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Singular Fractional Differential Equations With ψ-Caputo Operator And Modified Picard's Iterative Method*

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Abstract

In this paper, we study a class of Cauchy-type problem for a singular fractional differential equation involving a Caputo fractional derivative with respect to another function ψ. By using the modified Picard's iterative method, new existence and uniqueness results for the global solutions of Cauchy-type problem are established. In particular, the unique existence of a global solution is proved under the Lipschitz condition, without any constraints on the Lipschitz constant. The continuous dependence of solution of Cauchy-type problem is investigated via generalized Gronwall inequality. At the end, an illustrative example will be introduced to justify our results

1 Introduction

Over the decades, the fractional calculus has been building a great history and consolidating itself in several scientific areas such as: physics, mechanics chemistry, biology, engineering, among others. The emergence of various new definitions of fractional integrals and derivatives, makes the wide number of definitions becomes increasingly larger and clears its numerous applications. So in the literature several studies dealing with similar topics for different operators, for instance, Kilbas et al. in [10] introduced the properties of fractional integrals and fractional derivatives of a function with respect to another function. Also some of generalized fractional integral and differential operators and their properties were introduced by Agrawal in [3], and consequently, open a window for new applications. And over time, other types of new fractional derivatives and integrals arise and this makes the number of definitions wide, see [4, 8, 12, 13, 16].

Fractional differential equations have been proved to be new and valuable tools in the modeling of many phenomena in various fields of physics, engineering and economics. Recently, there are some works about the existence of solutions for a singular fractional differential equations, see [7, 9, 11, 17, 18, 19, 20] and the references therein, for example Lian et al. in [11], studied a class of singular fractional differential equations

$$\mathcal{D}_{0^+}^{\alpha}u(t) = f(t, u(t)),\ t \in (0, b], 0 < \alpha < 1, \tag{1}$$

$$u(0) = u_0, \tag{2}$$

where $\mathcal{D}_{0^+}^{\alpha}$ is the Caputo fractional derivative operator of order α, $u_0 \in \mathbb{R}$, $b > 0$ is a constant and the function f is defined on $\mathbb{R} \times \mathbb{R}$ with $\lim_{t\to 0^+} f(t, .) = \infty$.

As far as we know, there are few papers to discuss these problems for singular fractional differential equations, especially those that include generalized Caputo fractional derivatives with respect to another function ψ. In a very recent contribution, Almeida in [4] introduced a new type of fractional differentiation operator the so-called ψ-Caputo fractional derivative with respect to another function and extended the

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works introduced by Kilbas et al. [10, 14]. Almeida et al. [5] investigated the existence and uniqueness results for the Cauchy-type problem via fixed point theorem and Picard's iteration method. Sousa and de Oliveira in [15], discussed the existence, uniqueness and continuous dependence results of ψ-Hilfer fractional differential equations by means of fixed point theorem and generalized Gronwall inequality.

Motivated by the excellent results mentioned above and the methods used in [11] and [5], in this paper, we investigate the existence, uniqueness, and continuous dependence of global solution to the following singular fractional differential equation involving the left generalized Caputo fractional derivative with respect to another function ψ:

$$^{c}\mathcal{D}_{0^+}^{\alpha;\psi}u(t) = f(t,u(t)),\ t \in (0,b],\ b>0, \tag{3}$$

$$u(0) = u_0, \tag{4}$$

where $0<\alpha\leq 1$, $^{c}\mathcal{D}_{0^+}^{\alpha;\psi}$ is the ψ-Caputo fractional derivative introduced by Almeida in [4], $f:(0,b]\times\mathbb{R}\longrightarrow\mathbb{R}$ is given function with $\lim_{t\to 0^+} f(t,.)=\infty$ and satisfies some assumptions that will be specified in Section 3 and u_0 is a constant.

The remainder of the paper is organized as follows: In Section 2, we present some preliminaries and lemmas that will be used to prove our main results. In Section 3, we list the hypotheses and we also show that problem (3)–(4) is equivalent to a Volterra integral equation. Further, we discuss the existence and uniqueness of global solutions to the problem (3)–(4) via modified Picard's iterative method. Section 4 deals with the continuous dependence to such equations by means of generalized Gronwall inequality. Finally, an example is provided to illustrate our main results in section 5.

2 Preliminaries

For the convenience of the reader, we present here the necessary definitions and lemmas from fractional calculus theory. These preliminaries can be found in recent literature.

Definition 1 ([6]) *For $z\in\mathbb{C}$, $\mathcal{R}(z)>0$, the Euler gamma function is given by*

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt.$$

Moreover, $\Gamma(z+1) = z\Gamma(z)$.

Definition 2 ([6]) *Given $\mathcal{R}(p)>0$ and $\mathcal{R}(q)>0$. We define the Beta function (denoted $\mathcal{B}(p,q)$) by*

$$\mathcal{B}(p,q) = \int_0^1 u^{p-1}(1-u)^{q-1}du = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{5}$$

Definition 3 ([10]) *The left-sided ψ-Riemann-Liouville fractional integral and fractional derivative of order α $(n-1<\alpha<n)$ for an integrable function $h:[0,b]\to\mathbb{R}$ with respect to another function $\psi:[0,b]\to\mathbb{R}$ that is an increasing differentiable function such that $\psi'(t)\neq 0$, for all $t\in[0,b]$ are defined as follows*

$$\mathcal{I}_{0^+}^{\alpha,\psi}h(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}h(s)ds,\ t>0, \tag{6}$$

and

$$\mathcal{D}_{0^+}^{\alpha,\psi}h(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \mathcal{I}_{0^+}^{n-\alpha,\psi}h(t) \tag{7}$$

$$= \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_0^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1}h(s)ds,\ t>0, \tag{8}$$

respectively.

