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On The Gamma Gumbel Distribution[*]

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Abstract

In this paper, we study some mathematical properties of the gamma Gumbel distribution. We provide explicit expressions for the moments, quantile function, Rényi entropy, and order statistics. We also discuss the estimation of the model parameters using the maximum likelihood technique, method of moments and Bayesian method. We provide an application to a real data set which illustrates the usefulness of the model.

1 Introduction

The Gumbel distribution is one of the most widely applied statistical distributions in the engineering problems. A book written by Kotz and Nadarajah [21] lists over 50 applications of the Gumbel distribution. The probability density function (pdf) of this distribution is

$$f(x) = \frac{u}{\sigma} \exp\{-u\}, \quad x \in \mathbb{R}, \quad \sigma > 0,$$

where $u = \exp\left(-\frac{x-\mu}{\sigma}\right)$ and $\mu \in \mathbb{R}$. The corresponding cumulative distribution function (cdf) of this distribution is

$$F(x) = \exp\{-u\},$$

where μ and σ are the location and scale parameters.

Nadarajah and Kotz [27] introduced a generalization of the Gumbel distribution, called the beta Gumbel (BG) distribution, whose pdf is given by

$$f_{BG}(x) = \frac{u \exp(-\alpha u)\{1-\exp(-u)\}^{\beta-1}}{\sigma B(\alpha, \beta)}, \quad x \in \mathbb{R}, \quad \alpha,\, \beta,\, \sigma > 0, \tag{1}$$

where $u = \exp(-\frac{x-\mu}{\sigma})$, $\mu \in \mathbb{R}$, and $B(\cdot,\cdot)$ is the complete beta function. The corresponding cdf of the BG distribution is

$$F_{BG}(x) = I_{\exp(-u)}(\alpha, \beta), \quad -\infty < x < +\infty,$$

where $I_x(a,b) = \frac{1}{B(a,b)} \int_0^x t^{a-1}(1-t)^{b-1} \mathrm{d}t$ is the incomplete beta ratio function.

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A special case of the BG distribution is the exponentiated Gumbel (EG) distribution (set $\alpha = 1$ in (1)), discussed by [26], with the following pdf

$$f_{EG}(x) = \frac{\beta u}{\sigma} \exp(-u) \{1 - \exp(-u)\}^{\beta - 1}, \quad x \in \mathbb{R}, \quad \beta, \sigma > 0, \tag{2}$$

where u is defined as before and $\mu \in \mathbb{R}$. The corresponding cdf of the EG distribution is

$$F_{EG}(x) = 1 - \{1 - \exp(-u)\}^{\beta}.$$

Gholami [15] worked on the Bayesian estimation of the shape and scale parameters of the EG distribution (The location parameter was taken to be zero).

Cordeiro et al. [9] discussed the properties of the Kumaraswamy Gumbel (KG) distribution, proposed by [8], with the following pdf

$$f^*_{KG}(x) = \frac{\alpha\beta u}{\sigma} \exp(-\alpha u)\{1 - \exp(-\alpha u)\}^{\beta - 1}, \quad x \in \mathbb{R}, \quad \alpha, \beta, \sigma > 0, \tag{3}$$

where u is defined as before and $\mu \in \mathbb{R}$.

We can rewrite (3) as

$$\begin{aligned} f_{KG}(x) &= \frac{\beta}{\sigma} \alpha \exp\left(\frac{\mu}{\sigma}\right) \exp\left(\frac{-x}{\sigma}\right) \exp\left[-\alpha \exp\left(\frac{\mu}{\sigma}\right) \exp\left(\frac{-x}{\sigma}\right)\right] \\ &\times \left\{1 - \exp\left[-\alpha \exp\left(\frac{\mu}{\sigma}\right) \exp\left(\frac{-x}{\sigma}\right)\right]\right\}^{\beta - 1}. \end{aligned} \tag{4}$$

Take $\eta = \alpha \exp\left(\frac{\mu}{\sigma}\right)$ in (4), then we have

$$f_{KG}(x) = \frac{\beta\eta}{\sigma} \exp\left(\frac{-x}{\sigma}\right) \exp\left[-\eta \exp\left(\frac{-x}{\sigma}\right)\right] \left\{1 - \exp\left[-\eta \exp\left(\frac{-x}{\sigma}\right)\right]\right\}^{\beta - 1}.$$

Therefore $f^*_{KG}(x)$ is not identifiable and we may eliminate either parameter α or μ from (3) to attain an identifiable distribution. Setting $\alpha = 1$ in (3), we can eliminate parameter α from the model and then we arrive at the pdf of the EG distribution. Summing up, we conclude that the KG distribution is not a new distribution and it is in fact the EG distribution.

Another generalization of the Gumbel distribution is the exponentiated generalized Gumbel (EEG) distribution, that was introduced by [10]. The mathematical properties of the EEG distribution and estimation of its parameters were discussed by [2]. The pdf of the EEG is given by

$$f_{EEG}(x) = \frac{\alpha\beta u\, \mathrm{e}^{-u}}{\sigma} \{1 - \exp(-u)\}^{\alpha - 1} \{1 - (1 - \exp(-u))^{\alpha}\}^{\beta - 1}, \quad x \in \mathbb{R}, \ \alpha, \beta, \sigma > 0,$$

where u is defined as before and $\mu \in \mathbb{R}$.

