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Some New Hermite-Hadamard Type Inequalities Via k-Fractional Integrals Pertaining Differentiable Generalized Relative Semi-$\mathbf{m}$-$(r;h_1,h_2)$-Preinvex Mappings And Their Applications*

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Abstract

In this article, we first presented a new identity concerning differentiable mappings defined on m-invex set via k-fractional integrals. By using the notion of generalized relative semi-$\mathbf{m}$-$(r;h_1,h_2)$-preinvexity and the obtained identity as an auxiliary result, some new estimates with respect to Hermite-Hadamard type inequalities via k-fractional integrals are established. Some new special cases can be deduced from main results of the article. At the end, some applications to special means for positive real numbers are provided as well.

1 Introduction

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R}=(-\infty,+\infty)$ and I° to denote the interior of I. For any subset $K\subseteq\mathbb{R}^n$, K° is used to denote the interior of K. $\mathbb{R}^n$ is used to denote n-dimensional vector space.

The following double inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1 *Let $f:I\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ be a convex function on I and $a,b\in I$ with $a<b$. Then the following double inequality holds:*

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{2}. \tag{1}$$

In recent years, various generalizations, extensions and variants of such inequalities have been obtained. For other recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions interested readers are referred to [[1]-[7], [10]-[18], [20]-[28], [30], [33]-[35], [38]-[41], [44]-[48]]. Let us recall some special functions and evoke some basic definitions as follows.

Definition 1 *The Euler beta function is defined for $a,b>0$ as*

$$\beta(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}dt=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

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Definition 2 *The hypergeometric function* ${}_2F_1(a,b;c;z)$ *is defined by*

$$ {}_2F_1(a,b;c;z)=\frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt $$

for $c>b>0$ *and* $|z|<1$, *where* $\beta(x,y)$ *is the Euler beta function for all* $x,y>0$.

Definition 3 ([30]) *Let* $f\in L_1[a,b]$. *The Riemann-Liouville integrals* $J^{\alpha}_{a+}f$ *and* $J^{\alpha}_{b-}f$ *of order* $\alpha>0$ *with* $a\geq 0$ *are defined by*

$$ J^{\alpha}_{a+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt,\quad x>a $$

and

$$ J^{\alpha}_{b-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt,\quad b>x, $$

where $\Gamma(\alpha)=\int_0^{+\infty}e^{-u}u^{\alpha-1}du$. *Here* $J^0_{a+}f(x)=J^0_{b-}f(x)=f(x)$. *Note that* $\alpha=1$, *the fractional integral reduces to the classical integral.*

Due to the wide application of Riemann-Liouville fractional integrals, many authors extended to research Riemann-Liouville fractional inequalities via different classes of convex mappings: for generalizations, variations and new inequalities for them [32, 36].

In 2013, Sarikaya et al. [38] established the following theorem by utilizing Riemann-Liouville fractional integrals.

Theorem 2 *Let* $f:[a,b]\longrightarrow\mathbb{R}$ *be a positive function along with* $0\leq a<b$ *and let* $f\in L_1[a,b]$. *Suppose that* f *is a convex function on* $[a,b]$. *Then for* $\alpha>0$, *the following double inequalities for fractional integrals hold:*

$$ f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(1+\alpha)}{2(b-a)^{\alpha}}\Big[J^{\alpha}_{a+}f(b)+J^{\alpha}_{b-}f(a)\Big]\leq\frac{f(a)+f(b)}{2}. \tag{2} $$

For $\alpha=1$, *the double inequality (2) is reduced to (1).*

Definition 4 *For* $k\in\mathbb{R}^+$ *and* $x\in\mathbb{C}$, *the* k*-gamma function is defined by*

$$ \Gamma_k(x)=\lim_{n\longrightarrow\infty}\frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}. \tag{3} $$

Its integral representation is given by

$$ \Gamma_k(\alpha)=\int_0^{\infty}t^{\alpha-1}e^{-\frac{t^k}{k}}dt. \tag{4} $$

One can note that

$$ \Gamma_k(\alpha+k)=\alpha\Gamma_k(\alpha). $$

For $k=1$, (4) gives integral representation of gamma function.

Definition 5 ([32]) *Let* $f\in L_1[a,b]$. *Then* k*-fractional integrals of order* $\alpha,k>0$ *with* $a\geq 0$ *are defined as*

$$ I^{\alpha,k}_{a+}f(x)=\frac{1}{k\Gamma_k(\alpha)}\int_a^x (x-t)^{\frac{\alpha}{k}-1}f(t)dt,\quad x>a $$

and

$$ I^{\alpha,k}_{b-}f(x)=\frac{1}{k\Gamma_k(\alpha)}\int_x^b (t-x)^{\frac{\alpha}{k}-1}f(t)dt,\quad b>x. \tag{5} $$

For $k=1$, k-fractional integrals give Riemann-Liouville integrals.

Definition 6 ([8]) *A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y, x) \in K$ for every $x, y \in K$ and $t \in [0, 1]$.*

Definition 7 ([18]) *A set $K \subseteq \mathbb{R}^n$ is named as m-invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0, 1]$, if $mx + t\eta(y, mx) \in K$ holds for each $x, y \in K$ and any $t \in [0, 1]$.*

Remark 1 *In Definition 7, under certain conditions, the mapping $\eta(y, mx)$ could reduce to $\eta(y, x)$. For example when $m = 1$, then the m-invex set degenerates to an invex set on K.*

Definition 8 ([17]) *A non-negative function $f : I \subseteq \mathbb{R} \longrightarrow [0, +\infty)$ is said to be P-function, if*

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I,\ t \in [0, 1].$$

Definition 9 ([31]) *Let $h : [0, 1] \longrightarrow \mathbb{R}$ be a non-negative function and $h \neq 0$. The function f on the invex set K is said to be h-preinvex with respect to η, if*

$$f\big(x + t\eta(y, x)\big) \leq h(1-t)f(x) + h(t)f(y)$$

for each $x, y \in K$ and $t \in [0, 1]$ where $f(\cdot) > 0$.

Clearly, when putting $h(t) = t$ in Definition 9, f becomes a preinvex function [37]. If the mapping $\eta(y, x) = y - x$ in Definition 9, then the non-negative function f reduces to h-convex mappings [43].

Definition 10 ([42]) *Let $f : K \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a non-negative function, a function $f : K \longrightarrow \mathbb{R}$ is said to be a tgs-convex function on K if the inequality*

$$f\big((1-t)x + ty\big) \leq t(1-t)[f(x) + f(y)]$$

holds for all $x, y \in K$ and $t \in (0, 1)$.

Definition 11 ([29]) *A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be MT-convex functions, if it is non-negative and $\forall\, x, y \in I$ and $t \in (0, 1)$ satisfies the subsequent inequality:*

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{6}$$

Definition 12 ([35]) *A function: $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be m-MT-convex, if f is positive and for $\forall\, x, y \in I$ and $t \in (0, 1)$, among $m \in [0, 1]$, satisfies the following inequality*

$$f\big(tx + m(1-t)y\big) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{7}$$

Definition 13 ([36]) *Let $K \subseteq \mathbb{R}$ be an open m-invex set respecting $\eta : K \times K \longrightarrow \mathbb{R}$ and $h_1, h_2 : [0, 1] \longrightarrow [0, +\infty)$. A function $f : K \longrightarrow \mathbb{R}$ is said to be generalized (m, h_1, h_2)-preinvex, if*

$$f\big(mx + t\eta(y, mx)\big) \leq mh_1(t)f(x) + h_2(t)f(y) \tag{8}$$

is valid for all $x, y \in K$ and $t \in [0, 1]$, for some fixed $m \in (0, 1]$.

Motivated by the above literature, the main objective of this article is to establish in Section 2 some new estimates on Hermite-Hadamard type inequalities via k-fractional integrals associated with generalized relative semi-$\mathbf{m}$-$(r; h_1, h_2)$-preinvex mappings. It is pointed out that some new special cases will be deduced from main results of the article. In Section 3, some applications to special means for positive real numbers will be given.

2 Main Results

The following definitions will be used in this section.

Definition 14 *Let $\mathbf{m} : [0,1] \longrightarrow (0,1]$ be a function. A set $K \subseteq \mathbb{R}^n$ is named as $\mathbf{m}$-invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $\mathbf{m}(t)x + \xi\eta(y, \mathbf{m}(t)x) \in K$ holds for each $x, y \in K$ and any $t, \xi \in [0,1]$.*

Remark 2 *In Definition 14, under certain conditions, the mapping $\eta(y, \mathbf{m}(t)x)$ for any $t, \xi \in [0,1]$ could reduce to $\eta(y, mx)$. For example when $\mathbf{m}(t) = m$ for all $t \in [0,1]$, then the $\mathbf{m}$-invex set degenerates to an m-invex set on K.*

We next introduce the notion of generalized relative semi-$\mathbf{m}$-$(r; h_1, h_2)$-preinvex mappings.

