{2f(b)}{\eta(b,a)}\int_a^{a+\eta(b,a)}\left(\frac{x-b}{\eta(b,a)}\right)f(x)\mathrm{d}x\\ \leq\ &\frac{1}{\eta(b,a)}\int_a^b f^2(x)\mathrm{d}x+\frac{f^2(a)+f(a)f(b)+f^2(b)}{3}.\end{aligned}$$

Theorem 7 *Let $h_1, h_2 : J \subset \mathbb{R} \to \mathbb{R}$ be integrable and non-negative functions, where J is an interval such that $(0, 1) \subset J$ and $h_1, h_2 \neq 0$. Let $f : K_\eta = [a, a + \eta(b, a)] \subset \mathbb{R} \to \mathbb{R}$ be a non-negative (h_1, h_2)-preinvex function. Then*

$$\begin{aligned}&\left[h_1\left(\frac{1}{2}\right)+h_2\left(\frac{1}{2}\right)\right]\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x\\ \leq\ &\frac{1}{2}f\left(\frac{2a+\eta(b,a)}{2}\right)+\frac{h_1^2(\frac{1}{2})+h_2^2(\frac{1}{2})}{2f(\frac{2a+\eta(b,a)}{2})\eta(b,a)}\int_a^{a+\eta(b,a)}f^2(x)\mathrm{d}x\\ &+\frac{h_1(\frac{1}{2})h_2(\frac{1}{2})}{f(\frac{2a+\eta(b,a)}{2})}\left((f^2(a)+f^2(b))\int_0^1 h_2(t)h_1(t)\mathrm{d}t\right.\\ &\left.+(f(a)f(b))\left[\int_0^1 h_1^2(t)\mathrm{d}t+\int_0^1 h_2^2(t)\mathrm{d}t\right]\right).\end{aligned}$$

Proof. Let f be (h_1, h_2)-preinvex function on $[a, a+\eta(b, a)]$. Take $x = a+t\eta(b, a)$ and $y = a+(1-t)\eta(b, a)$ in (1) and using condition C, we have

$$f\left(\frac{2a+\eta(b,a)}{2}\right)\leq h_1\left(\frac{1}{2}\right)f(a+t\eta(b,a))+h_2\left(\frac{1}{2}\right)f(a+(1-t)\eta(b,a)).$$

Using the classical inequality $G(x, y) \leq A(x, y)$, we have

$$\begin{aligned}&4f\left(\frac{2a+\eta(b,a)}{2}\right)\left[h_1\left(\frac{1}{2}\right)f(a+t\eta(b,a))+h_2\left(\frac{1}{2}\right)f(a+(1-t)\eta(b,a))\right]\\ \leq\ &\left(f\left(\frac{2a+\eta(b,a)}{2}\right)+\left[h_1\left(\frac{1}{2}\right)f(a+t\eta(b,a))+h_2\left(\frac{1}{2}\right)f(a+(1-t)\eta(b,a))\right]\right)^2,\end{aligned}$$

and from this, we have

$$\begin{aligned}&2f\left(\frac{2a+\eta(b,a)}{2}\right)\left[h_1\left(\frac{1}{2}\right)f(a+t\eta(b,a))+h_2\left(\frac{1}{2}\right)f(a+(1-t)\eta(b,a))\right]\\ \leq\ &f^2\left(\frac{2a+\eta(b,a)}{2}\right)+h_1^2\left(\frac{1}{2}\right)f^2(a+t\eta(b,a))+h_2^2\left(\frac{1}{2}\right)f^2(a+(1-t)\eta(b,a))\\ &+2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)f(a+t\eta(b,a))f(a+(1-t)\eta(b,a))].\end{aligned}$$

Now applying the (h_1, h_2) preinvexity of the function f, we have

$$\begin{aligned}&2f\left(\frac{2a+\eta(b,a)}{2}\right)\left[h_1\left(\frac{1}{2}\right)f(a+t\eta(b,a))+h_2\left(\frac{1}{2}\right)f(a+(1-t)\eta(b,a))\right]\\ \leq\ &f^2\left(\frac{2a+\eta(b,a)}{2}\right)+h_1^2\left(\frac{1}{2}\right)f^2(a+t\eta(b,a))+h_2^2\left(\frac{1}{2}\right)f^2(a+(1-t)\eta(b,a))\\ &+2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)(h_1(t)f(a)+h_2(t)f(b))(h_2(t)f(a)+h_1(t)f(b)).\end{aligned}$$