The McDonald Gumbel (MG) distribution was also proposed by [11]. The pdf of this model is given by

$$f_{MG}(x; \mu, \sigma, \alpha, \beta, \lambda) = \frac{\lambda\, u \exp(-\alpha\, \lambda\, u)\{1 - \exp(-\lambda\, u)\}^{\beta - 1}}{\sigma B(\alpha, \beta)}, \quad x \in \mathbb{R}, \ \lambda, \alpha, \beta, \sigma > 0, \tag{5}$$

where u is defined as before and $\mu \in \mathbb{R}$.

We can rewrite (5) as

$$\begin{aligned} f^*_{MG}(x) &= \frac{1}{\sigma B(\alpha, \beta)} \lambda \exp\left(\frac{\mu}{\sigma}\right) \exp\left(\frac{-x}{\sigma}\right) \exp\left[-\alpha\, \lambda \exp\left(\frac{\mu}{\sigma}\right) \exp\left(\frac{-x}{\sigma}\right)\right] \\ &\times \left\{1 - \exp\left[-\alpha\, \lambda \exp\left(\frac{\mu}{\sigma}\right) \exp\left(\frac{-x}{\sigma}\right)\right]\right\}^{\beta - 1}. \end{aligned} \tag{6}$$

Take $\eta = \lambda \exp\left(\frac{\mu}{\sigma}\right)$ in (6), then we have

$$f^*_{MG}(x) = \frac{\eta \exp\left(\frac{-x}{\sigma}\right)}{\sigma B(\alpha,\beta)} \exp\left[-\alpha\eta \exp\left(\frac{-x}{\sigma}\right)\right] \left\{1 - \exp\left[-\alpha\eta \exp\left(\frac{-x}{\sigma}\right)\right]\right\}^{\beta-1}.$$

Therefore $f_{MG}(x)$ is not identifiable and either parameter λ or μ may be eliminated from (5) to obtain an identifiable distribution. We can eliminate parameter λ from the MG model by means of setting $\lambda = 1$ in (5) and then we arrive at the pdf of the BG distribution. Therefore, we conclude that the MG distribution is not new and it is in fact the BG distribution.

Moreover, Aryal and Tsokos [3] and Deka et al. [12] introduced the transmuted Gumbel (TG) distribution and the transmuted exponentiated Gumbel (TEG) distribution, respectively. The pdf and cdf of the TEG distribution are given by

$$f_{TEG}(x) = \frac{\alpha u\, e^{-u}}{\sigma}\{1-\exp(-u)\}^{\alpha-1}\{1-\beta+2\beta(1-\exp(-u))^{\alpha}\}, \quad x \in \mathbb{R},\ \alpha,\,\sigma > 0, \tag{7}$$

and

$$F_{TEG}(x) = 1 - (1-\exp(-u))^{\alpha}\{1-\beta+\beta(1-\exp(-u))^{\alpha}\},$$

respectively, where u is defined as before, $\mu \in \mathbb{R}$ and $-1 < \beta < 1$. The TG distribution is a special case of the TEG distribution, whose pdf is obtained by setting $\alpha = 1$ in (7). In addition, the five-parameter beta exponentiated Gumbel distribution was recently introduced by [28].

Recently, Amini et al. [1] discussed a family of distributions, called the *log-gamma generated family of distributions*. Suppose that $g(x)$ and $G(x)$ are the pdf and cdf of a random variable X, respectively, then the pdf of the log-gamma generated-G distribution is given by

$$f_G(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}[-\log(G(x))]^{\alpha-1}[G(x)]^{\beta-1}g(x), \quad \alpha,\,\beta > 0. \tag{8}$$

One member of the log-gamma generated-G family of distributions, as hinted by [1], is the gamma Gumbel (GG) distribution, that can be obtained by letting $g(x)$ and $G(x)$ in (8) to be the pdf and cdf of the Gumbel distribution, respectively. Consequently, the pdf of the GG distribution is given by

$$f^*(x) = \frac{\beta^{\alpha}}{\sigma\Gamma(\alpha)} \exp\left\{-\frac{\alpha(x-\mu)}{\sigma} - \beta\exp\left(-\frac{x-\mu}{\sigma}\right)\right\}, \quad x \in \mathbb{R}, \quad \alpha,\beta,\sigma > 0,\ \mu \in \mathbb{R}. \tag{9}$$

As $f^*(x)$ is not identifiable, we may eliminate parameter β from the model by setting $\beta = 1$ in (9) and therefore, we arrive at the following pdf

$$f(x) = \frac{1}{\sigma\Gamma(\alpha)} \exp\left\{-\frac{\alpha(x-\mu)}{\sigma} - \exp\left(-\frac{x-\mu}{\sigma}\right)\right\}, \qquad x \in \mathbb{R}, \quad \alpha,\sigma > 0,\ \mu \in \mathbb{R}. \tag{10}$$

We write $X \sim GG(\mu,\sigma,\alpha)$ if the pdf of X can be expressed as (10). The Gumbel distribution is recovered as a submodel of the GG distribution by setting $\alpha = 1$ in (10). As [1] stated, log-gamma generated-G family of distributions are suitable for modelling highly skewed and heavy-tailed data. Therefore, we hope that the GG distribution can be a proper model for skewed data sets that arise in many engineering applications. We note that Amini et al. [1] did not focus on the special case, the GG distribution and they studied several properties of the log-gamma generated-G family of distributions in general. In other words, the mathematical properties and application of the GG distribution were not discussed in detail by [1]. Here, we want to work on some properties of the GG distribution more specifically using both properties of the Gumbel distribution and the properties of the log-gamma generated-G family of distributions. In addition, Amini et al. [1] discussed the problem of parameter estimation for the log-gamma generated-G family of distributions with the help of two classical methods, namely the maximum likelihood (ML) approach and method of moments, in general. Here, we discuss the problem of estimation of the parameters of the GG distribution using a Bayesian approach as well as the ML technique and method of moments.