Definition 4 (**[4, 5]**) *Let $\alpha > 0$ and $\psi \in C^n[0,b]$ be a function such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in [0,b]$. Given $h \in C^{n-1}[0,b]$. The ψ-Caputo fractional derivative of h of order α is defined as follows*

$$^c\mathcal{D}_{0^+}^{\alpha,\psi}h(t) = \mathcal{D}_{0^+}^{\alpha,\psi}\left[h(t) - \sum_{k=0}^{n-1}\frac{h_\psi^{[k]}(0)}{k!}(\psi(t)-\psi(0))^k\right],$$

where $h_\psi^{[k]}(t) = \left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^k h(t)$, and $n = [\alpha]+1$ for $\alpha \notin \mathbb{N}$. Further, if $\alpha = n \in \mathbb{N}$, then $^c\mathcal{D}_{0^+}^{\alpha,\psi}h(t) = h_\psi^{[n]}(t)$. In particular, if $0<\alpha<1$, then $^c\mathcal{D}_{0^+}^{\alpha,\psi}h(t) = \mathcal{D}_{0^+}^{\alpha,\psi}[h(t)-h(0)]$. If $h \in C^n[0,b]$, then the left-sided ψ-Caputo fractional derivative of h of order α can be represented by the expression

$$\begin{aligned} ^c\mathcal{D}_{0^+}^{\alpha,\psi}h(t) &= \mathcal{I}_{0^+}^{n-\alpha;\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n h(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1}h_\psi^{[n]}(s)ds. \end{aligned}$$

Lemma 1 (**[4]**) *Let $\alpha > 0$, $\psi, h \in C^{n-1}[0,b]$ and $h^{(n)}$ exists almost everywhere on any bounded interval of $[0,b]$. Then*

$$\mathcal{I}_{0^+}^{\alpha,\psi}\ {}^c\mathcal{D}_{0^+}^{\alpha,\psi}h(t) = h(t) - \sum_{k=0}^{n-1}\frac{h_\psi^{[k]}(0)}{k!}(\psi(t)-\psi(0))^k.$$

In particular, if $0<\alpha<1$, we have $\mathcal{I}_{0^+}^{\alpha,\psi}\ {}^c\mathcal{D}_{0^+}^{\alpha,\psi}h(t) = h(t)-h(0)$.

Lemma 2 (**[10]**) *Let $\alpha > 0$ and $\beta > 0$. Then, we have the following semigroup property given by*

$$\mathcal{I}_{0^+}^{\alpha,\psi}\mathcal{I}_{0^+}^{\beta,\psi}h(t) = \mathcal{I}_{0^+}^{\alpha+\beta,\psi}h(t),\ \ t \in [0,b].$$

Lemma 3 (**[2]**) *Let $\alpha > 0$, $h \in C[0,b]$ and $\psi \in C^1[0,b]$. Then $\mathcal{I}_{0^+}^{\alpha;\psi}h \in C[0,b]$ and*

$$\mathcal{I}_{0^+}^{\alpha;\psi}h(0) = \lim_{t\to 0^+}\mathcal{I}_{0^+}^{\alpha;\psi}h(t) = 0.$$

The following generalization of Gronwall's lemma for singular kernels plays an important role in obtaining some of our main results.

Lemma 4 (**[15]**) *Let x, y, be two integrable functions and h continuous, with domain $[0,b]$. Let $\psi \in C[0,b]$ an increasing function such that $\psi'(t) \neq 0$, $\forall t \in [0,b]$. Assume that x and y are nonnegative and h is nonnegative and nondecreasing. If*

$$x(t) \leq y(t) + h(t)\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}x(s)ds,$$

then, for all $t \in [0,b]$, we have

$$x(t) \leq y(t) + \int_0^t \sum_{k=1}^{\infty}\frac{[h(t)\Gamma(\alpha)]^k}{\Gamma(\alpha k)}\psi'(s)(\psi(t)-\psi(s))^{\alpha k-1}y(s)ds. \tag{9}$$

Corollary 5 *Under the hypotheses of Lemma 4, let y be a nondecreasing function on $[a,b]$. Then, we have*

$$x(t) \leq y(t)E_\alpha(h(t)\Gamma(\alpha)\left[\psi(t)-\psi(0)\right]^\alpha),\ \ \forall t \in [0,b],$$

where $E_\alpha(\cdot)$ is the Mittag-Leffler function defined by $E_\alpha(t) = \sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k+1)}$.

3 Main Results

In this section, we show the equivalence between a Cauchy type problem (3)–(4) and the Volterra integral equation. Moreover, by using the method of modified Picard's iterative, we obtain the existence and uniqueness results of the given problem. Before proceeding for the main results, we introduce the lemmas needed in the sequel.

Lemma 6 *Let $0 < k < \alpha$. Then ψ-Riemann-Liouville fractional integral of a power function is given by*

$$\mathcal{I}_{0^+}^{\alpha,\psi}\left[\psi(t)-\psi(0)\right]^{-k} = \frac{\Gamma(\alpha)\Gamma(-k+1)}{\Gamma(\alpha-k+1)}\left[\psi(t)-\psi(0)\right]^{\alpha-k}. \tag{10}$$

Moreover, for all $k = \{0, 1, .., n-1\}$, $n \in \mathbb{N}$, we have

$${}^{c}\mathcal{D}_{0^+}^{\alpha,\psi}\left[\psi(t)-\psi(0)\right]^{k} = 0.$$

Proof. In view of Definition 3, we have

$$\mathcal{I}_{0^+}^{\alpha,\psi}\left[\psi(s)-\psi(0)\right]^{-k} = \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\left[\psi(s)-\psi(0)\right]^{-k} ds.$$

The integral is evaluated by the change of variable $\psi(s) = \psi(0) + z\left[\psi(t)-\psi(0)\right]$, and with the help of the Beta function defined by Eq.(5), we obtain

$$\begin{aligned}
\mathcal{I}_{0^+}^{\alpha,\psi}\left[\psi(s)-\psi(0)\right]^{-k} &= \frac{1}{\Gamma(\alpha)}\int_0^t \left[\psi(t)-\psi(s)\right]^{\alpha-1}\left[\psi(s)-\psi(0)\right]^{-k}\psi'(s)ds \\
&= \frac{1}{\Gamma(\alpha)}\left[\psi(t)-\psi(0)\right]^{\alpha-k}\int_0^1 (1-z)^{\alpha-1}z^{-k}dz \\
&= \frac{1}{\Gamma(\alpha)}\left[\psi(t)-\psi(0)\right]^{\alpha-k}\mathcal{B}(\alpha,-k+1) \\
&= \left[\psi(t)-\psi(0)\right]^{\alpha-k}\frac{\Gamma(-k+1)}{\Gamma(\alpha-k+1)}.
\end{aligned}$$