Integrate over the interval $[0,1]$, we have

$$\begin{aligned}
&2f\left(\frac{2a+\eta(b,a)}{2}\right)\left[h_1\left(\frac{1}{2}\right)\int_0^1 f(a+t\eta(b,a))\mathrm{d}t+h_2\left(\frac{1}{2}\right)\int_0^1 f(a+(1-t)\eta(b,a))\mathrm{d}t\right]\\
\leq\ &\int_0^1 f^2\left(\frac{2a+\eta(b,a)}{2}\right)\mathrm{d}t+\left[h_1^2\left(\frac{1}{2}\right)\int_0^1 f^2(a+t\eta(b,a))\mathrm{d}t+h_2^2\left(\frac{1}{2}\right)\int_0^1 f^2(a+(1-t)\eta(b,a))\mathrm{d}t\right]\\
&+2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left((f^2(a)+f^2(b))\int_0^1 h_2(t)h_1(t)\mathrm{d}t+(f(a)f(b))\left[\int_0^1 h_1^2(t)\mathrm{d}t+\int_0^1 h_2^2(t)\mathrm{d}t\right]\right).
\end{aligned}$$

Making the change of variable $a+t\eta(b,a)=x$ and dividing both sides by $2f(\frac{2a+\eta(b,a)}{2})$, we obtain

$$\begin{aligned}
&\left[h_1\left(\frac{1}{2}\right)+h_2\left(\frac{1}{2}\right)\right]\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x\\
\leq\ &\frac{1}{2}f\left(\frac{2a+\eta(b,a)}{2}\right)+\frac{h_1^2(\frac{1}{2})+h_2^2(\frac{1}{2})}{2f(\frac{2a+\eta(b,a)}{2})\eta(b,a)}\int_a^{a+\eta(b,a)} f^2(x)\mathrm{d}x\\
&+\frac{h_1(\frac{1}{2})h_2(\frac{1}{2})}{f(\frac{2a+\eta(b,a)}{2})}\left((f^2(a)+f^2(b))\int_0^1 h_2(t)h_1(t)\mathrm{d}t+(f(a)f(b))\left[\int_0^1 h_1^2(t)\mathrm{d}t+\int_0^1 h_2^2(t)\mathrm{d}t\right]\right),
\end{aligned}$$

which is the required result. ■

Now we discuss some special cases of Theorem 7.

I. If $h_1(t)=1-t$ and $h_2(t)=t$, then we have a new result for preinvex functions:

Corollary 8 *Let $f: K_\eta=[a,a+\eta(b,a)]\subset\mathbb{R}\to\mathbb{R}$ be a preinvex function. Then*

$$\begin{aligned}
&\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x\\
\leq\ &\frac{1}{2}f\left(\frac{2a+\eta(b,a)}{2}\right)+\frac{1}{4f(\frac{2a+\eta(b,a)}{2})\eta(b,a)}\int_a^{a+\eta(b,a)} f^2(x)\mathrm{d}x+\frac{f^2(a)+f^2(b)+4f(a)f(b)}{24f(\frac{2a+\eta(b,a)}{2})}.
\end{aligned}$$

Theorem 9 *Let $h_1,h_2: J\subset\mathbb{R}\to\mathbb{R}$ be integrable and non-negative functions, where J is an interval such that $(0,1)\subset J$ and $h_1,h_2\neq 0$. Let $f: K_\eta=[a,a+\eta(b,a)]\subset\mathbb{R}\to\mathbb{R}$ be a non-negative (h_1,h_2)-preinvex function. Then*

$$\begin{aligned}
&2\int_a^{a+\eta(b,a)}\int_a^{a+\eta(b,a)}\int_0^1 f(x+t\eta(y,x))[h_1(t)f(x)+h_2(t)f(y)]\mathrm{d}t\mathrm{d}y\mathrm{d}x\\
\leq\ &\int_a^{a+\eta(b,a)}\int_a^{a+\eta(b,a)}\int_0^1 f^2(x+t\eta(y,x))\mathrm{d}t\mathrm{d}y\mathrm{d}x\\
&+\eta(b,a)\int_a^{a+\eta(b,a)} f^2(x)\mathrm{d}x\left(\int_0^1 h_1^2(t)\mathrm{d}t+\int_0^1 h_2^2(t)\mathrm{d}t\right)\\
&+2\eta^2(b,a)\left(f(a)\int_0^1 h_1(t)\mathrm{d}t+f(b)\int_0^1 h_2(t)\mathrm{d}t\right)^2\left(\int_0^1 h_1(t)h_2(t)\mathrm{d}t\right).
\end{aligned}$$

