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Sharp Bounds For The Arc Lemniscate Sine Function*

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Abstract

The arc lemniscate sine function is given by

$$\operatorname{arcsl}(x) = \int_0^x \frac{1}{\sqrt{1-t^4}}dt.$$

In 2017, Mahmoud and Agarwal presented bounds for arcsl in terms of the Lerch zeta function

$$\Phi(z,s,a) = \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s}.$$

They proved

$$\frac{1}{8}\, x\, \Phi(x^4, 3/2, 1/4) < \operatorname{arcsl}(x) < \frac{1}{4}\, x\, \Phi(x^4, 3/2, 1/4) \qquad (0 < x < 1).$$

We show that the factor 1/4 can be replaced by $\operatorname{arcsl}(1)/\Phi(1, 3/2, 1/4) = 0.12836\ldots$. This constant is best possible.

1 Introduction and Statement of Result

Let F_1 and F_2 be two points in the plane, with distance $F_1F_2 = 2c$. The lemniscate of Bernoulli is the locus of all points P such that $PF_1 \cdot PF_2 = c^2$. It is named after the Swiss mathematician Jakob Bernoulli (1655-1705) who was the first who studied the lemniscate in detail. The arc length of the lemniscate curve L is given by the formula

$$L = 4\sqrt{2}c\, \operatorname{arcsl}(1),$$

where arcsl is the so-called arc lemniscate sine function, defined by

$$\operatorname{arcsl}(x) = \int_0^x \frac{1}{\sqrt{1-t^4}}dt \qquad (-1 \le x \le 1).$$

Many interesting information on this subject including historical comments can be found in Ayoub [1] and Langer & Singer [3].

This note is inspired by a remarkable paper published by Mahmoud and Agarwal [4] in 2017. Among others, the authors offered upper and lower bounds for arcsl in terms of the Lerch zeta function

$$\Phi(z,s,a) = \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s}.$$

They proved the elegant double-inequality

$$\frac{1}{8}\, x\, \Phi(x^4, 3/2, 1/4) < \operatorname{arcsl}(x) < \frac{1}{4}\, x\, \Phi(x^4, 3/2, 1/4) \qquad (0 < x < 1). \tag{1}$$

It is natural to ask whether the constant factors 1/8 and 1/4 are sharp. In this note, we refine the upper bound given in (1). Indeed, the constant 1/4 can be replaced by a smaller number as the following theorem reveals.

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Theorem 1 *For all $x \in (0,1)$ we have*

$$\alpha\, x\, \Phi(x^4, 3/2, 1/4) < \operatorname{arcsl}(x) < \beta\, x\, \Phi(x^4, 3/2, 1/4) \tag{2}$$

with the best possible constant factors

$$\alpha = \frac{1}{8} \quad \textit{and} \quad \beta = \frac{\operatorname{arcsl}(1)}{\Phi(1, 3/2, 1/4)} = 0.12836.... \tag{3}$$

In particular, we obtain that for all $x \in (0,1)$ the ratio $\operatorname{arcsl}(x)/\big(x\Phi(x^4, 3/2, 1/4)\big)$ lies between 1/8 and 1/7. The constant β can be expressed in terms of the Euler beta function and the Hurwitz zeta function, respectively, which are given by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \quad \text{and} \quad \zeta(s,a) = \Phi(1,s,a) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^s}.$$

The substitution $t = s^{1/4}$ gives

$$\operatorname{arcsl}(1) = \int_0^1 \frac{1}{\sqrt{1-t^4}}dt = \frac{1}{4}\int_0^1 s^{-3/4}(1-s)^{-1/2}ds = \frac{1}{4}B(1/4, 1/2).$$

Thus,

$$\beta = \frac{B(1/4,1/2)}{4\zeta(3/2,1/4)} = \frac{\Gamma(1/4)^2}{4\sqrt{2\pi}\zeta(3/2,1/4)}\,,$$

where Γ denotes the classical gamma function.

Schneider [7] proved in 1937 that the lemniscate constant arcsl(1) is a transcendental number; see also Todd [8].

2 Proof of Theorem 1

The following lemma plays an important role in the proof of our theorem. It is known in the literature as the monotone form of l'Hopital's rule; see Hardy et al. [2, p. 106] and Pinelis [5, 6].

Lemma 1 *Let $u, v : [a,b] \to R$ be continuous functions. Moreover, let u, v be differentiable on (a,b) and $v' \neq 0$ on (a,b). If u'/v' is strictly increasing on (a,b), then*

$$x \mapsto \frac{u(x)-u(a)}{v(x)-v(a)}$$

is strictly increasing on (a,b).

Proof of Theorem 1. Let

$$F(x) = \frac{\operatorname{arcsl}(x)}{x\Phi(x^4, 3/2, 1/4)}.$$

In order to prove that F is strictly increasing on $(0,1)$ we apply the lemma with

$$u(x) = \operatorname{arcsl}(x) \quad \text{and} \quad v(x) = x\Phi(x^4, 3/2, 1/4).$$

Let $x \in (0,1)$. We have

$$u(0) = v(0) = 0 \quad \text{and} \quad u'(x) = \frac{1}{\sqrt{1-x^4}}, \quad v'(x) = 8\sum_{k=0}^{\infty} \frac{x^{4k}}{\sqrt{4k+1}}.$$

It follows that

$$\frac{u'(x)}{v'(x)} = \frac{1}{8h(x^4)} \tag{4}$$

with

$$h(s) = \sqrt{1-s}\sum_{k=0}^{\infty} \frac{s^k}{\sqrt{4k+1}}.$$

Then,

$$2\sqrt{1-s}h'(s) = 2(1-s)\sum_{k=1}^{\infty} \frac{ks^{k-1}}{\sqrt{4k+1}} - \sum_{k=0}^{\infty} \frac{s^k}{\sqrt{4k+1}} = \sum_{k=0}^{\infty} a_k s^k,$$

where

$$a_k = \frac{2k+2}{\sqrt{4k+5}} - \frac{2k+1}{\sqrt{4k+1}} = \frac{-1}{(2k+2)(4k+1)\sqrt{4k+5}+(2k+1)(4k+5)\sqrt{4k+1}}.$$

Since $a_k < 0$ for $k = 0, 1, 2, ...$, we conclude that $h'(s) < 0$ for $s \in (0,1)$. Thus, h is strictly decreasing on $(0,1)$. Using (4) yields that u'/v' is strictly increasing on $(0,1)$, so that the lemma reveals that $F = u/v$ is strictly increasing on $(0,1)$. It follows that

$$F(0) < F(x) < F(1) \quad \text{for} \quad x \in (0,1). \tag{5}$$

We have

$$F(1) = \frac{\operatorname{arcsl}(1)}{\Phi(1, 3/2, 1/4)}. \tag{6}$$

Since

$$\lim_{x\to 0} \frac{\operatorname{arcsl}(x)}{x} = \frac{d}{dx}\operatorname{arcsl}(x)\Big|_{x=0} = 1 \quad \text{and} \quad \Phi(0, 3/2, 1/4) = 8,$$

we obtain

$$F(0) = \frac{1}{8}. \tag{7}$$

From (5), (6) and (7) we conclude that (2) is valid and that the constant factors α and β as given in (3) are best possible. This completes the proof of the theorem. ■

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