To prove that

$${}^{c}\mathcal{D}_{0^+}^{\alpha,\psi}\left[\psi(t)-\psi(0)\right]^{k} = 0. \tag{11}$$

We have from Definition 4,

$${}^{c}\mathcal{D}_{0^+}^{\alpha,\psi}\left[\psi(t)-\psi(0)\right]^{k} = \mathcal{I}_{0^+}^{n-\alpha;\psi}\,\mathcal{D}_{0^+}^{n,\psi}\left[\psi(t)-\psi(0)\right]^{k}. \tag{12}$$

Since $k < n \in \mathbb{N}$, then

$$\mathcal{D}_{0^+}^{n,\psi}\left[\psi(t)-\psi(0)\right]^{k} = \left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^n\left[\psi(t)-\psi(0)\right]^{k} = 0.$$

Replacing this last formula into Eq.(12), we conclude that the relation (11) holds. ■

Lemma 7 *For any constant function C, we have*

$${}^{c}\mathcal{D}_{0^+}^{\alpha,\psi}C = 0. \tag{13}$$

Proof. From the Definition 4, we have

$$\begin{aligned}
{}^{c}\mathcal{D}_{0^+}^{\alpha,\psi}C &= \mathcal{I}_{0^+}^{n-\alpha;\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n C \\
&= \mathcal{I}_{0^+}^{n-\alpha;\psi}\,D_{0^+}^{n;\psi}C \\
&= C\,\mathcal{I}_{0^+}^{n-\alpha;\psi}\,D_{0^+}^{n;\psi}[\psi(t)-\psi(0)]^0
\end{aligned}$$

since $n > 0$, $D_{0^+}^{n;\psi}[\psi(t)-\psi(0)]^0 = 0$. Therefore, the relation (13) holds. ■

Lemma 8 *Let $\alpha > 0$, ψ be as given in Definition 3 and $u \in C[0,b]$. Then*

$$^{c}\mathcal{D}_{0^+}^{\alpha,\psi}\mathcal{I}_{0^+}^{\alpha,\psi}u(t) = u(t), \text{ a.e.}$$

Proof. From the Definition 4, we observe that

$$^{c}\mathcal{D}_{0^+}^{\alpha,\psi}\mathcal{I}_{0^+}^{\alpha,\psi}u(t) = \mathcal{D}_{0^+}^{\alpha,\psi}\left[\mathcal{I}_{0^+}^{\alpha,\psi}u(t) - \sum_{k=0}^{n-1}\frac{\mathcal{I}_{0^+}^{\alpha,\psi}u_{\psi}^{[k]}(0)}{k!}\left[\psi(t)-\psi(0)\right]^{k}\right]. \tag{14}$$

Also, we have

$$\begin{aligned}
\mathcal{I}_{0^+}^{\alpha,\psi}u_{\psi}^{[k]}(t) &= \left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^{k}\mathcal{I}_{0^+}^{\alpha,\psi}u(t)\\
&= \mathcal{D}_{0^+}^{k,\psi}\mathcal{I}_{0^+}^{k,\psi}\mathcal{I}_{0^+}^{\alpha-k,\psi}u(t)\\
&= \frac{1}{\Gamma(\alpha-k)}\int_0^t \psi'(s)\left[\psi(t)-\psi(s)\right]^{\alpha-k-1}u(s)ds.
\end{aligned}$$

Since $u \in C[0,b]$, and by using Eq.(10), we find that

$$\left|\mathcal{I}_{0^+}^{\alpha,\psi}u_{\psi}^{[k]}(t)\right| \le \frac{\|u\|_C}{\Gamma(\alpha-k+1)}\left[\psi(t)-\psi(0)\right]^{\alpha-k}.$$

Consequently,

$$(\mathcal{I}_{0^+}^{\alpha,\psi}u)_{\psi}^{[k]}(0) = 0, \text{ for all } k = 0,1,...,n-1.$$

From the last equality with Eqs.(14), (7), and using Lemma 2, we obtain

$$\begin{aligned}
^{c}\mathcal{D}_{0^+}^{\alpha,\psi}\mathcal{I}_{0^+}^{\alpha,\psi}u(t) &= \mathcal{D}_{0^+}^{\alpha,\psi}\mathcal{I}_{0^+}^{\alpha,\psi}u(t)\\
&= \left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^{n}\mathcal{I}_{0^+}^{n-\alpha,\psi}\mathcal{I}_{0^+}^{\alpha,\psi}u(t)\\
&= \mathcal{D}_{0^+}^{n,\psi}\mathcal{I}_{0^+}^{n,\psi}u(t) = u(t).
\end{aligned}$$

■

Corollary 9 ([2]) *Let $\alpha > 0$, ψ be as given in Definition 3, and $h \in C[0,b]$. Then*

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}h(s)ds, \ t \in (0,b]. \tag{15}$$

is a solution for the Cauchy problem

$$\begin{cases} ^{c}\mathcal{D}_{0^+}^{\alpha,\psi}u(t) = h(t), \ a.e. t \in (0,b],\\ u(0) = u_0. \end{cases} \tag{16}$$

For the forthcoming analysis, we introduce additional conditions that will be used to show our main result.

(A1) $f : (0,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous with $\lim\limits_{t\to 0^+} f(t,u) = \infty$ and there exists a constant $0 < k < \alpha$ such that $\left[\psi(t)-\psi(0)\right]^{k} f(t,u)$ is a continuous function on $[0,b]\times\mathbb{R}$.

(A2) For the k above, there exists constant $L > 0$ such that

$$\left[\psi(t)-\psi(0)\right]^{k}\left|f(t,u_1)-f(t,u_2)\right| \le L\left|u_1-u_2\right| \tag{17}$$

for all $t \in [0,b]$ and for all $u_1, u_2 \in \mathbb{R}$.