Proof. Since f is (h_1,h_2)-preinvex function on $[a,a+\eta(b,a)]$, we have

$$f(x+t\eta(y,x))\leq h_1(t)f(x)+h_2(t)f(y),\quad \forall x,x+\eta(y,x)\in K_\eta,\ t\in[0,1].$$

Using $G(x,y)\leq A(x,y)$, we have

$$\begin{aligned}
&4f(x+t\eta(y,x))[h_1(t)f(x)+h_2(t)f(y)]\\
\leq\ &\big(f(x+t\eta(y,x))+[h_1(t)f(x)+h_2(t)f(y)]\big)^2,
\end{aligned}$$

from which we obtain

$$\begin{aligned}
& 2f(x+t\eta(y,x))[h_1(t)f(x)+h_2(t)f(y)] \\
\leq\ & f^2(x+t\eta(y,x))+h_1^2(t)f^2(x)+h_2^2(t)f^2(y)+2h_1(t)h_2(t)f(x)f(y).
\end{aligned}$$

Integrating the above inequality over the interval $[0,1]$, with respect to t, we have

$$\begin{aligned}
& 2\int_0^1 f(x+t\eta(y,x))[h_1(t)f(x)+h_2(t)f(y)]\mathrm{d}t \\
\leq\ & \int_0^1 f^2(x+t\eta(y,x))\mathrm{d}t+f^2(x)\int_0^1 h_1^2(t)\mathrm{d}t+f^2(y)\int_0^1 h_2^2(t)\mathrm{d}t \\
& +2f(x)f(y)\int_0^1 h_1(t)h_2(t)\mathrm{d}t.
\end{aligned}$$

Again integrating the above inequality with respect to y on $[a,a+\eta(b,a)]$ and x on $[a,a+\eta(b,a)]$, we have

$$\begin{aligned}
& 2\int_a^{a+\eta(b,a)}\int_a^{a+\eta(b,a)}\int_0^1 f(x+t\eta(y,x))[h_1(t)f(x)+h_2(t)f(y)]\mathrm{d}t\mathrm{d}y\mathrm{d}x \\
\leq\ & \int_a^{a+\eta(b,a)}\int_a^{a+\eta(b,a)}\int_0^1 f^2(x+t\eta(y,x))\mathrm{d}t\mathrm{d}y\mathrm{d}x \\
& +\eta(b,a)\int_a^{a+\eta(b,a)} f^2(x)\mathrm{d}x\bigg(\int_0^1 h_1^2(t)\mathrm{d}t+\int_0^1 h_2^2(t)\mathrm{d}t\bigg) \\
& +2\bigg(\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x\bigg)^2\bigg(\int_0^1 h_1(t)h_2(t)\mathrm{d}t\bigg).
\end{aligned}$$

Using the Hermite-Hadamard inequality (3) for (h_1,h_2) preinvex function, we have

$$\begin{aligned}
& 2\int_a^{a+\eta(b,a)}\int_a^{a+\eta(b,a)}\int_0^1 f(x+t\eta(y,x))[h_1(t)f(x)+h_2(t)f(y)]\mathrm{d}t\mathrm{d}y\mathrm{d}x \\
\leq\ & \int_a^{a+\eta(b,a)}\int_a^{a+\eta(b,a)}\int_0^1 f^2(x+t\eta(y,x))\mathrm{d}t\mathrm{d}y\mathrm{d}x \\
& +\eta(b,a)\int_a^{a+\eta(b,a)} f^2(x)\mathrm{d}x\bigg(\int_0^1 h_1^2(t)\mathrm{d}t+\int_0^1 h_2^2(t)\mathrm{d}t\bigg) \\
& +2\eta^2(b,a)\bigg(f(a)\int_0^1 h_1(t)\mathrm{d}t+f(b)\int_0^1 h_2(t)\mathrm{d}t\bigg)^2\bigg(\int_0^1 h_1(t)h_2(t)\mathrm{d}t\bigg),
\end{aligned}$$

which is the required result. ■

Now we discuss some special cases of Theorem 9.