Definition 15 *Let $K \subseteq \mathbb{R}$ be an open $\mathbf{m}$-invex set with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}$. Suppose $h_1, h_2 : [0,1] \longrightarrow [0,+\infty)$, $\varphi : I \longrightarrow K$ are continuous functions and $\mathbf{m} : [0,1] \longrightarrow (0,1]$. A mapping $f : K \longrightarrow (0,+\infty)$ is said to be generalized relative semi-$\mathbf{m}$-$(r; h_1, h_2)$-preinvex, if*

$$f\big(\mathbf{m}(t)\varphi(x) + \xi\eta(\varphi(y), \mathbf{m}(t)\varphi(x))\big) \leq \Big[\mathbf{m}(t)h_1(\xi)f^r(x) + h_2(\xi)f^r(y)\Big]^{\frac{1}{r}} \tag{9}$$

holds for all $x, y \in I$ and $t, \xi \in [0,1]$, where $r \neq 0$.

Remark 3 *In Definition 15, if we choose $\mathbf{m} = m = r = 1$, these definition reduces to the definition considered by Noor in [33] and Preda et al. in [19].*

Remark 4 *In Definition 15, if we choose $\mathbf{m} = m = r = 1$ and $\varphi(x) = x$, then we get Definition 13.*

Remark 5 *For $r = 1$, let us discuss some special cases in Definition 15 as follows.*

(I) If taking $h_1(t) = h_2(t) = 1$, then we get generalized relative semi-$(\mathbf{m}, P)$-preinvex mappings.

(II) If taking $h_1(t) = (1-t)^s$, $h_2(t) = t^s$ for $s \in (0,1]$, then we get generalized relative semi-$(\mathbf{m}, s)$-Breckner-preinvex mappings.

(III) If taking $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ for $s \in (0,1]$, then we get generalized relative semi-$(\mathbf{m}, s)$-Godunova-Levin-Dragomir-preinvex mappings.

(IV) If taking $h_1(t) = h(1-t)$, $h_2(t) = h(t)$, then we get generalized relative semi-$(\mathbf{m}, h)$-preinvex mappings.

(V) If taking $h_1(t) = h_2(t) = t(1-t)$, then we get generalized relative semi-$(\mathbf{m}, tgs)$-preinvex mappings.

(VI) If taking $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$, then we get generalized relative semi-$\mathbf{m}$-MT-preinvex mappings.

It is worth to mention here that to the best of our knowledge all the special cases discussed above are new in the literature. For the simplicities of notations, let

$$\delta(\alpha, \xi) = \int_0^1 |t^\alpha - \xi| dt, \quad \varrho(\alpha, \xi, p) = \int_0^1 |t^\alpha - \xi|^p dt. \tag{10}$$

Lemma 3 *For $0 \leq \xi \leq 1$, we have*

(a)

$$\delta(\alpha, \xi) := \begin{cases} \frac{1}{\alpha+1}, & \xi = 0; \\ \frac{2\alpha\xi^{1+\frac{1}{\alpha}}+1}{\alpha+1} - \xi, & 0 < \xi < 1; \\ \frac{\alpha}{\alpha+1}, & \xi = 1. \end{cases}$$

(b)

$$\varrho(\alpha,\xi,p):=\begin{cases}\frac{1}{p\alpha+1}, & \xi=0;\\ \frac{\xi^{p+\frac{1}{\alpha}}}{\alpha}\beta\left(\frac{1}{\alpha},p+1\right)+\frac{(1-\xi)^{p+1}}{\alpha(p+1)}\ {}_2F_1\left(1-\frac{1}{\alpha},1;p+2;1-\xi\right), & 0<\xi<1;\\ \frac{1}{\alpha}\beta\left(p+1,\frac{1}{\alpha}\right), & \xi=1.\end{cases}$$

Proof. These equalities follow from a straightfoward computation of definite integrals. ■

For establishing our main results regarding some new Hermite-Hadamard type integral inequalities associated with generalized relative semi-**m**-$(r;h_1,h_2)$-preinvexity via k-fractional integrals, we need the following lemma.

Lemma 4 *Let $\alpha,k>0$ and $r\geq 0$. Let $\varphi:I\longrightarrow K$ be a continuous function and $\mathbf{m}:[0,1]\longrightarrow(0,1]$. Assume that $f:K=[\mathbf{m}(t)\varphi(a),\mathbf{m}(t)\varphi(a)+\eta(\varphi(b),\mathbf{m}(t)\varphi(a))]\longrightarrow\mathbb{R}$ be a differentiable mapping on K° with respect to $\eta:K\times K\longrightarrow\mathbb{R}$ for $\eta(\varphi(b),\mathbf{m}(t)\varphi(a))>0$ and $\forall\, t\in[0,1]$. Then for any $\lambda,\mu\in[0,1]$, we have the following equality for k-fractional integrals:*

$$\begin{aligned}&\left(\frac{\eta(\varphi(b),\mathbf{m}(t)\varphi(a))}{r+1}\right)^{\frac{\alpha}{k}}\Bigg\{\lambda\Bigg[f\left(\mathbf{m}(t)\varphi(a)+\frac{r}{r+1}\eta(\varphi(b),\mathbf{m}(t)\varphi(a))\right)\\&-f\Big(\mathbf{m}(t)\varphi(a)+\eta(\varphi(b),\mathbf{m}(t)\varphi(a))\Big)\Bigg]\\&+\mu\left[f\left(\mathbf{m}(t)\varphi(a)+\frac{\eta(\varphi(b),\mathbf{m}(t)\varphi(a))}{r+1}\right)-f(\mathbf{m}(t)\varphi(a))\right]\\&+f(\mathbf{m}(t)\varphi(a))+f(\mathbf{m}(t)\varphi(a)+\eta(\varphi(b),\mathbf{m}(t)\varphi(a)))\Bigg\}\\&-\Gamma_k(\alpha+k)\times\Bigg[I^{\alpha,k}_{(\mathbf{m}(t)\varphi(a))^+}f\left(\mathbf{m}(t)\varphi(a)+\frac{\eta(\varphi(b),\mathbf{m}(t)\varphi(a))}{r+1}\right)\\&+I^{\alpha,k}_{(\mathbf{m}(t)\varphi(a)+\eta(\varphi(b),\mathbf{m}(t)\varphi(a)))^-}f\left(\mathbf{m}(t)\varphi(a)+\frac{r}{r+1}\eta(\varphi(b),\mathbf{m}(t)\varphi(a))\right)\Bigg]\\=\;&\left(\frac{\eta(\varphi(b),\mathbf{m}(t)\varphi(a))}{r+1}\right)^{\frac{\alpha}{k}+1}\Bigg\{\int_0^1\left(\xi^{\frac{\alpha}{k}}-\lambda\right)f'\left(\mathbf{m}(t)\varphi(a)+\left(\frac{r+\xi}{r+1}\right)\eta(\varphi(b),\mathbf{m}(t)\varphi(a))\right)d\xi\\&+\int_0^1\left(\mu-\xi^{\frac{\alpha}{k}}\right)f'\left(\mathbf{m}(t)\varphi(a)+\left(\frac{1-\xi}{r+1}\right)\eta(\varphi(b),\mathbf{m}(t)\varphi(a))\right)d\xi\Bigg\}.\end{aligned}\tag{11}$$

We denote

$$\begin{aligned}&I^{\alpha,k}_{f,\eta,\varphi,\mathbf{m}}(\lambda,\mu;r,a,b)\\:\;=&\left(\frac{\eta(\varphi(b),\mathbf{m}(t)\varphi(a))}{r+1}\right)^{\frac{\alpha}{k}+1}\Bigg\{\int_0^1\left(\xi^{\frac{\alpha}{k}}-\lambda\right)f'\left(\mathbf{m}(t)\varphi(a)+\left(\frac{r+\xi}{r+1}\right)\eta(\varphi(b),\mathbf{m}(t)\varphi(a))\right)d\xi\\&+\int_0^1\left(\mu-\xi^{\frac{\alpha}{k}}\right)f'\left(\mathbf{m}(t)\varphi(a)+\left(\frac{1-\xi}{r+1}\right)\eta(\varphi(b),\mathbf{m}(t)\varphi(a))\right)d\xi\Bigg\}.\end{aligned}\tag{12}$$