Lemma 10 *Assume that (A1) and (A2) are fulfilled. The function $u \in C[0,b]$ is a solution to Cauchy problem (3)–(4) if and only if u satisfies the following Volterra integral equation*

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, u(s))ds, \quad t \in (0, b]. \tag{18}$$

Proof. We first show that for every $u \in C[0,b]$, the integral

$$\int_0^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s))ds$$

is convergent and uniformly bounded for $t \in [0,b]$. In fact every $u \in C[0,b]$ is bounded. From the hypothesis (A2), we know that $(\psi(t) - \psi(0))^k f(t, u(t)) \in C[0,b]$. So there exists a constant $M > 0$ such that

$$\left|\psi(t) - \psi(0))^k f(t, u(t))\right| \leq M \implies |f(t, u(t))| \leq M \left[\psi(t) - \psi(0)\right]^{-k},$$

for all $t \in [0,b]$. Using Lemma 6, and definition of Beta function, we get

$$\begin{aligned} &\left|\int_0^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s))ds\right| \\ \leq\ & \frac{M}{\Gamma(\alpha)} \int_0^t \psi'(s) \left[\psi(t) - \psi(s)\right]^{\alpha-1} \left[\psi(s) - \psi(0)\right]^{-k} ds \\ =\ & \frac{M\Gamma(-k+1)}{\Gamma(\alpha - k + 1)} \left[\psi(t) - \psi(0)\right]^{\alpha-k} \\ \leq\ & \frac{M}{\Gamma(\alpha)} \mathcal{B}(\alpha, -k+1) \left[\psi(b) - \psi(0)\right]^{\alpha-k}. \end{aligned}$$

This means that $\int_0^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s))ds$ is convergent and uniformly bounded for $t \in [0,b]$.

The necessity is a consequence of Lemma 1. In view of Corollary 9, suppose that $u \in C[0,b]$ is a solution of Cauchy problem (3)–(4). Applying $\mathcal{I}_{0^+}^{\alpha;\psi}$ on both sides of Eq. (3) and according to Lemma 1, we deduce that

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, u(s))ds. \tag{19}$$

On the other hand, assume that $u \in C[0,b]$ satisfying the Volterra integral equation Eq. (18), and we prove that u also satisfies the nonlinear fractional differential equation Eq. (3). Applying the fractional derivative operator ${}^c\mathcal{D}_{0^+}^{\alpha;\psi}$ on both sides of Eq. (19) with using Lemma 8 and Eq. (13), we get

$$\begin{aligned} {}^c\mathcal{D}_{0^+}^{\alpha;\psi} u(t) &= {}^c\mathcal{D}_{0^+}^{\alpha;\psi} \left[u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, u(s))ds\right] \\ &= {}^c\mathcal{D}_{0^+}^{\alpha;\psi} u_0 + {}^c\mathcal{D}_{0^+}^{\alpha;\psi} \mathcal{I}_{a^+}^{\alpha;\psi} f(t, u(t)) \\ &= f(t, u(t)). \end{aligned} \tag{20}$$

Since $f(t,.) \in C[0,b]$, the Lemma 3 shows that $\mathcal{I}_{0^+}^{\alpha,\psi} f(t, u(t))$ in Eq.(19) vanishes at the initial point $t = 0$, and thus $u(0) = u_0$. So this completes the proof. ■

Next, we prove the existence and uniqueness of solution for the Cauchy problem (3)–(4) in $C[0,b]$ by means of the modified Picard's iterative.

Theorem 11 *Assume that the hypotheses (A1) and (A2) are fulfilled, then there exists a uniquely defined function $x \in C[0,b]$ solving the Cauchy problem (3)–(4).*

Proof. In view of Lemma 10, we know that it suffices to prove that the integral equation (3) has a unique solution. To this end, the equation (18) makes sense in any interval $[0, t_1] \subset [0, b], (t_1 = t_0 + h_0,\ h_0 > 0,\ t_1 < b\ )$. Thus, we choose t_1 such that the inequality

$$L\frac{\Gamma(-k+1)}{\Gamma(\alpha-k+1)}\left[\psi(t_1)-\psi(0)\right]^{\alpha-k} < \frac{1}{2} \tag{21}$$

holds, and then prove the existence of a unique solution $x \in C[0, t_1]$ to the equation (18) on the interval $[0, t_1]$. We define a function sequence by

$$x_0^{(1)}(t) = u_0,\ t \in [0, t_1] \tag{22}$$

and

$$x_n^{(1)}(t) = u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s, x_{n-1}^{(1)}(s))ds,\ t \in [0, t_1],\ n \in \mathbb{N}. \tag{23}$$

It is clear that $x_n^{(1)} \in C[0, t_1]$. Hence, it follows from the proof of Lemma 10 that the functions sequence $x_n^{(1)}$ are well-defined for all $n = 1, 2, \cdots$. Furthermore, according to hypotheses (A1), (A2) and Lemma 6, we conclude that

$$\begin{aligned}
&\left|x_n^{(1)}(t) - x_{n-1}^{(1)}(t)\right| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\left|f(s, x_{n-1}^{(1)}(s)) - f(s, x_{n-2}^{(1)}(s))\right| ds \\
\leq\ & \frac{L}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}(\psi(s)-\psi(0))^{-k}\left|x_{n-1}^{(1)}(s) - x_{n-2}^{(1)}(s)\right| ds \\
=\ & \frac{L}{\Gamma(\alpha)}\mathcal{B}(\alpha, -k+1)\left[\psi(t)-\psi(0)\right]^{\alpha-k}\left\|x_{n-1}^{(1)} - x_{n-2}^{(1)}\right\|_\infty \\
\leq\ & \frac{L\Gamma(-k+1)}{\Gamma(\alpha-k+1)}\left[\psi(t_1)-\psi(0)\right]^{\alpha-k}\left\|x_{n-1}^{(1)} - x_{n-2}^{(1)}\right\|_\infty .
\end{aligned}$$

Therefore, by Eq. (21), we get

$$\left|x_n^{(1)}(t) - x_{n-1}^{(1)}(t)\right| \leq \frac{1}{2}\left\|x_{n-1}^{(1)} - x_{n-2}^{(1)}\right\|_\infty .$$