I. If $h_1(t)=1-t$ and $h_2(t)=t$, then we have a new result for preinvex functions.

Corollary 10 *Let $f: K_\eta=[a,a+\eta(b,a)]\subset\mathbb{R}\to\mathbb{R}$ be a preinvex function. Then*

$$\begin{aligned}
& 2\int_a^{a+\eta(b,a)}\int_a^{a+\eta(b,a)}\int_0^1 f(x+t\eta(y,x))[(1-t)f(x)+tf(y)]\mathrm{d}t\mathrm{d}y\mathrm{d}x \\
\leq\ & \int_a^{a+\eta(b,a)}\int_a^{a+\eta(b,a)}\int_0^1 f^2(x+t\eta(y,x))\mathrm{d}t\mathrm{d}y\mathrm{d}x \\
& +\frac{2\eta(b,a)}{3}\int_a^{a+\eta(b,a)} f^2(x)\mathrm{d}x+12\eta^2(b,a)\frac{(f(a)+f(b))^2}{12}.
\end{aligned}$$

Theorem 11 *Let $h_1, h_2 : J \subset \mathbb{R} \to \mathbb{R}$ be integrable and non-negative functions, where J is an interval such that $(0,1) \subset J$ and $h_1, h_2 \neq 0$. Let $f, g : K_\eta = [a, a+\eta(b,a)] \subset \mathbb{R} \to \mathbb{R}$ be a non-negative (h_1, h_2)-preinvex functions. Then*

$$\begin{aligned}&\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}[f(x)+f(x)g(x)+g(x)]\mathrm{d}t\\ \leq\ & [f(a)g(a)]\int_0^1 h_1^2(t)\mathrm{d}t+[f(b)g(b)]\int_0^1 h_2^2(t)\mathrm{d}t+2[f(a)g(b)+g(a)f(b)]\int_0^1 h_1(t)h_2(t)\mathrm{d}t+1.\end{aligned}$$

Proof. Since f, g are (h_1, h_2)-preinvex functions on $[a, a+\eta(b,a)]$, we have

$$\begin{aligned}&f(a+t\eta(b,a)) \leq h_1(t)f(a)+h_2(t)f(b),\\ &g(a+t\eta(b,a)) \leq h_1(t)g(a)+h_2(t)g(b), \quad \forall a,b \in K_\eta,\ t\in[0,1].\end{aligned}$$

Using the inequality $xy+x+y \leq x^2+y^2+1$, $x, y \in \mathbb{R}$, we have

$$\begin{aligned}&f(a+t\eta(b,a))g(a+t\eta(b,a))+f(a+t\eta(b,a))+g(a+t\eta(b,a))\\ \leq\ & f^2(a+t\eta(b,a))+g^2(a+t\eta(b,a))+1.\end{aligned}$$

Now using the (h_1, h_2)-preinvexity of the functions f, g, we have

$$\begin{aligned}&f(a+t\eta(b,a))g(a+t\eta(b,a))+f(a+t\eta(b,a))+g(a+t\eta(b,a))\\ \leq\ & [f(a)g(a)]h_1^2(t)+[f(b)g(b)]h_2^2(t)+2[f(a)g(b)+g(a)f(b)]h_1(t)h_2(t)+1.\end{aligned}$$

Integrating the above inequality over the interval $[0,1]$, with respect to t, we have

$$\begin{aligned}&\int_0^1 f(a+t\eta(b,a))g(a+t\eta(b,a))\mathrm{d}t+\int_0^1 f(a+t\eta(b,a))\mathrm{d}t+\int_0^1 g(a+t\eta(b,a))\mathrm{d}t\\ \leq\ & [f(a)g(a)]\int_0^1 h_1^2(t)\mathrm{d}t+[f(b)g(b)]\int_0^1 h_2^2(t)\mathrm{d}t+2[f(a)g(b)+g(a)f(b)]\int_0^1 h_1(t)h_2(t)\mathrm{d}t+\int_0^1 1\mathrm{d}t.\end{aligned}$$

Making the change of variable, we have

$$\begin{aligned}&\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}[f(x)+f(x)g(x)+g(x)]\mathrm{d}t\\ \leq\ & [f(a)g(a)]\int_0^1 h_1^2(t)\mathrm{d}t+[f(b)g(b)]\int_0^1 h_2^2(t)\mathrm{d}t+2[f(a)g(b)+g(a)f(b)]\int_0^1 h_1(t)h_2(t)\mathrm{d}t+1,\end{aligned}$$

which is the required result. ■

Now we discuss some special cases of Theorem 11.