Proof. Integrating by parts (12), we get

$$\begin{aligned}
& I^{\alpha,k}_{f,\eta,\varphi,\mathbf{m}}(\lambda,\mu;r,a,b) \\
= & \left(\frac{\eta(\varphi(b),\mathbf{m}(t)\varphi(a))}{r+1}\right)^{\frac{\alpha}{k}+1} \left\{ \left[\left. \frac{(r+1)\left(\xi^{\frac{\alpha}{k}}-\lambda\right) f\left(\mathbf{m}(t)\varphi(a)+\left(\frac{r+\xi}{r+1}\right)\eta(\varphi(b),\mathbf{m}(t)\varphi(a))\right)}{\eta(\varphi(b),\mathbf{m}(t)\varphi(a))}\right|_0^1 \right.\right. \\
& \left. -\frac{\alpha(r+1)}{k\eta(\varphi(b),\mathbf{m}(t)\varphi(a))}\int_0^1 \xi^{\frac{\alpha}{k}-1} f\left(\mathbf{m}(t)\varphi(a)+\left(\frac{r+\xi}{r+1}\right)\eta(\varphi(b),\mathbf{m}(t)\varphi(a))\right)d\xi\right] \\
& +\left[\left.\frac{-(r+1)\left(\mu-\xi^{\frac{\alpha}{k}}\right) f\left(\mathbf{m}(t)\varphi(a)+\left(\frac{1-\xi}{r+1}\right)\eta(\varphi(b),\mathbf{m}(t)\varphi(a))\right)}{\eta(\varphi(b),\mathbf{m}(t)\varphi(a))}\right|_0^1\right. \\
& \left.\left. -\frac{\alpha(r+1)}{k\eta(\varphi(b),\mathbf{m}(t)\varphi(a))}\int_0^1 \xi^{\frac{\alpha}{k}-1} f\left(\mathbf{m}(t)\varphi(a)+\left(\frac{1-\xi}{r+1}\right)\eta(\varphi(b),\mathbf{m}(t)\varphi(a))\right)d\xi\right]\right\} \\
= & \left(\frac{\eta(\varphi(b),\mathbf{m}(t)\varphi(a))}{r+1}\right)^{\frac{\alpha}{k}} \\
& \times\left\{\lambda\left[f\left(\mathbf{m}(t)\varphi(a)+\frac{r}{r+1}\eta(\varphi(b),\mathbf{m}(t)\varphi(a))\right)-f(\mathbf{m}(t)\varphi(a)+\eta(\varphi(b),\mathbf{m}(t)\varphi(a)))\right]\right. \\
& +\mu\left[f\left(\mathbf{m}(t)\varphi(a)+\frac{\eta(\varphi(b),\mathbf{m}(t)\varphi(a))}{r+1}\right)-f(\mathbf{m}(t)\varphi(a))\right] \\
& \left. +f(\mathbf{m}(t)\varphi(a))+f(\mathbf{m}(t)\varphi(a)+\eta(\varphi(b),\mathbf{m}(t)\varphi(a)))\right\} \\
& -\Gamma_k(\alpha+k)\times\left[I^{\alpha,k}_{(\mathbf{m}(t)\varphi(a))^+} f\left(\mathbf{m}(t)\varphi(a)+\frac{\eta(\varphi(b),\mathbf{m}(t)\varphi(a))}{r+1}\right)\right. \\
& \left. +I^{\alpha,k}_{(\mathbf{m}(t)\varphi(a)+\eta(\varphi(b),\mathbf{m}(t)\varphi(a)))^-} f\left(\mathbf{m}(t)\varphi(a)+\frac{r}{r+1}\eta(\varphi(b),\mathbf{m}(t)\varphi(a))\right)\right].
\end{aligned}$$

This completes the proof of the lemma. ∎

Remark 6 *If we choose* $k=1$, $\eta(\varphi(b),\mathbf{m}(t)\varphi(a))=\varphi(b)-\mathbf{m}(t)\varphi(a)$ *and* $\mathbf{m}(t)\equiv 1$ *for all* $t\in[0,1]$, *we get the following equality for fractional integrals:*

$$\begin{aligned}
& \left(\frac{\varphi(b)-\varphi(a)}{r+1}\right)^{\alpha}\left\{\lambda\left[f\left(\varphi(a)+\frac{r}{r+1}(\varphi(b)-\varphi(a))\right)-f(\varphi(b))\right]\right. \\
& \left. +\mu\left[f\left(\varphi(a)+\frac{\varphi(b)-\varphi(a)}{r+1}\right)-f(\varphi(a))\right]+\left[f(\varphi(a))+f(\varphi(b))\right]\right\} \\
& -\Gamma(\alpha+1)\times\left[J^{\alpha}_{(\varphi(a))^+} f\left(\varphi(a)+\frac{\varphi(b)-\varphi(a)}{r+1}\right)+J^{\alpha}_{(\varphi(b))^-} f\left(\varphi(a)+\frac{r}{r+1}(\varphi(b)-\varphi(a))\right)\right] \\
= & \left(\frac{\varphi(b)-\varphi(a)}{r+1}\right)^{\alpha+1}\times\left\{\int_0^1(\xi^{\alpha}-\lambda)f'\left(\varphi(a)+\left(\frac{r+\xi}{r+1}\right)(\varphi(b)-\varphi(a))\right)d\xi\right. \\
& \left. +\int_0^1(\mu-\xi^{\alpha})f'\left(\varphi(a)+\left(\frac{1-\xi}{r+1}\right)(\varphi(b)-\varphi(a))\right)d\xi\right\}. \qquad (13)
\end{aligned}$$

Using Lemmas 3 and 4, we now state the following theorems for the corresponding version for power of first derivative.

Theorem 5 *Let $\alpha, k > 0$ and $0 < r \leq 1$. Suppose $h_1, h_2 : [0,1] \longrightarrow [0,+\infty)$, $\varphi : I \longrightarrow K$ are continuous functions and $\mathbf{m} : [0,1] \longrightarrow (0,1]$. Let $K = [\mathbf{m}(t)\varphi(a), \mathbf{m}(t)\varphi(a) + \eta(\varphi(b), \mathbf{m}(t)\varphi(a))] \subseteq \mathbb{R}$ be an open $\mathbf{m}$-invex subset with respect to $\eta : K \times K \longrightarrow \mathbb{R}$ and let $\eta(\varphi(b), \mathbf{m}(t)\varphi(a)) > 0$ for all $t \in [0,1]$. Assume that $f : K \longrightarrow (0,+\infty)$ be a differentiable mapping on K°. If $(f'^{q}$ is generalized relative semi-$\mathbf{m}$-$(r; h_1, h_2)$-preinvex mapping on K, $q > 1$, $p^{-1} + q^{-1} = 1$, then for any $\lambda, \mu \in [0,1]$ and $r_1 \geq 0$, the following inequality for k-fractional integrals holds:*

$$\begin{aligned} &\left|I^{\alpha,k}_{f,\eta,\varphi,\mathbf{m}}(\lambda,\mu;r_1,a,b)\right| \\ \leq &\left(\frac{\eta(\varphi(b),\mathbf{m}(t)\varphi(a))}{r_1+1}\right)^{\frac{\alpha}{k}+1} \\ &\times\left\{\sqrt[p]{\varrho\left(\frac{\alpha}{k},\lambda,p\right)} \times \sqrt[rq]{(f'(a))^{rq} I_1^r(h_1(\xi);\mathbf{m}(\xi),r,r_1) + (f'(b))^{rq} I_2^r(h_2(\xi);r,r_1)}\right. \\ &\left.+\sqrt[p]{\varrho\left(\frac{\alpha}{k},\mu,p\right)} \times \sqrt[rq]{(f'(a))^{rq}\,\overline{I_1}^r(h_1(\xi);\mathbf{m}(\xi),r,r_1) + (f'(b))^{rq}\,\overline{I_2}^r(h_2(\xi);r,r_1)}\right\}, \end{aligned} \tag{14}$$

where

$$I_1(h_1(\xi);\mathbf{m}(\xi),r,r_1) := \int_0^1 \mathbf{m}^{\frac{1}{r}}(\xi) h_1^{\frac{1}{r}}\left(\frac{r_1+\xi}{r_1+1}\right)d\xi, \quad I_2(h_2(\xi);r,r_1) := \int_0^1 h_2^{\frac{1}{r}}\left(\frac{r_1+\xi}{r_1+1}\right)d\xi;$$

$$\overline{I_1}(h_1(\xi);\mathbf{m}(\xi),r,r_1) := \int_0^1 \mathbf{m}^{\frac{1}{r}}(\xi) h_1^{\frac{1}{r}}\left(\frac{1-\xi}{r_1+1}\right)d\xi, \quad \overline{I_2}(h_2(\xi);r,r_1) := \int_0^1 h_2^{\frac{1}{r}}\left(\frac{1-\xi}{r_1+1}\right)d\xi$$

and $\varrho\left(\frac{\alpha}{k},\lambda,p\right)$, $\varrho\left(\frac{\alpha}{k},\mu,p\right)$ are defined as in Lemma 3.