The rest of the paper is organized as follows: The mathematical properties of the GG model are presented in Section 2. Section 3 is devoted to the estimation of the parameters of the GG distribution using the ML technique, method of moments and Bayesian procedure. An application of the GG distribution is given in Section 4.

2 Mathematical Properties

In this section, we derive various mathematical properties of the GG distribution.

2.1 Distribution Function

The cdf of the GG model is

$$F(x)=\frac{1}{\sigma\Gamma(\alpha)}\int_{-\infty}^{x}\exp\left(-\frac{\alpha(t-\mu)}{\sigma}\right)\exp\left(-\exp\big(-\frac{t-\mu}{\sigma}\big)\right)\mathrm{d}t.$$

Taking $z=\exp\left(-\frac{t-\mu}{\sigma}\right)$, we have

$$\begin{aligned}F(x) &= \frac{1}{\Gamma(\alpha)}\int_{u}^{\infty} z^{\alpha-1}\exp(-z)\mathrm{d}z = 1-\frac{1}{\Gamma(\alpha)}\int_{0}^{u} z^{\alpha-1}\exp(-z)\ \mathrm{d}z \\ &= 1-\frac{\Gamma(\alpha,u)}{\Gamma(\alpha)} = 1-\frac{1}{\Gamma(\alpha)}\sum_{m=0}^{\infty}\frac{(-1)^m u^{m+\alpha}}{m!(m+\alpha)},\end{aligned} \tag{11}$$

where u is defined as before and $\Gamma(a,x)=\int_0^x t^{a-1}e^{-t}\mathrm{d}t$ is the incomplete gamma function.

2.2 Shape Characteristics of the Pdf and Hazard Function

It follows from (10) that

$$\frac{\partial f(x)}{\partial x}=\frac{u^{\alpha}}{\sigma^2\Gamma(\alpha)}\exp(-u)\,(u-\alpha)\,. \tag{12}$$

Equating (12) with zero, we arrive at $x_0=\mu-\sigma\log(\alpha)$. Now, the second derivative of $f(x)$ with respect to x is

$$\frac{\partial^2 f(x)}{\partial x^2} = \frac{u^{\alpha}}{\sigma^3\Gamma(\alpha)}\exp\left(-u\right)\left[\alpha^2-2\alpha u-u+u^2\right].$$

Therefore we have

$$\left.\frac{\partial^2 f(x)}{\partial x^2}\right|_{x=x_0}=-\frac{\alpha^{\alpha+1}\exp(-\alpha)}{\sigma^3\Gamma(\alpha)}<0.$$

So we conclude that the pdf of the GG distribution is unimodal and attains its maximum at x_0.

Next, we consider the hazard rate function (hrf) that is defined by $h(x)=\frac{f(x)}{1-F(x)}$. For the GG distribution, we have

$$h(x)=\frac{u^{\alpha}\exp(-u)}{\sigma\Gamma(\alpha,u)}.$$

Figure 1 plots $f(x)$ and $h(x)$ for selected values α when $\mu=0$ and $\sigma=1$. From Figure 1, we can see that the pdf is unimodal and the hrf can be an increasing function of x.

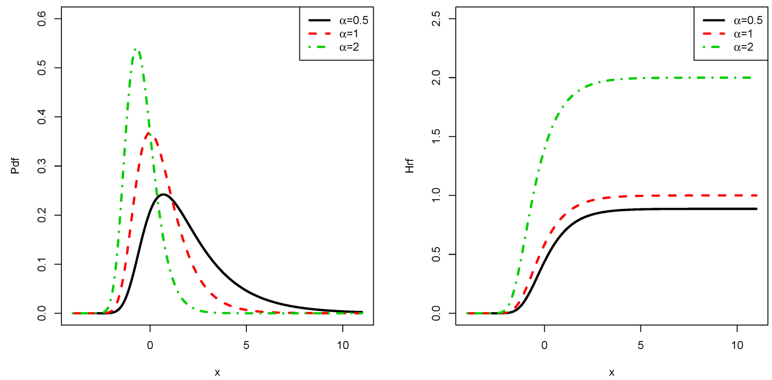

Figure 1: Pdfs (left) and hrfs (right) of the GG distribution for selected values α when $\mu = 0$ and $\sigma = 1$.