I. If $h_1(t) = 1-t$ and $h_2(t) = t$, then we have a new result for preinvex functions.

Corollary 12 *Let $f, g : K_\eta = [a, a+\eta(b,a)] \subset \mathbb{R} \to \mathbb{R}$ be preinvex functions. Then*

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}[f(x)+f(x)g(x)+g(x)]\mathrm{d}t \leq \frac{f^(a)g(a)+f(b)g(b)}{3}+\frac{f(a)g(b)+g(a)f(b)}{6}+1.$$

Theorem 13 *Let $h_1, h_2 : J \subset \mathbb{R} \to \mathbb{R}$ be integrable and non-negative functions, where J is an interval such that $(0,1) \subset J$ and $h_1, h_2 \neq 0$. Let $f, g : K_\eta = [a, a+\eta(b,a)] \subset \mathbb{R} \to \mathbb{R}$ be a non-negative and (h_1, h_2)-preinvex* *functions. If* $fg \in L[a,b]$, *then*

$$\left(\frac{g(a)}{\eta(b,a)}\int_a^{a+\eta(b,a)} h_1\left(\frac{x-a}{\eta(b,a)}\right)f(x)\mathrm{d}x + \frac{g(b)}{\eta(b,a)}\int_a^{a+\eta(b,a)} h_2\left(\frac{x-b}{\eta(b,a)}\right)f(x)\mathrm{d}x\right)$$
$$\times\left(\frac{f(a)}{\eta(b,a)}\int_a^{a+\eta(b,a)} h_1\left(\frac{x-a}{\eta(b,a)}\right)g(x)\mathrm{d}x + \frac{f(b)}{\eta(b,a)}\int_a^{a+\eta(b,a)} h_2\left(\frac{x-b}{\eta(b,a)}\right)g(x)\mathrm{d}x\right)$$
$$\leq f(a)g(a)\int_0^1 h_1^2(t)\mathrm{d}t + f(b)g(b)\int_0^1 h_2^2(t)\mathrm{d}t + [f(a)g(b)+f(b)g(a)]\int_0^1 h_1(t)h_2(t)\mathrm{d}t$$
$$+\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)\mathrm{d}x.$$

Proof. Since $f,\ g$ be (h_1,h_2)-preinvex functions, we have

$$f(a+t\eta(b,a)) \leq h_1(t)f(a)+h_2(t)f(b),$$
$$g(a+t\eta(b,a)) \leq h_1(t)g(a)+h_2(t)g(b), \quad \forall a,b\in K_\eta, t\in[0,1].$$

Now, using $\langle x_1-x_2, x_3-x_4\rangle \geq 0, (x_1,x_2,x_3,x_4\in\mathbb{R})$ and $x_1<x_2,\ x_3<x_4$, we have

$$f(a+t\eta(b,a))[h_1(t)g(a)+h_2(t)g(b)] + g(a+t\eta(b,a))[h_1(t)f(a)+h_2(t)f(b)]$$
$$\leq [h_1(t)f(a)+h_2(t)f(b)][h_1(t)g(a)+h_2(t)g(b)] + f(a+t\eta(b,a))g(a+t\eta(b,a))$$

and we obtain

$$g(a)h_1(t)f(a+t\eta(b,a)) + g(b)h_2(t)f(a+t\eta(b,a)) + f(a)h_1(t)g(a+t\eta(b,a)) + f(b)h_2(t)g(a+t\eta(b,a))$$
$$\leq h_1(t)^2f(a)g(a) + t^2f(b)g(b) + h_1(t)h_2(t)[f(a)g(b)+f(b)g(a)] + f(a+t\eta(b,a))g(a+t\eta(b,a)).$$