Proof. Suppose that $q > 1$, $r_1 \geq 0$ and $0 < r \leq 1$. From Lemmas 3 and 4, generalized relative semi-$\mathbf{m}$-$(r; h_1, h_2)$-preinvexity of $(f'^{q}$, Hölder inequality, Minkowski inequality and properties of the modulus, we have

$$\begin{aligned}
&\left|I^{\alpha,k}_{f,\eta,\varphi,\mathbf{m}}(\lambda,\mu;r_1,a,b)\right| \\
\leq\ & \left(\frac{\eta(\varphi(b),\mathbf{m}(t)\varphi(a))}{r_1+1}\right)^{\frac{\alpha}{k}+1}\left\{\int_0^1\left|\xi^{\frac{\alpha}{k}}-\lambda\right|\left|f'\left(\mathbf{m}(t)\varphi(a)+\left(\frac{r_1+\xi}{r_1+1}\right)\eta(\varphi(b),\mathbf{m}(t)\varphi(a))\right)\right|d\xi\right. \\
&\left.+\int_0^1\left|\mu-\xi^{\frac{\alpha}{k}}\right|\left|f'\left(\mathbf{m}(t)\varphi(a)+\left(\frac{1-\xi}{r_1+1}\right)\eta(\varphi(b),\mathbf{m}(t)\varphi(a))\right)\right|d\xi\right\} \\
\leq\ & \left(\frac{\eta(\varphi(b),\mathbf{m}(t)\varphi(a))}{r_1+1}\right)^{\frac{\alpha}{k}+1} \\
&\times\left\{\left(\int_0^1\left|\xi^{\frac{\alpha}{k}}-\lambda\right|^p d\xi\right)^{\frac{1}{p}}\left(\int_0^1\left(f'\left(\mathbf{m}(t)\varphi(a)+\left(\frac{r_1+\xi}{r_1+1}\right)\eta(\varphi(b),\mathbf{m}(t)\varphi(a))\right)\right)^q d\xi\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1\left|\mu-\xi^{\frac{\alpha}{k}}\right|^p d\xi\right)^{\frac{1}{p}}\left(\int_0^1\left(f'\left(\mathbf{m}(t)\varphi(a)+\left(\frac{1-\xi}{r_1+1}\right)\eta(\varphi(b),\mathbf{m}(t)\varphi(a))\right)\right)^q d\xi\right)^{\frac{1}{q}}\right\} \\
\leq\ & \left(\frac{\eta(\varphi(b),\mathbf{m}(t)\varphi(a))}{r_1+1}\right)^{\frac{\alpha}{k}+1} \\
&\times\left\{\sqrt[p]{\varrho\left(\frac{\alpha}{k},\lambda,p\right)}\left(\int_0^1\left[\mathbf{m}(\xi)h_1\left(\frac{r_1+\xi}{r_1+1}\right)(f'(a))^{rq}+h_2\left(\frac{r_1+\xi}{r_1+1}\right)(f'(b))^{rq}\right]^{\frac{1}{r}}d\xi\right)^{\frac{1}{q}}\right. \\
&\left.+\sqrt[p]{\varrho\left(\frac{\alpha}{k},\mu,p\right)}\left(\int_0^1\left[\mathbf{m}(\xi)h_1\left(\frac{1-\xi}{r_1+1}\right)(f'(a))^{rq}+h_2\left(\frac{1-\xi}{r_1+1}\right)(f'(b))^{rq}\right]^{\frac{1}{r}}d\xi\right)^{\frac{1}{q}}\right\} \\
\leq\ & \left(\frac{\eta(\varphi(b),\mathbf{m}(t)\varphi(a))}{r_1+1}\right)^{\frac{\alpha}{k}+1}\left\{\sqrt[p]{\varrho\left(\frac{\alpha}{k},\lambda,p\right)}\left[\left(\int_0^1\mathbf{m}^{\frac{1}{r}}(\xi)(f'(a))^q h_1^{\frac{1}{r}}\left(\frac{r_1+\xi}{r_1+1}\right)d\xi\right)^r\right.\right. \\
&\left.+\left(\int_0^1(f'(b))^q h_2^{\frac{1}{r}}\left(\frac{r_1+\xi}{r_1+1}\right)d\xi\right)^r\right]^{\frac{1}{rq}}+\sqrt[p]{\varrho\left(\frac{\alpha}{k},\mu,p\right)}\times\left[\left(\int_0^1\mathbf{m}^{\frac{1}{r}}(\xi)(f'(a))^q h_1^{\frac{1}{r}}\left(\frac{1-\xi}{r_1+1}\right)d\xi\right)^r\right. \\
&\left.\left.+\left(\int_0^1(f'(b))^q h_2^{\frac{1}{r}}\left(\frac{1-\xi}{r_1+1}\right)d\xi\right)^r\right]^{\frac{1}{rq}}\right\} \\
=\ & \left(\frac{\eta(\varphi(b),\mathbf{m}(t)\varphi(a))}{r_1+1}\right)^{\frac{\alpha}{k}+1}\left\{\sqrt[p]{\varrho\left(\frac{\alpha}{k},\lambda,p\right)}\times\sqrt[rq]{(f'(a))^{rq}I_1^r(h_1(\xi);\mathbf{m}(\xi),r,r_1)+(f'(b))^{rq}I_2^r(h_2(\xi);r,r_1)}\right. \\
&\left.+\sqrt[p]{\varrho\left(\frac{\alpha}{k},\mu,p\right)}\times\sqrt[rq]{(f'(a))^{rq}\,\overline{I_1}^r(h_1(\xi);\mathbf{m}(\xi),r,r_1)+(f'(b))^{rq}\,\overline{I_2}^r(h_2(\xi);r,r_1)}\right\}.
\end{aligned}$$

So, the proof of this theorem is complete. ∎

We point out some special cases of Theorem 5.

Corollary 6 *In Theorem 5 for $p = q = 2$, we get the following inequality:*

$$\begin{aligned} &\left| I^{\alpha,k}_{f,\eta,\varphi,\mathbf{m}}(\lambda,\mu;r_1,a,b)\right| \\ \leq\ & \left(\frac{\eta(\varphi(b),\mathbf{m}(t)\varphi(a))}{r_1+1}\right)^{\frac{\alpha}{k}+1} \\ &\times\left\{\sqrt{\varrho\left(\frac{\alpha}{k},\lambda,2\right)}\times\sqrt[2r]{(f'(a))^{2r}\,I^r_1(h_1(\xi);\mathbf{m}(\xi),r,r_1)+(f'(b))^{2r}\,I^r_2(h_2(\xi);r,r_1)}\right. \\ &\left.+\sqrt{\varrho\left(\frac{\alpha}{k},\mu,2\right)}\times\sqrt[2r]{(f'(a))^{2r}\,\overline{I_1}^r(h_1(\xi);\mathbf{m}(\xi),r,r_1)+(f'(b))^{2r}\,\overline{I_2}^r(h_2(\xi);r,r_1)}\right\}. \end{aligned} \tag{15}$$

Corollary 7 *In Theorem 5 if we choose $\lambda = \mu = 0$, $\eta(\varphi(b),\mathbf{m}(t)\varphi(a)) = \varphi(b) - \mathbf{m}(t)\varphi(a)$ and $\mathbf{m}(t) = m \in (0,1]$ for all $t \in [0,1]$, we get*

$$\begin{aligned} &\left| I^{\alpha,k}_{f,\varphi,m}(0,0;r_1,a,b)\right| \\ =\ & \left|\left(\frac{\varphi(b)-m\varphi(a)}{r_1+1}\right)^{\frac{\alpha}{k}}\big[f(m\varphi(a))+f(\varphi(b))\big]-\Gamma_k(\alpha+k)\times\left[I^{\alpha,k}_{(m\varphi(a))^+}f\left(m\varphi(a)+\frac{\varphi(b)-m\varphi(a)}{r_1+1}\right)\right.\right. \\ &\left.\left.+I^{\alpha,k}_{(\varphi(b))^-}f\left(m\varphi(a)+\frac{r_1}{r_1+1}(\varphi(b)-m\varphi(a))\right)\right]\right| \\ \leq\ & \sqrt[p]{\frac{k}{k+p\alpha}}\left(\frac{\varphi(b)-m\varphi(a)}{r_1+1}\right)^{\frac{\alpha}{k}+1}\left\{\sqrt[rq]{m\,(f'(a))^{rq}\,I^r_2(h_1(\xi);r,r_1)+(f'(b))^{rq}\,I^r_2(h_2(\xi);r,r_1)}\right. \\ &\left.+\sqrt[rq]{m\,(f'(a))^{rq}\,\overline{I_2}^r(h_1(\xi);r,r_1)+(f'(b))^{rq}\,\overline{I_2}^r(h_2(\xi);r,r_1)}\right\}. \end{aligned} \tag{16}$$

Corollary 8 *In Theorem 5 if we choose $\lambda = \mu = 1$, $\eta(\varphi(b),\mathbf{m}(t)\varphi(a)) = \varphi(b) - \mathbf{m}(t)\varphi(a)$ and $\mathbf{m}(t) = m \in (0,1]$ for all $t \in [0,1]$, we get*

$$\begin{aligned} &\left| I^{\alpha,k}_{f,\varphi,m}(1,1;r_1,a,b)\right| \\ =\ & \left|\left(\frac{\varphi(b)-m\varphi(a)}{r_1+1}\right)^{\frac{\alpha}{k}}\left[f\left(m\varphi(a)+\frac{r_1}{r_1+1}(\varphi(b)-m\varphi(a))\right)+f\left(m\varphi(a)+\frac{\varphi(b)-m\varphi(a)}{r_1+1}\right)\right]\right. \\ &-\Gamma_k(\alpha+k)\times\left[I^{\alpha,k}_{(m\varphi(a))^+}f\left(m\varphi(a)+\frac{\varphi(b)-m\varphi(a)}{r_1+1}\right)\right. \\ &\left.\left.+I^{\alpha,k}_{(\varphi(b))^-}f\left(m\varphi(a)+\frac{r_1}{r_1+1}(\varphi(b)-m\varphi(a))\right)\right]\right| \\ \leq\ & \sqrt[p]{\frac{k}{\alpha}\beta\left(p+1,\frac{k}{\alpha}\right)}\left(\frac{\varphi(b)-m\varphi(a)}{r_1+1}\right)^{\frac{\alpha}{k}+1}\left\{\sqrt[rq]{m\,(f'(a))^{rq}\,I^r_2(h_1(\xi);r,r_1)+(f'(b))^{rq}\,I^r_2(h_2(\xi);r,r_1)}\right. \\ &\left.+\sqrt[rq]{m\,(f'(a))^{rq}\,\overline{I_2}^r(h_1(\xi);r,r_1)+(f'(b))^{rq}\,\overline{I_2}^r(h_2(\xi);r,r_1)}\right\}. \end{aligned} \tag{17}$$