It follows,

$$\left|x_n^{(1)}(t) - x_{n-1}^{(1)}(t)\right| \leq \frac{1}{2^{n-1}}\left\|x_1^{(1)} - x_0^{(1)}\right\|_\infty,\ \text{for all } n = 2, 3, \cdots.$$

Thus, we have that the series $\sum_{n=1}^{\infty}\left[x_n^{(1)}(t) - x_{n-1}^{(1)}(t)\right]$ is uniformly convergent on the interval $[0, t_1]$. It then follows that $\left\{x_n^{(1)}(t)\right\}_{n=1}^{\infty}$ is uniformly convergent on $[0, t_1]$. Denote $x^{(1)}(t) = \lim_{n\to\infty} x_n^{(1)}(t)$. Then $x^{(1)} \in C[0, t_1]$, since $x_n^{(1)}$ is continuous on $[0, t_1]$ for all n. Now, we show that $x^{(1)}$ is the unique continuous solution of Cauchy problem (3)–(4) on the interval $[0, t_1]$. From (A2), we see that the expression

$$\left[\psi(t)-\psi(0)\right]^k\left|f(t, x_n^{(1)}(t)) - f(t, x^{(1)}(t))\right| \leq L\left|x_n^{(1)}(t) - x^{(1)}(t)\right| \to 0$$

uniformly as $n \to \infty$ on the interval $[0, t_1]$. Hence by Eqs.(22) and (23), we obtain

$$\begin{aligned} x^{(1)}(t) &= \lim_{n\to\infty} x_n^{(1)}(t) \\ &= \lim_{n\to\infty} \left[x_0^{(1)}(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s, x_{n-1}^{(1)}(s)) ds \right] \\ &= x_0^{(1)}(t) + \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^t \Big(\psi'(s)(\psi(t)-\psi(s))^{\alpha-1} (\psi(s)-\psi(0))^{-k} \\ &\quad \times (\psi(s)-\psi(0))^{k} f(s, x_{n-1}^{(1)}(s)) \Big) ds \\ &= x_0^{(1)}(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \Big(\psi'(s)(\psi(t)-\psi(s))^{\alpha-1} (\psi(s)-\psi(0))^{-k} \\ &\quad \times \lim_{n\to\infty} (\psi(s)-\psi(0))^{k} f(s, x_{n-1}^{(1)}(s)) \Big) ds \\ &= x_0^{(1)}(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \Big(\psi'(s)(\psi(t)-\psi(s))^{\alpha-1} (\psi(s)-\psi(0))^{-k} \\ &\quad \times (\psi(s)-\psi(0))^{k} f(s, x^{(1)}(s)) \Big) ds \\ &= x_0^{(1)}(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s, x^{(1)}(s)) ds. \end{aligned}$$

This means that Eq.(18) holds. From Lemma 10, we conclude that $x^{(1)}$ is a continuous solution of the problem (3)–(4) on $[0, t_1]$. Assume that $y \in C[0, t_1]$ is also a solution of the problem (3)–(4). Then for all $t \in [0, t_1]$,

$$y(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s, y(s)) ds.$$

In light of hypothesis (A2), and Lemma 6, then for $t \in [0, t_1]$, we have

$$\begin{aligned} \left| x^{(1)}(t) - y(t) \right| &\le \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \left| f(s, x^{(1)}(s)) - f(s, y(s)) \right| ds \\ &\le \frac{L}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} [\psi(s)-\psi(0)]^{-k} \left| x^{(1)}(s) - y(s) \right| ds \\ &\le \frac{L}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} [\psi(s)-\psi(0)]^{-k} \left\| x^{(1)} - y \right\|_\infty ds \\ &= \frac{L}{\Gamma(\alpha)} \mathcal{B}(\alpha, -k+1) [\psi(t)-\psi(0)]^{\alpha-k} \left\| x^{(1)} - y \right\|_\infty \\ &\le \frac{L\Gamma(-k+1)}{\Gamma(\alpha-k+1)} [\psi(t_1)-\psi(0)]^{\alpha-k} \left\| x^{(1)} - y \right\|_\infty . \end{aligned}$$

The last inequality with Eq.(21) lead us to

$$\left\| x^{(1)} - y \right\|_\infty \le \frac{1}{2} \left\| x^{(1)} - y \right\|_\infty .$$

This is a contradiction, and hence $x^{(1)} \equiv y$ on $[0, t_1]$. Therefore, $x^{(1)}$ is the unique continuous solution of the problem (3)-(4) on $[0, t_1]$.

Next, we discuss the solution on the interval $[t_1, t_2]$ where $t_2 = t_1 + h_1$, $h_1 > 0$ and $t_2 < b$. Thus, we choose t_2 such that

$$L \frac{\Gamma(-k+1)}{\Gamma(\alpha-k+1)} [\psi(t_2)-\psi(t_1)]^{\alpha-k} < \frac{1}{2} \tag{24}$$

holds and we prove the existence of unique solution $x \in C[t_1, t_2]$. Set

$$\phi(t) = u_0 + \int_0^{t_1} \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s, x^{(1)}(s))ds, \ t \in [t_1, t_2].$$

For $n = 1, 2, \cdots$, we define

$$x_0^{(2)}(t) = \phi(t), \ t \in [t_1, t_2]$$

and

$$x_n^{(2)}(t) = \phi(t) + \int_{t_1}^{t} \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s, x_{n-1}^{(2)}(s))ds, \quad t \in [t_1, t_2].$$

As seen above, $x^{(1)}$ is uniquely defined and continuous on $[0, t_1]$, ϕ is uniquely defined on $[t_1, t_2]$ and $\phi(t_1) = x^{(1)}(t_1)$. Moreover, one can easily conclude from (A1)–(A2) that ϕ is bounded on $[t_1, t_2]$, and hence $x_0^{(2)}$ is bounded on $[t_1, t_2]$. Now assume that $x_{n-1}^{(2)}$ is bounded on $[t_1, t_2]$, i.e. there exists a constant $L_1 \geq 0$ such that $\left|x_{n-1}^{(2)}(t)\right| \leq L_1$, for every $t \in [t_1, t_2]$. Since f is continuous on $[t_1, t_2] \times [-L_1, L_1]$, then $\left|f(t, x_{n-1}^{(2)}(t))\right| \leq L^*$ for some constant $L^* > 0$ and for all $t \in [t_1, t_2]$. Therefore, for every $t \in [t_1, t_2]$

$$\begin{aligned} \left|x_n^{(2)}(t)\right| &\leq |\phi(t)| + \int_{t_1}^{t} \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \left|f(s, x_{n-1}^{(2)}(s))\right| ds \\ &\leq \|\phi\|_\infty + \frac{[\psi(t_2)-\psi(t_1)]^\alpha}{\Gamma(\alpha+1)} L^* := \ell. \end{aligned}$$