2.3 Moments

Let $X \sim GG(\mu, \sigma, \alpha)$. The moment generating function of X is

$$\begin{aligned} M_X(t) &= \frac{1}{\sigma\Gamma(\alpha)} \int_{-\infty}^{\infty} \exp(tx) \exp\left(\frac{-\alpha(x-\mu)}{\sigma} - \exp\big(-\frac{x-\mu}{\sigma}\big)\right) \mathrm{d}x \\ &= \frac{\exp(t\mu)}{\Gamma(\alpha)} \int_0^{\infty} u^{\alpha - t\sigma - 1} \exp(-u) \mathrm{d}u \\ &= \frac{\exp(t\mu)}{\Gamma(\alpha)} \Gamma(\alpha - t\sigma), \qquad t < \frac{\alpha}{\sigma}. \end{aligned}$$

The n-th moment of X can be derived as

$$\begin{aligned} E(X^n) &= \frac{1}{\sigma\Gamma(\alpha)} \int_0^{\infty} x^n \exp\left(\frac{-\alpha(x-\mu)}{\sigma}\right) \exp\left(-\exp(-\frac{x-\mu}{\sigma})\right) \mathrm{d}x \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{\infty} (\mu - \sigma \ln u)^n u^{\alpha-1} \exp(-u) \mathrm{d}u. \end{aligned}$$

If n is an integer value, then using the binomial expansion, we have

$$\begin{aligned} E(X^n) &= \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \binom{n}{k} \mu^{n-k} (-\sigma)^k \int_0^{\infty} (\ln u)^k u^{\alpha-1} \exp(-u) \mathrm{d}u \\ &= \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \binom{n}{k} \mu^{n-k} (-\sigma)^k \frac{\partial^k}{\partial \alpha^k} \{\Gamma(\alpha)\}, \end{aligned}$$

where the last equality is obtained from Equation (4.358.5) in [18]. The first, second, third and fourth moments of X are simplified as

$$E(X) = \mu - \sigma\Psi(\alpha), \tag{13}$$

$$E(X^2) = \mu^2 - 2\,\mu\,\sigma\Psi(\alpha) + \sigma^2 \frac{\Gamma''(\alpha)}{\Gamma(\alpha)}, \tag{14}$$

$$E(X^3) = \mu^3 - 3\,\mu^2\,\sigma\Psi(\alpha) + 3\,\mu\,\sigma^2 \frac{\Gamma''(\alpha)}{\Gamma(\alpha)} - \sigma^3 \frac{\Gamma'''(\alpha)}{\Gamma(\alpha)}, \tag{15}$$

$$E(X^4) = \mu^4 - 4\,\mu^3\,\sigma\Psi(\alpha) + 6\,\mu^2\,\sigma^2\frac{\Gamma''(\alpha)}{\Gamma(\alpha)} - 4\,\mu\,\sigma^3\frac{\Gamma'''(\alpha)}{\Gamma(\alpha)} + \sigma^4\frac{\Gamma''''(\alpha)}{\Gamma(\alpha)},$$

respectively, where $\Psi(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}$ is the digamma function and $\Gamma'(\alpha), \Gamma''(\alpha), \Gamma'''(\alpha)$ and $\Gamma''''(\alpha)$ are the first, second, third and fourth derivatives of the gamma function, respectively.

The variance, skewness and kurtosis of $X \sim GG(\mu, \sigma, \alpha)$ can be computed using the following relations:

$$\begin{aligned} Var(X) &= E(X^2) - E^2(X), \\ Skewness(X) &= \frac{E(X^3) - 3E(X)E(X^2) + 2E^3(X)}{Var^{\frac{3}{2}}(X)}, \\ Kurtosis(X) &= \frac{E(X^4) - 4E(X)E(X^3) + 6E(X^2)E^2(X) - 3E^4(X)}{Var^2(X)}. \end{aligned}$$

2.4 Quantile Function

The quantile function of X can be determined as

$$F^{-1}(z) = \mu - \sigma \log\left(Q^{-1}(1-z;\alpha)\right),$$

for $0 < z < 1$, where $Q^{-1}(y;\alpha)$ is the inverse function of $Q(y;\alpha) = \frac{\Gamma(\alpha,y)}{\Gamma(\alpha)}$.

2.5 Entropies

The entropy represents a measure of uncertainty of a random variable. The Rényi entropy of a random variable X is defined as

$$I_R(\gamma) = \frac{1}{1-\gamma}\log\int_{-\infty}^{\infty} f^{\gamma}(x)\mathrm{d}x,$$

where $\gamma > 0$ and $\gamma \neq 1$. For the GG distribution, we have

$$\begin{aligned} \int_{-\infty}^{\infty} f^{\gamma}(x) &= \frac{1}{\sigma^{\gamma}\left(\Gamma(\alpha)\right)^{\gamma}}\int_{-\infty}^{\infty}\exp\left(-\frac{\alpha\gamma(x-\mu)}{\sigma}\right)\exp\left(-\gamma\exp(-\frac{x-\mu}{\sigma})\right)\mathrm{d}x \\ &= \frac{1}{\sigma^{\gamma-1}\left(\Gamma(\alpha)\right)^{\gamma}}\int_{0}^{\infty} u^{\alpha\gamma-1}\exp(-\gamma u)\mathrm{d}u \\ &= \frac{\Gamma(\alpha\gamma)}{\gamma^{\alpha\gamma}\left(\Gamma(\alpha)\right)^{\gamma}\sigma^{\gamma-1}}. \end{aligned}$$

Therefore, the Rényi entropy of $X \sim GG(\mu, \sigma, \alpha)$ is

$$I_R(\gamma) = \frac{1}{1-\gamma}\left(\log[\Gamma(\alpha\gamma)] - \alpha\gamma\log\gamma - \gamma\log[\Gamma(\alpha)] - (\gamma-1)\log\sigma\right).$$