Corollary 9 *In Theorem 5 for $h_1(t) = h_2(t) = 1$ and $\mathbf{m}(t) = m \in (0,1]$ for all $t \in [0,1]$, we get the following* *inequality for generalized relative semi-(m,P)-preinvex mappings:*

$$\begin{aligned} &\left|I^{\alpha,k}_{f,\eta,\varphi,m}(\lambda,\mu;r_1,a,b)\right| \\ \leq &\left(\frac{\eta(\varphi(b),m\varphi(a))}{r_1+1}\right)^{\frac{\alpha}{k}+1}\left[\sqrt[p]{\varrho\left(\frac{\alpha}{k},\lambda,p\right)}+\sqrt[p]{\varrho\left(\frac{\alpha}{k},\mu,p\right)}\right]\sqrt[rq]{m\left(f'(a)\right)^{rq}+\left(f'(b)\right)^{rq}}. \end{aligned} \tag{18}$$

Corollary 10 *In Theorem 5 for $h_1(t)=h(1-t)$, $h_2(t)=h(t)$ and $\mathbf{m}(t)=m\in(0,1]$ for all $t\in[0,1]$, we get the following inequality for generalized relative semi-(m,h)-preinvex mappings:*

$$\begin{aligned} &\left|I^{\alpha,k}_{f,\eta,\varphi,m}(\lambda,\mu;r_1,a,b)\right| \\ \leq &\left(\frac{\eta(\varphi(b),m\varphi(a))}{r_1+1}\right)^{\frac{\alpha}{k}+1}\left\{\sqrt[p]{\varrho\left(\frac{\alpha}{k},\lambda,p\right)}\sqrt[rq]{m\left(f'(a)\right)^{rq}I_2^r(h(1-\xi);r,r_1)+\left(f'(b)\right)^{rq}I_2^r(h(\xi);r,r_1)}\right. \\ &\left.+\sqrt[p]{\varrho\left(\frac{\alpha}{k},\mu,p\right)}\sqrt[rq]{m\left(f'(a)\right)^{rq}\overline{I_2}^r(h(1-\xi);r,r_1)+\left(f'(b)\right)^{rq}\overline{I_2}^r(h(\xi);r,r_1)}\right\}. \end{aligned} \tag{19}$$

Corollary 11 *In Corollary 10 for $h_1(t)=(1-t)^s$ and $h_2(t)=t^s$, we get the following inequality for generalized relative semi-(m,s)-Breckner-preinvex mappings:*

$$\begin{aligned} &\left|I^{\alpha,k}_{f,\eta,\varphi,m}(\lambda,\mu;r_1,a,b)\right| \\ \leq &\sqrt[q]{\frac{r}{r+s}}\left(\frac{1}{r_1+1}\right)^{\frac{s}{rq}}\left(\frac{\eta(\varphi(b),m\varphi(a))}{r_1+1}\right)^{\frac{\alpha}{k}+1}. \\ &\times\left[\sqrt[p]{\varrho\left(\frac{\alpha}{k},\lambda,p\right)}\sqrt[q]{(r_1+1)^{\frac{s}{r}+1}-r_1^{\frac{s}{r}+1}}+\sqrt[p]{\varrho\left(\frac{\alpha}{k},\mu,p\right)}\right]. \end{aligned} \tag{20}$$

Corollary 12 *In Corollary 10 for $h_1(t)=(1-t)^{-s}$, $h_2(t)=t^{-s}$ and $0<s<r$, we get the following inequality for generalized relative semi-(m,s)-Godunova-Levin-Dragomir-preinvex mappings:*

$$\begin{aligned} &\left|I^{\alpha,k}_{f,\eta,\varphi,m}(\lambda,\mu;r_1,a,b)\right| \\ \leq &\sqrt[q]{\frac{r}{r-s}}\left(\frac{1}{r_1+1}\right)^{-\frac{s}{rq}}\left(\frac{\eta(\varphi(b),m\varphi(a))}{r_1+1}\right)^{\frac{\alpha}{k}+1}. \\ &\times\left[\sqrt[p]{\varrho\left(\frac{\alpha}{k},\lambda,p\right)}\sqrt[q]{(r_1+1)^{1-\frac{s}{r}}-r_1^{1-\frac{s}{r}}}+\sqrt[p]{\varrho\left(\frac{\alpha}{k},\mu,p\right)}\right]. \end{aligned} \tag{21}$$

Corollary 13 *In Theorem 5 for $h_1(t)=h_2(t)=t(1-t)$ and $\mathbf{m}(t)=m\in(0,1]$ for all $t\in[0,1]$, we get the following inequality for generalized relative semi-(m,tgs)-preinvex mappings:*

$$\begin{aligned} &\left|I^{\alpha,k}_{f,\eta,\varphi,m}(\lambda,\mu;r_1,a,b)\right| \\ \leq &\sqrt[q]{\frac{C(r,r_1)}{(r_1+1)^2}}\left(\frac{\eta(\varphi(b),m\varphi(a))}{r_1+1}\right)^{\frac{\alpha}{k}+1}\left[\sqrt[p]{\varrho\left(\frac{\alpha}{k},\lambda,p\right)}+\sqrt[p]{\varrho\left(\frac{\alpha}{k},\mu,p\right)}\right] \\ &\times\sqrt[rq]{m\left(f'(a)\right)^{rq}+\left(f'(b)\right)^{rq}}, \end{aligned} \tag{22}$$

where

$$C(r,r_1):=\int_0^1(1-\xi)^{\frac{1}{r}}(r_1+\xi)^{\frac{1}{r}}d\xi.$$

Corollary 14 *In Corollary 10 for* $h_1(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$ *and* $h_2(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$, *we get the following inequality for generalized relative semi-m-MT-preinvex mappings:*

$$\begin{aligned}
&\left|I^{\alpha,k}_{f,\eta,\varphi,m}(\lambda,\mu;r_1,a,b)\right| \\
\leq\ & \sqrt[rq]{\frac{1}{2}}\left(\frac{\eta(\varphi(b),m\varphi(a))}{r_1+1}\right)^{\frac{\alpha}{k}+1}\left\{\sqrt[p]{\varrho\left(\frac{\alpha}{k},\lambda,p\right)}\times\sqrt[rq]{m\left(f'(a)\right)^{rq}D^r(r,r_1)+\left(f'(b)\right)^{rq}E^r(r,r_1)}\right. \\
&\left.+\sqrt[p]{\varrho\left(\frac{\alpha}{k},\mu,p\right)}\times\sqrt[rq]{m\left(f'(a)\right)^{rq}E^r(r,r_1)+\left(f'(b)\right)^{rq}D^r(r,r_1)}\right\},
\end{aligned}\tag{23}$$

where

$$D(r,r_1) := \int_0^1 \sqrt[r]{\frac{1-\xi}{r_1+\xi}}d\xi, \quad E(r,r_1) := \int_0^1 \sqrt[r]{\frac{r_1+\xi}{1-\xi}}d\xi.$$

Theorem 15 *Let* $\alpha, k > 0$ *and* $0 < r \leq 1$. *Suppose* $h_1, h_2 : [0,1] \longrightarrow [0,+\infty)$, $\varphi : I \longrightarrow K$ *are continuous functions and* $\mathbf{m} : [0,1] \longrightarrow (0,1]$. *Let* $K = [\mathbf{m}(t)\varphi(a), \mathbf{m}(t)\varphi(a) + \eta(\varphi(b),\mathbf{m}(t)\varphi(a))] \subseteq \mathbb{R}$ *be an open* $\mathbf{m}$*-invex subset with respect to* $\eta : K \times K \longrightarrow \mathbb{R}$ *and let* $\eta(\varphi(b),\mathbf{m}(t)\varphi(a)) > 0$ *for all* $t \in [0,1]$. *Assume that* $f : K \longrightarrow (0,+\infty)$ *be a differentiable mapping on* K°. *If* $(f'^q$ *is generalized relative semi-*$\mathbf{m}$*-*$(r;h_1,h_2)$*-preinvex mapping on* K, $q \geq 1$, *then for any* $\lambda, \mu \in [0,1]$ *and* $r_1 \geq 0$, *the following inequality for k-fractional integrals holds:*

