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A Real Eigenvector Of Circulant Matrices And A Conjecture Of Ryser*

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Abstract

We prove that there is no circulant Hadamard matrix H with first row $[h_1, \ldots, h_n]$ of order $n > 4$, under a condition about a sum of scalar products of rows of two other circulant matrices of size $n/2$ associated to H.

1 Introduction

A matrix of order n is a square matrix with n rows. A *circulant* matrix $A := \text{circ}(a_1, \ldots, a_n)$ of order n is a matrix of order n of first row $[a_1, \ldots, a_n]$ in which each row after the first is obtained by a cyclic shift of its predecessor by one position. For example, the second row of A is $[a_n, a_1, \ldots, a_{n-1}]$. A *Hadamard* matrix H of order n is a matrix of order n with entries in $\{-1, 1\}$ such that $K := \frac{H}{\sqrt{n}}$ is an orthogonal matrix. A *circulant Hadamard* matrix of order n is a circulant matrix that is Hadamard. Besides the two trivial matrices of order 1 $H_1 := \text{circ}(1)$ and $H_2 := -H_1$ the remaining 8 known circulant Hadamard matrices are $H_3 := \text{circ}(1, -1, -1, -1), H_4 := -H_3, H_5 := \text{circ}(-1, 1, -1, -1), H_6 := -H_5,\ H_7 := \text{circ}(-1, -1, 1, -1), H_8 := -H_7, H_9 := \text{circ}(-1, -1, -1, 1), H_{10} := -H_9$.

If $H = \text{circ}(h_1, \ldots, h_n)$ is a circulant Hadamard matrix of order n then its *representer* polynomial is the polynomial $R(x) := h_1 + h_2x + \cdots + h_nx^{n-1}$.

No one has been able to discover any other circulant Hadamard matrix. Ryser proposed in 1963 (see [12], [1, p. 97]) the conjecture of the non-existence of these matrices when $n > 4$. Preceding work on the conjecture includes [2, 3, 5, 6, 7, 8, 10, 11, 13].

The object of the present paper is to prove the conjecture, under a mild condition, in a new special case related to some properties of the real eigenvector v with all entries equal to 1 of any circulant matrix. The condition holds for the 8 circulant Hadamard matrices of order 4.

In fact the object of the present paper is to prove the following theorem.

Theorem 1 *Let $H = \text{circ}(h_1, \ldots, h_n)$ be a circulant Hadamard matrix of order $n \geq 4$. Let $E_1 := circ(h_1, h_3, h_5, \ldots, h_{n-1})$ and let $E_2 := circ(h_2, h_4, h_6, \ldots, h_n)$. Then $n = 4$ provided E_1 and E_2 are non singular and*

$$\sum_{j\neq 1, 1\leq j\leq n/2} \langle R_1, R_j\rangle + \sum_{j\neq 1, 1\leq j\leq n/2} \langle S_1, S_j\rangle = \sum_{j\neq 1, 1\leq j\leq n} \langle T_1, T_j\rangle = 0, \tag{1}$$

where R_j, (respectively S_j, T_j is the j-th row of the matrix E_1 (respectively of the matrices, E_2 and H) and the $\langle , \rangle$ is the usual scalar product.

Remark 1 *When $n = 4$, (1) holds for all 8 circulant Hadamard matrices $H_3, \ldots, H_{10}$.*

Remark 2 *The condition (1) on Theorem 1 can be proved heuristically as follows: Observe that, by writing explicitly, say, the first three rows T_1, T_2, and T_3 of H, we obtain*

$$\langle R_1, R_2\rangle + \langle S_1, S_2\rangle = \langle T_1, T_3\rangle.$$

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But H is Hadamard, thus $\langle T_1, T_3 \rangle = 0$. Continuing in this manner we might eventually obtain the condition (if it is true).

The necessary tools for the proof of the theorem are given in Section 2. The proof of Theorem 1 is presented in Section 3.

2 Tools

Our first tool is well known ([1]) and easy to prove.

Lemma 2 *Let $C := circ(c_1, \ldots, c_k)$ be a circulant matrix of order k. Let $v := [1, \ldots, 1] \in \mathbb{R}^k$. Then, v is an eigenvector of C with asociated eigenvalue $\lambda := c_1 + \cdots + c_k$.*

The following is well known. See, e.g., [4, p. 1193], [9, p. 234], [13, pp. 329-330].

Lemma 3 *Let H be a regular Hadamard matrix of order $n \geq 4$, i.e., a Hadamard matrix whose row and column sums are all equal. Then $n = 4h^2$ for some positive integer h. Moreover, the row and column sums are all equal to $\pm 2h$ and each row has $2h^2 \pm h$ positive entries and $2h^2 \mp h$ negative entries. Finally, if H is circulant then h is odd.*

Lemma 4 *Let H be a circulant Hadamard matrix of order n, let $w = \exp(2\pi i/n)$ and let $R(x)$ be its representer polynomial. Then*

(a) *all the eigenvalues $R(s)$ of H, where $s \in \{1, w, w^2, \ldots, w^{n-1}\}$, satisfy*

$$|R(s)| = \sqrt{n}.$$

(b) *the vector $v := [1, \ldots, 1] \in \mathbb{R}^n$ is an eigenvector of H with associated eigenvalue $\lambda = \sqrt{n}$.*

3 Proof of Theorem 1

Assume that $n > 4$. By Lemma 3, n is even. Put

$$a := \sum_{j \neq 1, 1 \leq j \leq n/2} \langle R_1, R_j \rangle \text{ and } b := \sum_{j \neq 1, 1 \leq j \leq n/2} \langle S_1, S_j \rangle,$$

put also $\lambda_1 := h_1 + h_3 + \cdots + h_{n-1}$, the real eigenvalue of E_1 associated to the eigevector $v_0 := [1, \ldots, 1] \in \mathbb{R}^{n/2}$, and $\lambda_2 := h_2 + h_4 + \cdots + h_n$, the real eigenvalue of E_2 associated to the same eigenvector v_0, (see Lemma 2). Since all $h_j^2 = 1$ we get $\langle R_1, R_1 \rangle = n/2 = \langle S_1, S_1 \rangle$. Thus

$$a + n/2 = \langle R_1, \sum_{1 \leq j \leq n/2} R_j \rangle = \langle R_1, \lambda_1 v_0 \rangle = \lambda_1^2, \tag{2}$$

and

$$b + n/2 = \langle S_1, \sum_{1 \leq j \leq n/2} S_j \rangle = \langle S_1, \lambda_2 v_0 \rangle = \lambda_2^2. \tag{3}$$

Now, our condition (1) says that

$$a + b = 0 \tag{4}$$

It follows then from (4) together with (2) and (3) that one has indeed

$$\lambda_1^2 + \lambda_2^2 = n. \tag{5}$$

But, it follows from Lemma 4 that

$$\lambda_1 + \lambda_2 = h_1 + h_2 + h_3 + + \cdots + h_n \in \{\sqrt{n}, -\sqrt{n}\}. \tag{6}$$

Clearly, we deduce from (5) and (6) the contradiction

$$\lambda_1 \lambda_2 = 0, \tag{7}$$

thereby finishing the proof of the theorem.

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