Integrating over $[0,1]$, and making the change of variable, we obtain

$$\left(\frac{g(a)}{\eta(b,a)}\int_a^{a+\eta(b,a)} h_1\left(\frac{x-a}{\eta(b,a)}\right)f(x)\mathrm{d}x + \frac{g(b)}{\eta(b,a)}\int_a^{a+\eta(b,a)} h_2\left(\frac{x-b}{\eta(b,a)}\right)f(x)\mathrm{d}x\right)$$
$$\times\left(\frac{f(a)}{\eta(b,a)}\int_a^{a+\eta(b,a)} h_1\left(\frac{x-a}{\eta(b,a)}\right)g(x)\mathrm{d}x + \frac{f(b)}{\eta(b,a)}\int_a^{a+\eta(b,a)} h_2\left(\frac{x-b}{\eta(b,a)}\right)g(x)\mathrm{d}x\right)$$
$$\leq f(a)g(a)\int_0^1 h_1^2(t)\mathrm{d}t + f(b)g(b)\int_0^1 h_2^2(t)\mathrm{d}t + [f(a)g(b)+f(b)g(a)]\int_0^1 h_1(t)h_2(t)\mathrm{d}t$$
$$+\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)\mathrm{d}x,$$

which is the required result. ■

Now we discuss some special cases of Theorem 13.

I. If $h_1(t)=1-t$ and $h_2(t)=t$, then we have a new result for preinvex functions.

Corollary 14 *Let* $f,g: K_\eta=[a,a+\eta(b,a)]\subset\mathbb{R}\to\mathbb{R}$ *be preinvex functions. Then*

$$\left(\frac{g(a)}{\eta(b,a)}\int_a^{a+\eta(b,a)} \left(\frac{x-a}{\eta(b,a)}\right)f(x)\mathrm{d}x + \frac{g(b)}{\eta(b,a)}\int_a^{a+\eta(b,a)} \left(\frac{x-b}{\eta(b,a)}\right)f(x)\mathrm{d}x\right)$$
$$\times\left(\frac{f(a)}{\eta(b,a)}\int_a^{a+\eta(b,a)} \left(\frac{x-a}{\eta(b,a)}\right)g(x)\mathrm{d}x + \frac{f(b)}{\eta(b,a)}\int_a^{a+\eta(b,a)} \left(\frac{x-b}{\eta(b,a)}\right)g(x)\mathrm{d}x\right)$$
$$\leq \frac{f(a)g(a)+f(b)g(b)}{3} + \frac{f(a)g(b)+f(b)g(a)}{6} + \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)\mathrm{d}x.$$

Now we establish some integral inequalities related to (h_1, h_2)-log-preinvex functions.

Theorem 15 *Let $h_1, h_2 : J \subset \mathbb{R} \to \mathbb{R}$ be integrable and non-negative functions, where J is an interval such that $(0,1) \subset J$ and $h_1, h_2 \neq 0$. Let $f : K_\eta = [a, a+\eta(b,a)] \subset \mathbb{R} \to \mathbb{R}$ be a non-negative (h_1, h_2)-log-preinvex function. If $f \in L[a,b]$, then*

$$\begin{aligned}\frac{1}{\left[h_1\left(\frac{1}{2}\right)+h_2\left(\frac{1}{2}\right)\right]}\log f\left(\frac{2a+\eta(b,a)}{2}\right) &\leq \int_a^{a+\eta(b,a)} \log f(x)\mathrm{d}x \\ &= \log f(a)\int_0^1 h_1(t)\mathrm{d}t + \log f(b)\int_0^1 h_2(t)\mathrm{d}t.\end{aligned}$$

Proof. Let f be (h_1, h_2)-log-preinvex function. With $t = \frac{1}{2}$ and letting $x = a + t\eta(b,a)$ and $y = a + (1-t)\eta(b,a)$ in (3), we have

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \left[f(a+t\eta(b,a))\right]^{h_1\left(\frac{1}{2}\right)}\left[f(a+(1-t)\eta(b,a))\right]^{h_2\left(\frac{1}{2}\right)}.$$

From this it follows that

$$\log f\left(\frac{2a+\eta(b,a)}{2}\right) \leq h_1\left(\frac{1}{2}\right)\log f(a+t\eta(b,a)) + h_2\left(\frac{1}{2}\right)\log f(a+(1-t)\eta(b,a)).$$

Integrating above inequality with respect to t on $[0,1]$, we have

$$\begin{aligned}\log f\left(\frac{2a+\eta(b,a)}{2}\right) &\leq h_1\left(\frac{1}{2}\right)\int_0^1 \log f(a+t\eta(b,a))\mathrm{d}t + h_2\left(\frac{1}{2}\right)\int_0^1 \log f(a+(1-t)\eta(b,a))\mathrm{d}t \\ &= \left[h_1\left(\frac{1}{2}\right)+h_2\left(\frac{1}{2}\right)\right]\int_a^{a+\eta(b,a)} \log f(x)\mathrm{d}x.\end{aligned}$$

This implies

$$\begin{aligned}\frac{1}{\left[h_1\left(\frac{1}{2}\right)+h_2\left(\frac{1}{2}\right)\right]}\log f\left(\frac{2a+\eta(b,a)}{2}\right) &\leq \int_a^{a+\eta(b,a)} \log f(x)\mathrm{d}x \\ &= \log f(a)\int_0^1 h_1(t)\mathrm{d}t + \log f(b)\int_0^1 h_2(t)\mathrm{d}t,\end{aligned}$$

which is the required result. ∎

Now we discuss some special cases of Theorem 15.