This means that $x_n^{(2)}$ is also bounded on interval $[t_1, t_2]$. So we can conclude by mathematical inductive that $x_n^{(2)}$ is bounded and continuous on $[t_1, t_2]$ for every $n = 1, 2, \cdots$, and hence the function sequence $\left\{x_n^{(2)}\right\}$ is well-defined. We now show that $\left\{x_n^{(2)}\right\}$ is convergent uniformly for each $t \in [t_1, t_2]$. In fact, since

$$\begin{aligned} & \left|x_n^{(2)}(t) - x_{n-1}^{(2)}(t)\right| \\ \leq\ & \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \left|f(s, x_{n-1}^{(2)}(s)) - f(s, x_{n-2}^{(2)}(s))\right| ds \\ \leq\ & \frac{L}{\Gamma(\alpha)} \int_{t_1}^{t} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} [\psi(s)-\psi(0)]^{-k} \left|x_{n-1}^{(2)}(s) - x_{n-2}^{(2)}(s)\right| ds \\ \leq\ & \frac{L}{\Gamma(\alpha)} \int_{t_1}^{t} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} [\psi(s)-\psi(0)]^{-k} \left\|x_{n-1}^{(2)} - x_{n-2}^{(2)}\right\|_\infty ds \\ \leq\ & \frac{L\Gamma(-k+1)}{\Gamma(\alpha-k+1)} [\psi(t_2)-\psi(t_1)]^{\alpha-k} \left\|x_{n-1}^{(2)} - x_{n-2}^{(2)}\right\|_\infty \\ \leq\ & \frac{1}{2} \left\|x_{n-1}^{(2)} - x_{n-2}^{(2)}\right\|_\infty, \end{aligned}$$

which implies that

$$\left|x_n^{(2)}(t) - x_{n-1}^{(2)}(t)\right| \leq \frac{1}{2^{n-1}} \left\|x_1^{(1)} - x_0^{(1)}\right\|_\infty, \text{ for all } n = 2, 3, \cdots.$$

Thus, we have that the series $\sum_{n=1}^{\infty} \left[x_n^{(2)}(t) - x_{n-1}^{(2)}(t)\right]$ is uniformly convergent on the interval $[t_1, t_2]$, similar with the proof of interval $[0, t_1]$, we know that $\left\{x_n^{(2)}(t)\right\}_{n=1}^{\infty}$ are uniformly convergent on $[t_1, t_2]$. Hence, we denote by $x^{(2)}(t) = \lim_{n\to\infty} x_n^{(2)}(t)$. Then $x^{(2)} \in C[t_1, t_2]$, since $x_n^{(2)} \in C[t_1, t_2]$ for all $n = 1, 2, \cdots$. Using the same arguments as above, we can deduce that $x^{(2)}$ is the unique continuous function satisfying

$$\begin{aligned} x^{(2)}(t) &= \phi(t)+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}f(s,x^{(2)}(s))ds \\ &= u_0+\frac{1}{\Gamma(\alpha)}\int_{0}^{t_1}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}f(s,x^{(1)}(s))ds \\ &\quad +\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}f(s,x^{(2)}(s))ds, \end{aligned}$$

for $t\in[t_1,t_2]$, which is the unique solution to (3)–(4) on $[t_1,t_2]$.

Taking the following interval $[t_2,t_3]$, where $t_3=t_2+h_2$, $h_2>0$ such that $t_3<b$. By reiterating this proceeding, we conclude that there exists a unique solution $x^{(i)}$ to the equation (18) on each interval $[t_{i-1},t_i]$,. Let us we set

$$x(t)=\begin{cases} x^{(1)}(t);\ t\in[0,t_1], \\ x^{(2)}(t);\ t\in[t_1,t_2], \\ \quad\vdots \\ x^{(N)}(t);\ t\in[t_{N-1},t_N]. \end{cases}$$

Since $x^{(i)}\in C[t_{i-1},t_i]$ (for $i=1,2,\cdots,N$, and $0=t_0<t_1<t_2<\cdots<t_N=b$) and by definition of $x^{(i)}$, $i=1,2,\cdots,N$, we notice that $x(t)$ is the unique continuous solution of the problem (3)-(4) on $[0,b]$. This completes the proof. ■

Corollary 12 *Assume that the hypotheses (A1)–(A2) hold. Then the Cauchy problem (3)–(4) has a unique solution on* $[0,\infty)$.

4 Continuous Dependence

In this section, we study the data continuous dependence of the fractional differential equation including ψ-Caputo derivative via the generalized Gronwall inequality. To this end, under conditions of Theorem 11, we consider that Cauchy problem

$$^{c}D_{0^+}^{\alpha-\epsilon,\psi}u^*(t)=f(t,u^*(t)), \tag{25}$$

$$u^*(0)=u_0^*, \tag{26}$$

has a unique solution

$$u^*(t)=u_0^*+\frac{1}{\Gamma(\alpha-\epsilon)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-\epsilon-1}f(s,u^*(s))ds,$$

where $0<\alpha-\epsilon<\alpha<1$.