The Shannon entropy is a special case of the Rényi entropy when $\gamma \uparrow 1$ that is defined for a random variable X as $E[-\log f(X)]$. For the GG distribution, we can write

$$\begin{aligned} E[-\log f(X)] &= \log\sigma + \log[\Gamma(\alpha)] + \frac{\alpha}{\sigma}(E(X) - \mu) + E\left[\exp\left(-\frac{X-\mu}{\sigma}\right)\right] \\ &= \log\sigma + \log[\Gamma(\alpha)] + \alpha(1 - \Psi(\alpha)). \end{aligned}$$

2.6 Order Statistics

Let $X_1, \cdots, X_n$ be a random sample from the GG distribution with parameters μ, σ and α, and let $X_{i:n}$ denote the ith order statistic, then its pdf is given by

$$\begin{aligned} f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} f(x) \left(1-\overline{F}(x)\right)^{i-1} \left(\overline{F}(x)\right)^{n-i} \\ &= \frac{n!}{(i-1)!(n-i)!} f(x) \sum_{j=0}^{i-1} \binom{i-1}{j} (-1)^j \left(\overline{F}(x)\right)^{n+j-i}. \end{aligned}$$

From (11), we find

$$f_{i:n}(x) = \frac{n! \exp(-u)}{(i-1)!(n-i)!\sigma\Gamma(\alpha)} \sum_{j=0}^{i-1} \binom{i-1}{j} (-1)^j \frac{u^{(n+j-i+1)\alpha}}{[\Gamma(\alpha)]^{n+j-i}} \left(\sum_{m=0}^{\infty} \frac{(-1)^m u^m}{m!(m+\alpha)} \right)^{n+j-i}.$$

For any positive integer q, we have (see [18], section 0.314)

$$\left(\sum_{i=0}^{\infty} a_i x^i \right)^q = \sum_{i=0}^{\infty} c_{q,i} x^i,$$

where the coefficients $c_{q,i}$ (for $i = 1, 2, ...$) are easily obtained from the following recurrence equation:

$$c_{q,i} = (i a_0)^{-1} \sum_{k=1}^{i} (kq - i + k) a_k c_{q,i-k},$$

with $c_{q,0} = a_0^q$. Therefore, we can write

$$\left(\sum_{m=0}^{\infty} \frac{(-1)^m u^m}{m!(m+\alpha)} \right)^{n+j-i} = \sum_{m=0}^{\infty} c_{n+j-i,m} (-u)^m,$$

where

$$c_{n+j-i,m} = \frac{\alpha}{m} \sum_{k=1}^{m} \frac{k(n+j-i+1) - m}{k!(k+\alpha)} c_{n+j-i,m-k},$$

and $c_{n+j-i,0} = \alpha^{-(n+j-i)}$. The quantity $c_{n+j-i,m}$ can be obtained from $c_{n+j-i,0}, ..., c_{n+j-i,m-1}$ and then from $w_0, ..., w_m$ where $w_r = 1/[r!(r+\alpha)]$ for $r = 0, ..., m$. Combining the terms, we have

$$f_{i:n}(x) = \frac{n! \exp(-u)}{(i-1)!(n-i)!\,\sigma} \sum_{j=0}^{i-1} \sum_{m=0}^{\infty} \frac{\binom{i-1}{j}(-1)^{j+m}}{[\Gamma(\alpha)]^{n+j-i+1}} c_{n+j-i,m} u^{(n+j-i+1)\alpha+m},$$

where $u = \exp\left(-\frac{x-\mu}{\sigma}\right)$ and $x \in \mathbb{R}$.

3 Estimation of the Parameters

In this section, we discuss how to estimate the parameters of the GG distribution by means of employing three well-known methods, namely the ML method, method of moments and Bayesian method.

3.1 Maximum Likelihood Estimation

In this subsection, we consider the estimation of the unknown parameters using the ML method. Let $\boldsymbol{x} = (x_1, x_2, ..., x_n)$ be a set of an observed random sample of size n from the GG distribution with the set of parameters $\boldsymbol{\theta} = (\mu, \sigma, \alpha)$. Then the likelihood function of $\boldsymbol{\theta}$, given $\boldsymbol{x}$, is

$$L(\boldsymbol{\theta}|\boldsymbol{x}) = \sigma^{-n}\left[\Gamma(\alpha)\right]^{-n} \exp\left\{-\frac{n\alpha(\bar{x}-\mu)}{\sigma} - \sum_{i=1}^{n} \exp\left(-\frac{x_i-\mu}{\sigma}\right)\right\}. \tag{16}$$

The log-likelihood function of $\boldsymbol{\theta}$ based on the given sample is also given by

$$\ell(\boldsymbol{\theta}) = -n\log\sigma - n\log\left(\Gamma(\alpha)\right) - \frac{\alpha}{\sigma}\sum_{i=1}^{n} x_i + \frac{n\alpha\mu}{\sigma} - \sum_{i=1}^{n}\exp\left(-\frac{x_i-\mu}{\sigma}\right). \tag{17}$$

The ML estimates of μ, σ and α are obtained by maximizing (17) with respect to the parameters. In this paper, we used the Maximize function in the package Optimization in Maple 17 to find the ML estimates. We note that the Maximize function does not need initial values to maximize real-valued functions.