I. If $h_1(t) = 1-t$ and $h_2(t) = t$, then we have a result for log-preinvex functions.

Corollary 16 *Let $f, g : K_\eta = [a, a+\eta(b,a)] \subset \mathbb{R} \to \mathbb{R}$ be preinvex functions. Then*

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \exp^{\int_a^{a+\eta(b,a)} \log f(x)\mathrm{d}x} = \sqrt{f(a)f(b)}.$$

Theorem 17 *Let $h_1, h_2 : J \subset \mathbb{R} \to \mathbb{R}$ be integrable and non-negative functions, where J is an interval such that $(0,1) \subset J$ and $h_1, h_2 \neq 0$. Let $f : K_\eta = [a, a+\eta(b,a)] \subset \mathbb{R} \to \mathbb{R}$ be a non-negative increasing (h_1, h_2)-log-preinvex function. If $f \in L[a,b]$, then*

$$\frac{8}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x \int_0^1 [f(a)]^{h_1(t)}[f(b)]^{h_2(t)}\mathrm{d}t \leq \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f^4(x)\mathrm{d}x + \int_0^1 [f(a)]^{4h_1(t)}[f(b)]^{4h_2(t)}\mathrm{d}t.$$

Proof. Let f be (h_1, h_2)-log-preinvex function. Then

$$f(a + t\eta(b,a)) \leq [f(a)]^{h_1(t)}[f(b)]^{h_2(t)}.$$

Using the inequality $8xy \leq x^4 + y^4 + 8$ for all $x, y \in \mathbb{R}$, we have

$$8f(a + t\eta(b,a)) \cdot [f(a)]^{h_1(t)}[f(b)]^{h_2(t)} \leq f^4(a + t\eta(b,a)) + [f(a)]^{4h_1(t)}[f(b)]^{4h_2(t)}.$$

Integrating the above inequality, we obtain

$$\begin{aligned} & 8\int_0^1 f(a + t\eta(b,a))\mathrm{d}t \int_0^1 [f(a)]^{h_1(t)}[f(b)]^{h_2(t)}\mathrm{d}t \\ \leq\ & 8\int_0^1 f(a + t\eta(b,a)) \cdot [f(a)]^{h_1(t)}[f(b)]^{h_2(t)}\mathrm{d}t \\ \leq\ & \int_0^1 f^4(a + t\eta(b,a))\mathrm{d}t + \int_0^1 [f(a)]^{4h_1(t)}[f(b)]^{4h_2(t)}\mathrm{d}t. \end{aligned}$$

Making the change of variable $a + t\eta(b,a) = x$, we obtain the required result. ■

Now we discuss a special case of Theorem 17.

I. If $h_1(t) = 1 - t$ and $h_2(t) = t$, then we have a result for log-preinvex functions.

Corollary 18 *Let $f, g : K_\eta = [a, a + \eta(b,a)] \subset \mathbb{R} \to \mathbb{R}$ be preinvex functions. Then*

$$\begin{aligned} & \frac{8}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\mathrm{d}x L(f(a), f(b)) \\ \leq\ & \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f^4(x)\mathrm{d}x + \frac{1}{8}\frac{f^2(a) + f^2(b)}{2}\frac{f(a) + f(b)}{2}\frac{f(b) - f(a)}{\log f(b) - \log f(a)} + 1. \end{aligned}$$

Conclusion

In this paper, we have introduced and investigated some new classes of preinvex functions involving two non-negative functions. It has been shown that generalized preinvex functions include several new and known classes of convex and preinvex functions as special cases. Several new integral inequalities for generalized preinvex functions have been established. Some applications of the obtained results are discussed. The idea and techniques of this paper may be starting point for further research.

Acknowledgment. The authors would like to thank Rector, COMSATS Institute of Information Technology, Pakistan, for providing excellent research and academic environment.

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