Theorem 13 *Let $\psi,f\in C([0,b],\mathbb{R})$ two functions such that ψ is increasing function and $\psi'(t)\neq 0$, for all $t\in[0,b]$ and f satisfying Lipschitz condition Eq.(17) in $\mathbb{R}$. Let $\alpha>0$, $\epsilon>0$ such that $0<\alpha-\epsilon<\alpha\leq 1$. Assume that u is the solution of Cauchy problem (3)–(4) and u^* is the solution of Cauchy problem (25)–(26). Then for $0<t\leq b$,*

$$\begin{aligned} &|u^*(t)-u(t)| \\ \leq\ & A(t)+\int_0^t\left[\sum_{k=1}^{\infty}\left(\frac{L\rho\Gamma(\alpha-\epsilon)}{\Gamma(\alpha)}\right)^k\frac{\psi'(s)(\psi(t)-\psi(s))^{(\alpha-\epsilon)k-1}}{\Gamma((\alpha-\epsilon)k)}A(s)\right]ds, \end{aligned}$$

where

$$\rho:=\frac{\Gamma(\alpha-k+1)}{2L\Gamma(-k+1)},$$

$$A(t) = |u_0^* - u_0| + \|f\| \left| \frac{(\psi(t)-\psi(0))^{\alpha-\epsilon}}{\Gamma(\alpha-\epsilon+1)} - \frac{(\psi(t)-\psi(0))^{\alpha-\epsilon}}{\Gamma(\alpha)\Gamma(\alpha-\epsilon)} \right| + \|f\| \left| \frac{(\psi(t)-\psi(0))^{\alpha-\epsilon}}{\Gamma(\alpha)\Gamma(\alpha-\epsilon)} - \frac{(\psi(t)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} \right|, \quad (27)$$

and $\|f\| = \sup_{(t,u)\in(0,b]\times\mathbb{R}} |f(t,u)|$.

Proof. The problems (3)–(4) and (25)–(26), have similar integral solutions and are given by

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s,u(s))ds, \ t>0$$

and

$$u^*(t) = u_0^* + \frac{1}{\Gamma(\alpha-\epsilon)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-\epsilon-1} f(s,u^*(s))ds, \ t>0,$$

respectively. It follows that

$$\begin{aligned}
|u^*(t)-u(t)| \leq\ & |u_0^* - u_0| \\
& + \left| \frac{1}{\Gamma(\alpha-\epsilon)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-\epsilon-1} f(s,u^*(s))ds \right. \\
& \left. - \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s,u(s))ds \right| \\
\leq\ & |u_0^* - u_0| \\
& + \left| \int_0^t \psi'(s) \left[\frac{(\psi(t)-\psi(s))^{\alpha-\epsilon-1}}{\Gamma(\alpha-\epsilon)} - \frac{(\psi(t)-\psi(s))^{\alpha-\epsilon-1}}{\Gamma(\alpha)} \right] f(s,u^*(s))ds \right. \\
& + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-\epsilon-1} \left[f(s,u^*(s)) - f(s,u(s)) \right] ds \\
& \left. + \int_0^t \psi'(s) \left[\frac{(\psi(t)-\psi(s))^{\alpha-\epsilon-1}}{\Gamma(\alpha)} - \frac{(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \right] f(s,u(s))ds \right| \\
\leq\ & |u_0^* - u_0| + \|f\| \left| \frac{(\psi(t)-\psi(0))^{\alpha-\epsilon}}{\Gamma(\alpha-\epsilon+1)} - \frac{(\psi(t)-\psi(0))^{\alpha-\epsilon}}{\Gamma(\alpha)\Gamma(\alpha-\epsilon)} \right| \\
& + \frac{L}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-\epsilon-1} \left[\psi(s)-\psi(0)\right]^{-k} |u^*(s)-u(s)|\, ds \\
& + \|f\| \left| \frac{(\psi(t)-\psi(0))^{\alpha-\epsilon}}{\Gamma(\alpha)\Gamma(\alpha-\epsilon)} - \frac{(\psi(t)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} \right|.
\end{aligned}$$

From Eq.(21) we note that, for $s \in (0,b]$,

$$\left[\psi(s)-\psi(0)\right]^{-k} < \left[\psi(s)-\psi(0)\right]^{\alpha-k} < \frac{\Gamma(\alpha-k+1)}{2L\Gamma(-k+1)} := \rho.$$

Hence

$$|u^*(t)-u(t)| \leq A(t) + \frac{L\rho}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-\epsilon-1} |u^*(s)-u(s)|\, ds,$$

where $A(t)$ is defined as in Eq.(27). In view of Lemma 4, we conclude that

$$\begin{aligned}
& |u^*(t)-u(t)| \\
\leq\ & A(t) + \int_0^t \left[\sum_{k=1}^{\infty} \left(\frac{L\rho\Gamma(\alpha-\epsilon)}{\Gamma(\alpha)} \right)^k \frac{\psi'(s)(\psi(t)-\psi(s))^{(\alpha-\epsilon)k-1}}{\Gamma((\alpha-\epsilon)k)} A(s) \right] ds.
\end{aligned}$$

■

Next, we discuss the continuous dependence of solution of (3)–(4) with small change in the initial condition. Consider the following fractional differential equation

$$ {}^{c}\mathcal{D}_{0^+}^{\alpha,\psi}u(t) = f(t,u(t)), \quad t>0, \tag{28} $$

$$ u(0) = u_0 + \delta, \tag{29} $$

where δ is an arbitrary positive constant.

Theorem 14 *Assume that hypotheses of Theorem 11 hold. Let u and u^* are solutions of the problems (3)–(4) and (28)–(29) respectively. Then*

$$ |u(t)-u^*(t)| \le |\delta|\, E_{\alpha-k}\left(\Gamma(-k+1)L\,[\psi(t)-\psi(0)]^{\alpha-k}\right), \ t\in[0,b]. $$

Proof. In view of Theorem 11, we have $u(t)=\lim_{n\to\infty}u_n(t)$ with

$$ u_0(t) = u_0 \tag{30} $$

and

$$ \begin{aligned} u_n(t) &= u_0(t) + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s,u_{n-1}(s))ds. \\ &= u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s,u_{n-1}(s))ds. \end{aligned} \tag{31} $$

Clearly, we can write $u^*(t)=\lim_{n\to\infty}u_n^*(t)$ with

$$ u_0^*(t) = u_0 + \delta, \tag{32} $$

and

$$ \begin{aligned} u_n^*(t) &= u_0^*(t) + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s,u_{n-1}^*(s))ds \\ &= u_0 + \delta + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s,u_{n-1}^*(s))ds, \end{aligned} \tag{33} $$