In addition, upon differentiating the log-likelihood function with respect to the parameters and equating them with zero, we have

$$\begin{aligned}
\frac{\partial\ell(\boldsymbol{\theta})}{\partial\mu} &= \frac{n\alpha}{\sigma} - \frac{1}{\sigma}\sum_{i=1}^{n}\exp\left(-\frac{x_i-\mu}{\sigma}\right) = 0,\\
\frac{\partial\ell(\boldsymbol{\theta})}{\partial\sigma} &= \frac{\alpha}{\sigma^2}\sum_{i=1}^{n} x_i - \frac{n}{\sigma} - \frac{n\alpha\mu}{\sigma^2} - \sum_{i=1}^{n}\frac{(x_i-\mu)}{\sigma^2}\exp\left(-\frac{x_i-\mu}{\sigma}\right) = 0,\\
\frac{\partial\ell(\boldsymbol{\theta})}{\partial\alpha} &= -n\Psi(\alpha) - \frac{1}{\sigma}\sum_{i=1}^{n} x_i + \frac{n\mu}{\sigma} = 0,
\end{aligned}$$

where $\Psi(\alpha) = \frac{\partial(\log\Gamma(\alpha))}{\partial\alpha}$ is the digamma function. Solving the above nonlinear equations numerically can help us find the ML estimates of the parameters as well. One can use the nleqslv function in the R contributed package nleqslv [19] to solve the above equations. A system of nonlinear equations can be solved by either a Broyden or a full Newton method using the nleqslv function, see [19] for more details.

3.2 Method of Moments

Here, we consider another well-known method of estimation, namely the method of moments (MM). Given the random sample $x_1, x_2, ..., x_n$ from the GG distribution with the set of parameters $\boldsymbol{\theta} = (\mu, \sigma, \alpha)$, the r-th sample moment is given by $\frac{1}{n}\sum_{i=1}^{n} x_i^r$. The MM estimates of the parameters of the three-parameter GG distribution are obtained by equating the first three theoretical moments with the first three sample moments. Therefore from (13), (14) and (15), the MM estimates of the parameters will be obtained by solving the following nonlinear equations

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} x_i &= \mu - \sigma\Psi(\alpha),\\
\frac{1}{n}\sum_{i=1}^{n} x_i^2 &= \mu^2 - 2\,\mu\,\sigma\,\Psi(\alpha) + \sigma^2\frac{\Gamma''(\alpha)}{\Gamma(\alpha)},\\
\frac{1}{n}\sum_{i=1}^{n} x_i^3 &= \mu^3 - 3\,\mu^2\,\sigma\,\Psi(\alpha) + 3\,\mu\,\sigma^2\,\frac{\Gamma''(\alpha)}{\Gamma(\alpha)} - \sigma^3\frac{\Gamma'''(\alpha)}{\Gamma(\alpha)}.
\end{aligned}$$

We can use the nleqslv function in the R contributed package nleqslv to solve the above equations, see [19].

3.3 Bayesian Estimation

In this section, we perform a Bayesian analysis for the GG model. To do so, one should adopt a prior density for the parameters, denoted as $\pi(\boldsymbol{\theta})$, and then combine it with the likelihood function, $L(\boldsymbol{\theta}|\boldsymbol{x})$. Then, the posterior density of the parameters can be determined as follows

$$\pi(\boldsymbol{\theta}|\boldsymbol{x}) \propto L(\boldsymbol{\theta}|\boldsymbol{x})\pi(\boldsymbol{\theta}).$$

The remaining of the estimation process is achieved based on the obtained posterior distribution. Depending on the considered loss function, one can use the mean, median and other measures of the posterior distribution to estimate the parameters, see for example [7, 14, 22] for more details.

Considering the nature of parameters, namely μ is the location parameter, σ is the scale parameter and α is the shape parameter, we set the priors to be

$$\mu \sim \mathcal{N}(\mu_0, \sigma_0^2), \quad \alpha \sim \mathcal{G}a(a_1, b_1),, \quad \sigma \sim \mathcal{IG}(a_2, b_2), \tag{18}$$

where $\mathcal{N}$, $\mathcal{G}a$ and $\mathcal{IG}$ stand for the normal, gamma and inverse-gamma distributions, respectively. The parameters of the prior density are called *hyperparameters* which can be considered to be fixed before the analysis. In order to guarantee that the posterior distribution is a valid distribution and the priors have an ignorable effect on the posterior distribution, following many authors (see for example [22]), we fix the hyperparameters as $\mu_0 = 0$, $\sigma_0^2 = 10^4$, $a_1 = b_1 = 0.00001$ and $a_2 = b_2 = 0.001$. These values produce priors with large variances and consequently, they have a small impact on the resulting posterior distribution. Furthermore, we suppose that the parameters are apriori independent. From (16) and (18), the posterior density of the parameters is given by

$$\begin{aligned}\pi(\alpha, \sigma, \mu|\boldsymbol{x}) \quad \propto \quad & \exp\left\{-\sum_{i=1}^{n} \exp\left(-\frac{x_i - \mu}{\sigma}\right)\right\} \alpha^{a_1-1}\sigma^{-n-a_2-1}[\Gamma(\alpha)]^{-n} \\ & \times \exp\left\{-b_1\alpha - \frac{b_2 + n\alpha(\bar{x} - \mu)}{\sigma} - \frac{(\mu - \mu_0)^2}{2\sigma_0^2}\right\}.\end{aligned}$$

This posterior distribution does not admit a closed form and therefore it cannot be employed in the inference directly. Thus, we may perform an approximate analysis by means of obtaining random samples from this distribution via Gibbs sampler. Gibbs sampler draws samples from the desired distribution of the parameters using a process of sequential sampling from the full conditional distributions of the parameters, see [31] for more details.