$$\begin{aligned}
&\left|I^{\alpha,k}_{f,\eta,\varphi,\mathbf{m}}(\lambda,\mu;r_1,a,b)\right| \\
\leq\ & \left(\frac{\eta(\varphi(b),\mathbf{m}(t)\varphi(a))}{r_1+1}\right)^{\frac{\alpha}{k}+1} \\
&\times\left\{\delta^{1-\frac{1}{q}}\left(\frac{\alpha}{k},\lambda\right)\sqrt[rq]{\left(f'(a)\right)^{rq}I_1^r(h_1(\xi);\mathbf{m}(\xi),r,r_1,\lambda,\alpha,k)+\left(f'(b)\right)^{rq}I_2^r(h_2(\xi);r,r_1,\lambda,\alpha,k)}\right. \\
&\left.+\delta^{1-\frac{1}{q}}\left(\frac{\alpha}{k},\mu\right)\sqrt[rq]{\left(f'(a)\right)^{rq}\overline{I_1}^r(h_1(\xi);\mathbf{m}(\xi),r,r_1,\mu,\alpha,k)+\left(f'(b)\right)^{rq}\overline{I_2}^r(h_2(\xi);r,r_1,\mu,\alpha,k)}\right\},
\end{aligned}\tag{24}$$

where

$$I_1(h_1(\xi);\mathbf{m}(\xi),r,r_1,\lambda,\alpha,k) := \int_0^1 \mathbf{m}^{\frac{1}{r}}(\xi)\left|\xi^{\frac{\alpha}{k}}-\lambda\right|h_1^{\frac{1}{r}}\left(\frac{r_1+\xi}{r_1+1}\right)d\xi;$$

$$I_2(h_2(\xi);r,r_1,\lambda,\alpha,k) := \int_0^1 \left|\xi^{\frac{\alpha}{k}}-\lambda\right|h_2^{\frac{1}{r}}\left(\frac{r_1+\xi}{r_1+1}\right)d\xi;$$

$$\overline{I_1}(h_1(\xi);\mathbf{m}(\xi),r,r_1,\mu,\alpha,k) := \int_0^1 \mathbf{m}^{\frac{1}{r}}(\xi)\left|\mu-\xi^{\frac{\alpha}{k}}\right|h_1^{\frac{1}{r}}\left(\frac{1-\xi}{r_1+1}\right)d\xi;$$

$$\overline{I_2}(h_2(\xi);r,r_1,\mu,\alpha,k) := \int_0^1 \left|\mu-\xi^{\frac{\alpha}{k}}\right|h_2^{\frac{1}{r}}\left(\frac{1-\xi}{r_1+1}\right)d\xi$$

and $\delta\left(\frac{\alpha}{k},\lambda\right)$, $\delta\left(\frac{\alpha}{k},\mu\right)$ *are defined as in Lemma 3.*

Proof. Suppose that $q \geq 1$, $r_1 \geq 0$ and $0 < r \leq 1$. From Lemmas 3 and 4, generalized relative semi-$\mathbf{m}$-$(r;h_1,h_2)$-preinvexity of $(f'^q$, the well-known power mean inequality, Minkowski inequality and properties of the modulus, we have

$$
\begin{aligned}
&\left|I_{f,\eta,\varphi,\mathbf{m}}^{\alpha,k}(\lambda,\mu;r_1,a,b)\right| \\
\leq\ & \left(\frac{\eta(\varphi(b),\mathbf{m}(t)\varphi(a))}{r_1+1}\right)^{\frac{\alpha}{k}+1}\left\{\int_0^1\left|\xi^{\frac{\alpha}{k}}-\lambda\right|\left|f'\left(\mathbf{m}(t)\varphi(a)+\left(\frac{r_1+\xi}{r_1+1}\right)\eta(\varphi(b),\mathbf{m}(t)\varphi(a))\right)\right|d\xi\right. \\
&\left.+\int_0^1\left|\mu-\xi^{\frac{\alpha}{k}}\right|\left|f'\left(\mathbf{m}(t)\varphi(a)+\left(\frac{1-\xi}{r_1+1}\right)\eta(\varphi(b),\mathbf{m}(t)\varphi(a))\right)\right|d\xi\right\} \\
\leq\ & \left(\frac{\eta(\varphi(b),\mathbf{m}(t)\varphi(a))}{r_1+1}\right)^{\frac{\alpha}{k}+1}\left\{\left(\int_0^1\left|\xi^{\frac{\alpha}{k}}-\lambda\right|d\xi\right)^{1-\frac{1}{q}}\right. \\
&\times\left(\int_0^1\left|\xi^{\frac{\alpha}{k}}-\lambda\right|\left(f'\left(\mathbf{m}(t)\varphi(a)+\left(\frac{r_1+\xi}{r_1+1}\right)\eta(\varphi(b),\mathbf{m}(t)\varphi(a))\right)\right)^q d\xi\right)^{\frac{1}{q}} \\
&\left.+\left(\int_0^1\left|\mu-\xi^{\frac{\alpha}{k}}\right|d\xi\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\mu-\xi^{\frac{\alpha}{k}}\right|\left(f'\left(\mathbf{m}(t)\varphi(a)+\left(\frac{1-\xi}{r_1+1}\right)\eta(\varphi(b),\mathbf{m}(t)\varphi(a))\right)\right)^q d\xi\right)^{\frac{1}{q}}\right\} \\
\leq\ & \left(\frac{\eta(\varphi(b),\mathbf{m}(t)\varphi(a))}{r_1+1}\right)^{\frac{\alpha}{k}+1}\left\{\delta^{1-\frac{1}{q}}\left(\frac{\alpha}{k},\lambda\right)\right. \\
&\times\left(\int_0^1\left|\xi^{\frac{\alpha}{k}}-\lambda\right|\left[\mathbf{m}(\xi)h_1\left(\frac{r_1+\xi}{r_1+1}\right)(f'(a))^{rq}+h_2\left(\frac{r_1+\xi}{r_1+1}\right)(f'(b))^{rq}\right]^{\frac{1}{r}}d\xi\right)^{\frac{1}{q}} \\
&\left.+\delta^{1-\frac{1}{q}}\left(\frac{\alpha}{k},\mu\right)\left(\int_0^1\left|\mu-\xi^{\frac{\alpha}{k}}\right|\left[\mathbf{m}(\xi)h_1\left(\frac{1-\xi}{r_1+1}\right)(f'(a))^{rq}+h_2\left(\frac{1-\xi}{r_1+1}\right)(f'(b))^{rq}\right]^{\frac{1}{r}}d\xi\right)^{\frac{1}{q}}\right\} \\
\leq\ & \left(\frac{\eta(\varphi(b),\mathbf{m}(t)\varphi(a))}{r_1+1}\right)^{\frac{\alpha}{k}+1}\left\{\delta^{1-\frac{1}{q}}\left(\frac{\alpha}{k},\lambda\right)\left[\left(\int_0^1\mathbf{m}^{\frac{1}{r}}(\xi)\,(f'(a))^q\left|\xi^{\frac{\alpha}{k}}-\lambda\right|h_1^{\frac{1}{r}}\left(\frac{r_1+\xi}{r_1+1}\right)d\xi\right)^r\right.\right. \\
&\left.+\left(\int_0^1(f'(b))^q\left|\xi^{\frac{\alpha}{k}}-\lambda\right|h_2^{\frac{1}{r}}\left(\frac{r_1+\xi}{r_1+1}\right)d\xi\right)^r\right]^{\frac{1}{rq}} \\
&+\delta^{1-\frac{1}{q}}\left(\frac{\alpha}{k},\mu\right)\times\left[\left(\int_0^1\mathbf{m}^{\frac{1}{r}}(\xi)\,(f'(a))^q\left|\mu-\xi^{\frac{\alpha}{k}}\right|h_1^{\frac{1}{r}}\left(\frac{1-\xi}{r_1+1}\right)d\xi\right)^r\right. \\
&\left.\left.+\left(\int_0^1(f'(b))^q\left|\mu-\xi^{\frac{\alpha}{k}}\right|h_2^{\frac{1}{r}}\left(\frac{1-\xi}{r_1+1}\right)d\xi\right)^r\right]^{\frac{1}{rq}}\right\} \\
=\ & \left(\frac{\eta(\varphi(b),\mathbf{m}(t)\varphi(a))}{r_1+1}\right)^{\frac{\alpha}{k}+1}\left\{\delta^{1-\frac{1}{q}}\left(\frac{\alpha}{k},\lambda\right)\right. \\
&\times\sqrt[rq]{(f'(a))^{rq}\,I_1^r(h_1(\xi);\mathbf{m}(\xi),r,r_1,\lambda,\alpha,k)+(f'(b))^{rq}\,I_2^r(h_2(\xi);r,r_1,\lambda,\alpha,k)} \\
&\left.+\delta^{1-\frac{1}{q}}\left(\frac{\alpha}{k},\mu\right)\sqrt[rq]{(f'(a))^{rq}\,\overline{I_1}^r(h_1(\xi);\mathbf{m}(\xi),r,r_1,\mu,\alpha,k)+(f'(b))^{rq}\,\overline{I_2}^r(h_2(\xi);r,r_1,\mu,\alpha,k)}\right\}.
\end{aligned}
$$

So, the proof of this theorem is complete. ■

We point out some special cases of Theorem 15.