By Eq.(30) and Eq.(32) we get

$$ \begin{aligned} |u_0(t)-u_0^*(t)| &= |u_0-u_0-\delta| \\ &\le |\delta|. \end{aligned} \tag{34} $$

Using relations Eqs.(30), (31), (32), (33), the Lipschitz condition Eq.(17) and the inequality Eq.(34), we get

$$ \begin{aligned} & |u_1(t)-u_1^*(t)| \\ \le\ & |\delta| + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\,|f(s,u_0(s))-f(s,u_0^*(s)|\,ds \\ \le\ & |\delta| + \frac{L}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\,[\psi(s)-\psi(0)]^{-k}\,|u_0(t)-u_0^*(t)|\,ds \\ \le\ & |\delta| + L\,|\delta|\,\frac{\mathcal{B}(\alpha,-k+1)}{\Gamma(\alpha)}\,[\psi(t)-\psi(0)]^{\alpha-k} \\ =\ & |\delta| + L\,|\delta|\,\Gamma(-k+1)\frac{[\psi(t)-\psi(0)]^{\alpha-k}}{\Gamma(\alpha-k+1)}. \end{aligned} $$

Hence,

$$|u_1(t) - u_1^*(t)| \leq |\delta| \sum_{i=0}^{1} (\Gamma(-k+1)L)^i \frac{[\psi(t) - \psi(0)]^{(\alpha-k)i}}{\Gamma((\alpha-k)i+1)}. \tag{35}$$

On the other hand, we have

$$\begin{aligned}
& |u_2(t) - u_2^*(t)| \\
\leq \;& |\delta| \\
& + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} |f(s, u_1(s)) - f(s, u_1^*(s)| \, ds \\
\leq \;& |\delta| + \frac{L}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} [\psi(s) - \psi(0)]^{-k} |u_1(t) - u_1^*(t)| \, ds \\
\leq \;& |\delta| + \frac{L}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} [\psi(s) - \psi(0)]^{-k} \\
& \times \left[|\delta| + \mathcal{B}(\alpha, -k+1) \frac{L\,|\delta|}{\Gamma(\alpha)} [\psi(s) - \psi(0)]^{\alpha-k} \right] ds \\
= \;& |\delta| + \frac{L\,|\delta|}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} [\psi(s) - \psi(0)]^{-k} \, ds \\
& + \frac{\mathcal{B}(\alpha, -k+1)}{\Gamma(\alpha)} \frac{L^2\,|\delta|}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \left[[\psi(s) - \psi(0)]^{\alpha-2k} \right] ds \\
= \;& |\delta| + L\,|\delta| \frac{\Gamma(-k+1)}{\Gamma((\alpha-k+1)} [\psi(t) - \psi(0)]^{\alpha-k} + \left[\frac{\Gamma(-k+1)}{\Gamma((\alpha-k+1)} \right]^2 L^2\,|\delta| \, [\psi(t) - \psi(0)]^{2(\alpha-k)} \\
= \;& |\delta| \sum_{i=0}^{2} (\Gamma(-k+1)L)^i \frac{[\psi(t) - \psi(0)]^{(\alpha-k)i}}{\Gamma((\alpha-k)i+1)}.
\end{aligned}$$

Using the induction, we get

$$|u_n(t) - u_n^*(t)| \leq |\delta| \sum_{i=0}^{n} (\Gamma(-k+1)L)^i \frac{[\psi(t) - \psi(0)]^{(\alpha-k)i}}{\Gamma((\alpha-k)i+1)} \tag{36}$$

Taking the limit $n \to \infty$ in Eq.(36), we obtain

$$|u(t) - u^*(t)| \leq |\delta| \, E_{\alpha-k} \left(\Gamma(-k+1)L \, [\psi(t) - \psi(0)]^{\alpha-k} \right).$$

■

5 An Example

Fix a kernel $\psi : [0, 1] \to \mathbb{R}$. Consider the singular fractional differential equation

$${}^{c}D_{0^+}^{\frac{2}{3},\psi} u(t) = [\psi(t) - \psi(0)]^{-\frac{1}{2}} (1 + \frac{1}{9} u(t)), \quad , t \in (0, 10], \tag{37}$$

with the initial condition

$$u(0) = 4, \tag{38}$$

where $\alpha = \frac{2}{3}$, $u_0 = 4$, $f(t, u) = [\psi(t) - \psi(0)]^{-\frac{1}{2}} (1 + \frac{1}{9}u)$, for $(t, u) \in (0, 10] \times \mathbb{R}$, and $\lim_{t \to 0^+} f(t, .) = \infty$. Set $k = \frac{1}{2}$, then $[\psi(t) - \psi(0)]^k f(t, u) = (1 + \frac{1}{9}u)$ is continuous on $[0, 10]$. So the hypothesis (A1) is satisfied.

For all $u, u^* \in \mathbb{R}$, and $t \in (0, 10]$, we have

$$
\begin{aligned}
|f(t,u) - f(t,u^*)| &= [\psi(t) - \psi(0)]^{\frac{-1}{2}} \left|(1 + \frac{1}{9}u) - (1 + \frac{1}{9}u^*)\right| \\
&= \frac{1}{9} [\psi(t) - \psi(0)]^{\frac{-1}{2}} |u - u^*| .
\end{aligned}
$$

Consider $\psi(t) = \sqrt{t+1}$, for $t \in [0, 10]$, we get

$$
|f(t,u) - f(t,u^*)| = \frac{1}{9} \left[\sqrt{t+1} - 1\right]^{-\frac{1}{2}} |u - u^*| .
$$

So, the hypothesis (A2) is also satisfied with $L = \frac{1}{9}$ and $k = \frac{1}{2}$. Moreover,

$$
\frac{L\Gamma(-k+1)}{\Gamma(\alpha - k + 1)} [\psi(b) - \psi(0)]^{\alpha - k} = \frac{\sqrt{\pi}\left[\sqrt{11} - 1\right]^{\frac{1}{6}}}{9\Gamma(\frac{7}{6})} < \frac{1}{2}.
$$

Therefore, all the assumptions in Theorem 1 are fulfilled. This implies that the Cauchy problem (37)-(38) has a uniquely continuous solution on $[0, 10]$.

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