The full conditional distribution of α can be derived as

$$\pi(\alpha|\sigma, \mu, \boldsymbol{x}) \quad \propto \quad [\Gamma(\alpha)]^{-n}\, \alpha^{a_1-1} \, \exp\left\{-\alpha\left(b_1 + \frac{n(\bar{x} - \mu)}{\sigma}\right)\right\}, \tag{19}$$

which is not a convenient density and needs more delicate methods to sample. Depending on different values of parameters, (19) could be either log-concave or log-convex. Here, we use the adaptive Metropolis rejection sampling (ARMS) method, stated in [16], in order to sample from (19). The ARMS method is used to sample from complicated univariate distributions efficiently. This method is a generalization of the adaptive rejection sampling (ARS) method (see [17]) which involves a Metropolis step [24] to consider non-concavity in the log density. The ARMS method constructs an envelope function for the log of the target density, like the ARS method, which will be used in the rejection sampling. If a proposed sample is rejected by the ARS method, then the envelope is updated to be more close to the true log density. Whenever the log-concavity is not obtained, the ARMS method performs a Metropolis step on each point accepted at an ARS rejection step. For more details, see [16].

The joint conditional posterior density of σ and μ is given by

$$\pi(\sigma, \mu|\alpha, \boldsymbol{x}) \propto \sigma^{-n-a_2-1} \exp\left\{-\frac{b_2 + n\alpha(\bar{x} - \mu)}{\sigma} - \frac{(\mu - \mu_0)^2}{2\sigma_0^2} - \sum_{i=1}^{n} \exp\left(-\frac{x_i - \mu}{\sigma}\right)\right\}. \tag{20}$$

Model	μ	σ	α	β	W^*	A^*	K-S	p-value
GG(μ, σ, α)	3.5971	1.8757	2.7207	—	0.0395	0.2846	0.0728	0.8681
Gumbel(μ, σ)	1.4937	1.0594	—	—	0.0725	0.4631	0.0981	0.5469
EG(μ, σ, β)	2.3329	1.5614	—	2.1094	0.0407	0.2905	0.0734	0.8616
TEG$(\mu, \sigma, \alpha, \beta)$	1.8258	1.8372	2.4558	-1.0000	0.0415	0.2951	0.0743	0.8523

Table 1: Parameter ML estimates and the goodness-of-fit test statistics for the windshield data.

To sample from the bivariate distribution (20), we use the adaptive Metropolis-Hastings algorithm in which the covariance matrix of the Gaussian proposal distribution is updated adaptively for efficient mixing of the algorithm, see for example [32] for more details. Therefore, we can generate the required samples with the help of running a two-step Gibbs sampler and then approximate the posterior distribution.

4 Application

Each proposed distribution should be compared with its competitive (rival) distributions and its submodels as well. It is customary to select several other generalizations of the same distribution (usually with the same number of parameters) as well as the submodels and then compare them with each other. Here, we chose two generalizations of the Gumbel distribution such as the EG and TEG distributions with pdfs (2) and (7), respectively, (the TEG distribution has 4 parameters) as well as the Gumbel distribution (the submodel of the GG distribution) as the competitive distributions. We expect that as the number of parameters increases, the flexibility of the distribution gets more improved. But as we will see later, the 3-parameter GG distribution possesses a better fit than the 4-parameter TEG distribution.

Here, we consider the service times for a particular model windshield given in Table 16.11 of [25] and studied later by [30]. The data were measured in 1000 hours and are as follows:
0.046, 1.436, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.719, 2.717, 0.280, 1.794, 2.819, 0.313, 1.915, 2.820, 0.389, 1.920, 2.878, 0.487, 1.963, 2.950, 0.622, 1.978, 3.003, 0.900, 2.053, 3.102, 0.952, 2.065, 3.304, 0.996, 2.117, 3.483, 1.003, 2.137, 3.500, 1.010, 2.141, 3.622, 1.085, 2.163, 3.665, 1.092, 2.183, 3.695, 1.152, 2.240, 4.015, 1.183, 2.341, 4.628, 1.244, 2.435, 4.806, 1.249, 2.464, 4.881, 1.262, 2.543, 5.140.
Table 4.1 includes the ML estimates of the parameters, the values of Cramer-Von Mises (W^*) and Anderson-Darling (A^*) statistics (see [6] for more details regarding the W^* and A^* statistics), the values of Kolmogorov-Smirnov (K-S) test statistics with the corresponding p-values determined by fitting the models. The unknown parameters of each model were estimated by the ML method using the Maximize function in the package Optimization in Maple 17. The goodness of fit statistics were computed using the goodness.fit function in the R contributed package AdequacyModel, see [23] and [29].