Corollary 16 *In Theorem 15 for $q=1$, we get the following inequality:*

$$\begin{aligned}
&\left|I_{f,\eta,\varphi,\mathbf{m}}^{\alpha,k}(\lambda,\mu;r_1,a,b)\right| \\
\leq\ & \left(\frac{\eta(\varphi(b),\mathbf{m}(t)\varphi(a))}{r_1+1}\right)^{\frac{\alpha}{k}+1}\left\{\sqrt[r]{(f'(a))^r\, I_1^r(h_1(\xi);\mathbf{m}(\xi),r,r_1,\lambda,\alpha,k)+(f'(b))^r\, I_2^r(h_2(\xi);r,r_1,\lambda,\alpha,k)}\right. \\
&\left.+\sqrt[r]{(f'(a))^r\, \overline{I_1}^r(h_1(\xi);\mathbf{m}(\xi),r,r_1,\mu,\alpha,k)+(f'(b))^r\, \overline{I_2}^r(h_2(\xi);r,r_1,\mu,\alpha,k)}\right\}. \qquad (25)
\end{aligned}$$

Corollary 17 *In Theorem 15 if we choose* $\lambda=\mu=0$, $\eta(\varphi(b),\mathbf{m}(t)\varphi(a))=\varphi(b)-\mathbf{m}(t)\varphi(a)$ *and* $\mathbf{m}(t)=m\in(0,1]$ *for all* $t\in[0,1]$, *we get*

$$\begin{aligned}
&\left|I_{f,\varphi,m}^{\alpha,k}(0,0;r_1,a,b)\right| \\
\leq\ & \left(\frac{k}{k+\alpha}\right)^{1-\frac{1}{q}}\left(\frac{\varphi(b)-m\varphi(a)}{r_1+1}\right)^{\frac{\alpha}{k}+1} \\
&\times\left\{\sqrt[rq]{m\,(f'(a))^{rq}\, I_2^r(h_1(\xi);r,r_1,0,\alpha,k)+(f'(b))^{rq}\, I_2^r(h_2(\xi);r,r_1,0,\alpha,k)}\right. \\
&\left.+\sqrt[rq]{m\,(f'(a))^{rq}\, \overline{I_2}^r(h_1(\xi);r,r_1,0,\alpha,k)+(f'(b))^{rq}\, \overline{I_2}^r(h_2(\xi);r,r_1,0,\alpha,k)}\right\}. \qquad (26)
\end{aligned}$$

Corollary 18 *In Theorem 15 if we choose* $\lambda=\mu=1$, $\eta(\varphi(b),\mathbf{m}(t)\varphi(a))=\varphi(b)-\mathbf{m}(t)\varphi(a)$ *and* $\mathbf{m}(t)=m\in(0,1]$ *for all* $t\in[0,1]$, *we get*

$$\begin{aligned}
&\left|I_{f,\varphi,m}^{\alpha,k}(1,1;r_1,a,b)\right| \\
\leq\ & \left(\frac{\alpha}{k+\alpha}\right)^{1-\frac{1}{q}}\left(\frac{\varphi(b)-m\varphi(a)}{r_1+1}\right)^{\frac{\alpha}{k}+1} \\
&\times\left\{\sqrt[rq]{m\,(f'(a))^{rq}\, I_2^r(h_1(\xi);r,r_1,1,\alpha,k)+(f'(b))^{rq}\, I_2^r(h_2(\xi);r,r_1,1,\alpha,k)}\right. \\
&\left.+\sqrt[rq]{m\,(f'(a))^{rq}\, \overline{I_2}^r(h_1(\xi);r,r_1,1,\alpha,k)+(f'(b))^{rq}\, \overline{I_2}^r(h_2(\xi);r,r_1,1,\alpha,k)}\right\}. \qquad (27)
\end{aligned}$$

Corollary 19 *In Theorem 15 for* $h_1(t)=h_2(t)=1$ *and* $\mathbf{m}(t)=m\in(0,1]$ *for all* $t\in[0,1]$, *we get the following inequality for generalized relative semi-*(m,P)*-preinvex mappings:*

$$\begin{aligned}
&\left|I_{f,\eta,\varphi,m}^{\alpha,k}(\lambda,\mu;r_1,a,b)\right| \\
\leq\ & \left(\frac{\eta(\varphi(b),m\varphi(a))}{r_1+1}\right)^{\frac{\alpha}{k}+1}\left[\delta\left(\frac{\alpha}{k},\lambda\right)+\delta\left(\frac{\alpha}{k},\mu\right)\right]\sqrt[rq]{m\,(f'(a))^{rq}+(f'(b))^{rq}}. \qquad (28)
\end{aligned}$$

Corollary 20 *In Theorem 15 for* $h_1(t)=h(1-t)$, $h_2(t)=h(t)$ *and* $\mathbf{m}(t)=m\in(0,1]$ *for all* $t\in[0,1]$, *we* *get the following inequality for generalized relative semi-(m,h)-preinvex mappings:*

$$\begin{aligned}
&\left|I_{f,\eta,\varphi,m}^{\alpha,k}(\lambda,\mu;r_1,a,b)\right| \\
\leq\ &\left(\frac{\eta(\varphi(b),m\varphi(a))}{r_1+1}\right)^{\frac{\alpha}{k}+1}\left\{\delta^{1-\frac{1}{q}}\left(\frac{\alpha}{k},\lambda\right)\right. \\
&\times\sqrt[rq]{m\left(f'(a)\right)^{rq}I_2^r(h(1-\xi);r,r_1,\lambda,\alpha,k)+\left(f'(b)\right)^{rq}I_2^r(h(\xi);r,r_1,\lambda,\alpha,k)} \\
&+\delta^{1-\frac{1}{q}}\left(\frac{\alpha}{k},\mu\right) \\
&\left.\times\sqrt[rq]{m\left(f'(a)\right)^{rq}\overline{I_2}^r(h(1-\xi);\mathbf{m}(\xi),r,r_1,\mu,\alpha,k)+\left(f'(b)\right)^{rq}\overline{I_2}^r(h(\xi);r,r_1,\mu,\alpha,k)}\right\}. \qquad (29)
\end{aligned}$$

Corollary 21 *In Corollary 20 for $h_1(t)=(1-t)^s$ and $h_2(t)=t^s$, we get the following inequality for generalized relative semi-(m,s)-Breckner-preinvex mappings:*

$$\begin{aligned}
&\left|I_{f,\eta,\varphi,m}^{\alpha,k}(\lambda,\mu;r_1,a,b)\right| \\
\leq\ &\left(\frac{\eta(\varphi(b),m\varphi(a))}{r_1+1}\right)^{\frac{\alpha}{k}+1}\left\{\delta^{1-\frac{1}{q}}\left(\frac{\alpha}{k},\lambda\right)\right. \\
&\times\sqrt[rq]{m\left(f'(a)\right)^{rq}I_2^r(\xi^s;r,r_1,\lambda,\alpha,k)+\left(f'(b)\right)^{rq}I_2^r((1-\xi)^s;r,r_1,\lambda,\alpha,k)} \\
&\left.+\delta^{1-\frac{1}{q}}\left(\frac{\alpha}{k},\mu\right)\sqrt[rq]{m\left(f'(a)\right)^{rq}\overline{I_2}^r(\xi^s;r,r_1,\mu,\alpha,k)+\left(f'(b)\right)^{rq}\overline{I_2}^r((1-\xi)^s;r,r_1,\mu,\alpha,k)}\right\}. \qquad (30)
\end{aligned}$$

Corollary 22 *In Corollary 20 for $h_1(t)=(1-t)^{-s}$ and $h_2(t)=t^{-s}$, we get the following inequality for generalized relative semi-(m,s)-Godunova-Levin-Dragomir-preinvex mappings:*

$$\begin{aligned}
&\left|I_{f,\eta,\varphi,m}^{\alpha,k}(\lambda,\mu;r_1,a,b)\right| \\
\leq\ &\left(\frac{\eta(\varphi(b),m\varphi(a))}{r_1+1}\right)^{\frac{\alpha}{k}+1}\left\{\delta^{1-\frac{1}{q}}\left(\frac{\alpha}{k},\lambda\right)\right. \\
&\times\sqrt[rq]{m\left(f'(a)\right)^{rq}I_2^r(\xi^{-s};r,r_1,\lambda,\alpha,k)+\left(f'(b)\right)^{rq}I_2^r((1-\xi)^{-s};r,r_1,\lambda,\alpha,k)} \\
&+\delta^{1-\frac{1}{q}}\left(\frac{\alpha}{k},\mu\right) \\
&\left.\times\sqrt[rq]{m\left(f'(a)\right)^{rq}\overline{I_2}^r(\xi^{-s};r,r_1,\mu,\alpha,k)+\left(f'(b)\right)^{rq}\overline{I_2}^r((1-\xi)^{-s};r,r_1,\mu,\alpha,k)}\right\}. \qquad (31)
\end{aligned}$$

Corollary 23 *In Theorem 15 for $h_1(t)=h_2(t)=t(1-t)$ and $\mathbf{m}(t)=m\in(0,1]$ for all $t\in[0,1]$, we get the following inequality for generalized relative semi-(m,tgs)-preinvex mappings:*

$$\begin{aligned}
&\left|I_{f,\eta,\varphi,m}^{\alpha,k}(\lambda,\mu;r_1,a,b)\right| \\
\leq\ &\left(\frac{\eta(\varphi(b),m\varphi(a))}{r_1+1}\right)^{\frac{\alpha}{k}+1} \\
&\times\left\{\delta^{1-\frac{1}{q}}\left(\frac{\alpha}{k},\lambda\right)\sqrt[rq]{m\left(f'(a)\right)^{rq}I_2^r(\xi(1-\xi);r,r_1,\lambda,\alpha,k)+\left(f'(b)\right)^{rq}I_2^r(\xi(1-\xi);r,r_1,\lambda,\alpha,k)}\right. \\
&\left.+\delta^{1-\frac{1}{q}}\left(\frac{\alpha}{k},\mu\right)\sqrt[rq]{m\left(f'(a)\right)^{rq}\overline{I_2}^r(\xi(1-\xi);r,r_1,\mu,\alpha,k)+\left(f'(b)\right)^{rq}\overline{I_2}^r(\xi(1-\xi);r,r_1,\mu,\alpha,k)}\right\}. \\
&\qquad (32)
\end{aligned}$$

Corollary 24 *In Corollary 20 for $h_1(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$ and $h_2(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$, we get the following inequality for generalized relative semi-m-MT-preinvex mappings:*

$$\begin{aligned}
&\left|I^{\alpha,k}_{f,\eta,\varphi,m}(\lambda,\mu;r_1,a,b)\right| \\
\leq\ & \left(\frac{\eta(\varphi(b),m\varphi(a))}{r_1+1}\right)^{\frac{\alpha}{k}+1}\left\{\delta^{1-\frac{1}{q}}\left(\frac{\alpha}{k},\lambda\right)\right. \\
&\times\sqrt[rq]{m\left(f'(a)\right)^{rq} I_2^r\left(\frac{\sqrt{1-\xi}}{2\sqrt{\xi}};r,r_1,\lambda,\alpha,k\right)+\left(f'(b)\right)^{rq} I_2^r\left(\frac{\sqrt{\xi}}{2\sqrt{1-\xi}};r,r_1,\lambda,\alpha,k\right)} \\
&+\delta^{1-\frac{1}{q}}\left(\frac{\alpha}{k},\mu\right) \\
&\left.\times\sqrt[rq]{m\left(f'(a)\right)^{rq} \overline{I_2}^r\left(\frac{\sqrt{1-\xi}}{2\sqrt{\xi}};r,r_1,\mu,\alpha,k\right)+\left(f'(b)\right)^{rq} \overline{I_2}^r\left(\frac{\sqrt{\xi}}{2\sqrt{1-\xi}};r,r_1,\mu,\alpha,k\right)}\right\}. \qquad (33)
\end{aligned}$$

Remark 7 *For $k = 1$, by our Theorems 5 and 15, we can get some new Hermite-Hadamard type inequalities associated with generalized relative semi-$\mathbf{m}$-$(r;h_1,h_2)$-preinvex functions via fractional integrals. For $\alpha = k = 1$, by our Theorems 5 and 15, we can get some new Hermite-Hadamard type inequalities associated with generalized relative semi-$\mathbf{m}$-$(r;h_1,h_2)$-preinvex functions via classical integrals.*

Remark 8 *Also, applying our Theorems 5 and 15, for different values of $\lambda,\mu \in (0,1)$ and if $f'(x) \leq K$, for all $x \in I$, we can get some new Hermite-Hadamard type inequalities for fractional integrals associated with generalized relative semi-$\mathbf{m}$-$(r;h_1,h_2)$-preinvex mappings.*

3 Applications to Special Means

Definition 16 *[9] A function $M : \mathbb{R}^2_+ \longrightarrow \mathbb{R}_+$, is called a Mean function if it has the following properties:*

1. *Homogeneity: $M(ax,ay) = aM(x,y)$, for all $a > 0$,*
2. *Symmetry: $M(x,y) = M(y,x)$,*
3. *Reflexivity: $M(x,x) = x$,*
4. *Monotonicity: If $x \leq x'$ and $y \leq y'$, then $M(x,y) \leq M(x',y')$,*
5. *Internality: $\min\{x,y\} \leq M(x,y) \leq \max\{x,y\}$.*

We consider some means for arbitrary positive real numbers α,β $(\alpha \neq \beta)$.

1. The arithmetic mean:
$$A := A(\alpha,\beta) = \frac{\alpha+\beta}{2}.$$
2. The geometric mean:
$$G := G(\alpha,\beta) = \sqrt{\alpha\beta}.$$
3. The harmonic mean:
$$H := H(\alpha,\beta) = \frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}}.$$
4. The power mean:
$$P_r := P_r(\alpha,\beta) = \left(\frac{\alpha^r+\beta^r}{2}\right)^{\frac{1}{r}}, \quad r \geq 1.$$

5. The identric mean:
$$I := I(\alpha,\beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right), & \alpha \neq \beta; \\ \alpha, & \alpha = \beta. \end{cases}$$

6. The logarithmic mean:
$$L := L(\alpha,\beta) = \frac{\beta-\alpha}{\log(\beta)-\log(\alpha)}, \quad \alpha \neq \beta.$$

7. The generalized log-mean:
$$L_p := L_p(\alpha,\beta) = \left[\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right]^{\frac{1}{p}}; \quad p \in \mathbb{R}\setminus\{-1,0\}, \quad \alpha \neq \beta.$$

8. The weighted p-power mean:
$$M_p\begin{pmatrix} \alpha_1, & \alpha_2, & \cdots & ,\alpha_n \\ u_1, & u_2, & \cdots & ,u_n \end{pmatrix} = \left(\sum_{i=1}^{n}\alpha_i u_i^p\right)^{\frac{1}{p}}$$

   where $0 \leq \alpha_i \leq 1$, $u_i > 0$ $(i = 1,2,\ldots,n)$ with $\sum_{i=1}^{n}\alpha_i = 1$.

Now, let a and b be positive real numbers such that $a < b$. Consider the function $\overline{M} := M(\varphi(a),\varphi(b)) : [\varphi(a),\varphi(a)+\eta(\varphi(b),\varphi(a))] \times [\varphi(a),\varphi(a)+\eta(\varphi(b),\varphi(a))] \longrightarrow \mathbb{R}_+$, which is one of the above mentioned means, therefore one can obtain various inequalities using the results of Section 2 for these means as follows: Replace $\eta(\varphi(y),\mathbf{m}(t)\varphi(x))$ with $\eta(\varphi(y),\varphi(x))$ where $\mathbf{m}(t) \equiv 1$ for all $t \in [0,1]$ and setting $\eta(\varphi(b),\varphi(a)) = M(\varphi(a),\varphi(b))$ in (14) and (24), one can obtain the following interesting inequalities involving means:

$$\begin{aligned} &\left|I^{\alpha,k}_{f,M(\cdot,\cdot),\varphi}(\lambda,\mu;r_1,a,b)\right| \\ \leq\; &\left(\frac{\overline{M}}{r_1+1}\right)^{\frac{\alpha}{k}+1}\left\{\sqrt[p]{\varrho\left(\frac{\alpha}{k},\lambda,p\right)}\sqrt[rq]{\left(f'(a)\right)^{rq} I_2^r(h_1(\xi);r,r_1) + \left(f'(b)\right)^{rq} I_2^r(h_2(\xi);r,r_1)}\right. \\ &\left.+\sqrt[p]{\varrho\left(\frac{\alpha}{k},\mu,p\right)}\sqrt[rq]{\left(f'(a)\right)^{rq}\overline{I_2}^r(h_1(\xi);r,r_1) + \left(f'(b)\right)^{rq}\overline{I_2}^r(h_2(\xi);r,r_1)}\right\}, \end{aligned} \tag{34}$$

$$\begin{aligned} &\left|I^{\alpha,k}_{f,M(\cdot,\cdot),\varphi}(\lambda,\mu;r_1,a,b)\right| \\ \leq\; &\left(\frac{\overline{M}}{r_1+1}\right)^{\frac{\alpha}{k}+1}\left\{\delta^{1-\frac{1}{q}}\left(\frac{\alpha}{k},\lambda\right)\right. \\ &\times\sqrt[rq]{\left(f'(a)\right)^{rq} I_2^r(h_1(\xi);r,r_1,\lambda,\alpha,k) + \left(f'(b)\right)^{rq} I_2^r(h_2(\xi);r,r_1,\lambda,\alpha,k)} \\ &\left.+\delta^{1-\frac{1}{q}}\left(\frac{\alpha}{k},\mu\right)\sqrt[rq]{\left(f'(a)\right)^{rq}\overline{I_2}^r(h_1(\xi);r,r_1,\mu,\alpha,k) + \left(f'(b)\right)^{rq}\overline{I_2}^r(h_2(\xi);r,r_1,\mu,\alpha,k)}\right\}. \end{aligned} \tag{35}$$

Letting $\overline{M} := A, G, H, P_r, I, L, L_p, M_p$ in (34) and (35), we get the inequalities involving means for a particular choices of $(f'^q$ that are generalized relative semi-1-$(r;h_1,h_2)$-preinvex mappings.

Remark 9 *Also, applying our Theorems 5 and 15 for appropriate choices of functions h_1 and h_2 such that $(f'^q$ to be generalized relative semi-1-$(r;h_1,h_2)$-preinvex mappings, we can deduce some new inequalities using above special means. The details are left to the interested reader.*

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