From Table 4.1, we see that the GG distribution has the smallest values of W^*, A^* and K-S statistics, and the largest p-value related to the K-S test and therefore it provides the best fit among the other considered models. The plots of the fitted densities and the probability plots shown in Figures 4.1 and 4.2, respectively, confirm this conclusion.

Next, we focus on estimating the parameters of the GG distribution using the method of moments and Bayesian procedure. We obtained the MM estimates using the nleqslv function in the R contributed package nleqslv, see [29] and [19]. In the context of Bayesian estimation, the analysis was performed using the Gibbs sampling with $200,000$ iterations. We discarded $100,000$ of the generated samples to reduce the impact of warming up effects of the Markov chains and then followed sampling steps of sizes 20,000 from the remaining of the chain trajectories to reduce the correlation, see [31] for more details on the theory and application of the Markov chain Monte Carlo method and Gibbs sampling. We considered the absolute error loss function and approximated the medians of the marginal posterior densities of the parameters as the approximate Bayes

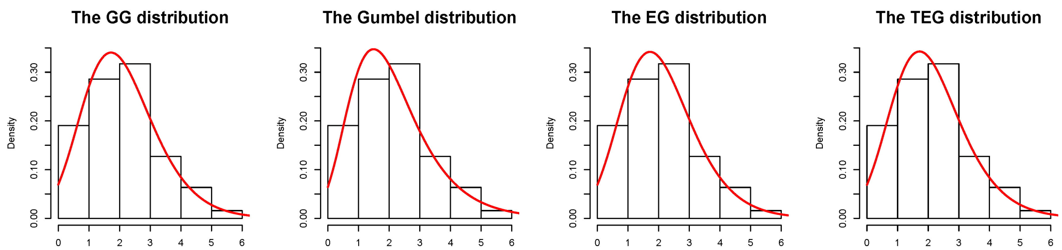

Figure 2: Histogram of the windshield data and the fitted pdfs of the GG, Gumbel, EG and TEG distributions.

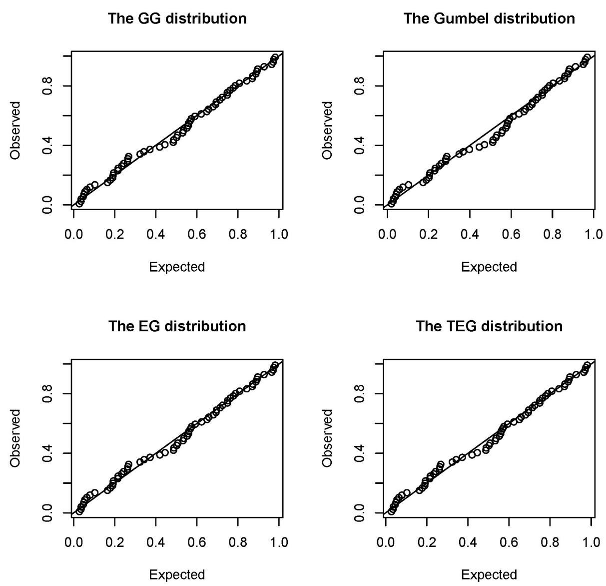

Figure 3: Probability plots for the fits of the GG, Gumbel, EG and TEG distributions.

(AB) estimates. Figure 4 shows the trajectories of the Gibbs sampling and convergence of the estimates. Figure 4 indicates that the produced chains mix very well. The ML, MM and AB estimates of the parameters of the GG model for the windshield data are given in Table 4.2. The computations related to the Bayesian estimation were done in R [29] using the arms function in the package armspp, see [4] (see also [13]) and the Metro_Hastings function in the package MHadaptive, see [5]. We also used the R contributed package invgamma, see [20]. Summing up, we may conclude that the GG distribution is a flexible model for the windshield data.

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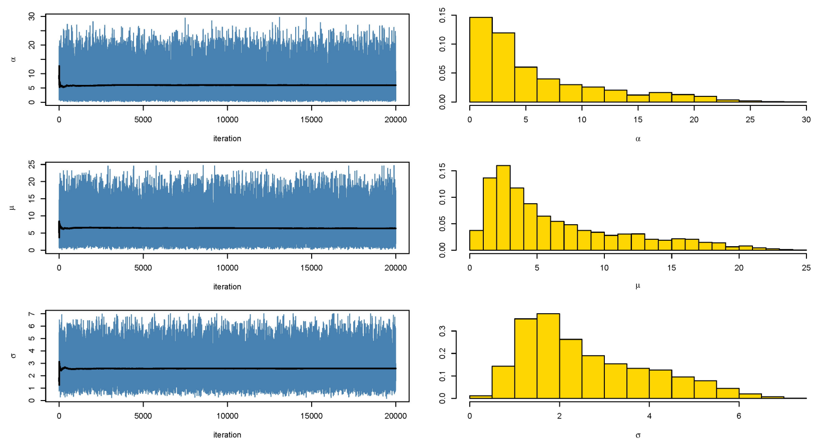

Figure 4: Evolution of the Gibbs sampling with convergence of the estimates (*left column*) along with the corresponding histograms (*right column*) for the produced chains for μ, σ and α.

ML			MM			AB		
μ	σ	α	μ	σ	α	μ	σ	α
3.5971	1.8757	2.7207	6.6687	2.8015	5.6270	4.4920	2.1867	3.6066

Table 2: The ML, MM and AB estimates of the parameters of the GG distribution for the windshield data.

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