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Perturbed Partial Functional Differential Equations On Unbounded Domains With Finite Delay*

Mohamed Helal[†]

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Abstract

In this paper we investigate the existence of solutions of perturbed partial hyperbolic differential equations of fractional order with finite delay and Caputo's fractional derivative by using a nonlinear alternative of Avramescu on Fréchet spaces.

1 Introduction

In this paper we are concerned with the existence of solutions to fractional order initial value problem (IVP for short), for the system

$$({}^cD_0^r u)(t,x) = f(t,x,u_{(t,x)}) + g(t,x,u_{(t,x)}),\ \text{if } (t,x)\in J, \tag{1}$$

$$u(t,x) = \phi(t,x),\ \text{if } (t,x)\in \tilde{J}, \tag{2}$$

$$u(t,0) = \varphi(t),\ u(0,x) = \psi(x),\ (t,x)\in J, \tag{3}$$

where $\varphi(0) = \psi(0)$, $J := [0,\infty)\times[0,\infty)$, $\tilde{J} := [-\alpha,+\infty)\times[-\beta,+\infty)\backslash[0,\infty)\times[0,\infty)$, ${}^cD_0^r$ is the standard Caputo's fractional derivative of order $r = (r_1, r_2)\in(0,1]\times(0,1]$, $f, g : J\times C\to\mathbb{R}^n$ are given functions, $\phi\in C := C([-\alpha,0]\times[-\beta,0],\mathbb{R}^n)$ is a given continuous function with $\phi(t,0) = \varphi(t)$, $\phi(0,x) = \psi(x)$ for each $(t,x)\in J$, $\varphi : [0,\infty)\to\mathbb{R}^n$, $\psi : [0,\infty)\to\mathbb{R}^n$ are given absolutely continuous functions and C is the space of continuous functions on $[-\alpha,0]\times[-\beta,0]$.

We denote by $u_{(t,x)}$ the element of C defined by

$$u_{(t,x)}(s,\tau) = u(t+s, x+\tau);\ (s,\tau)\in[-\alpha,0]\times[-\beta,0],$$

here $u_{(t,x)}(.,.)$ represents the history of the state u.

In recent years, fractional differential and partial differential equations have become more important in some mathematical models of real phenomena, especially in control, biological and medical domains. In these models, the investigated simulating processes and phenomena usually are subject to short-term perturbations whose duration is negligible in comparison with the duration of the process. We can find numerous applications of differential equations of fractional order in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [7, 17, 23, 24, 26]). There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [2], Aissani *et al.* [5], Kilbas *et al.* [19], Lakshmikantham *et al.* [21], and the papers by Agarwal *et al* [3, 4], Belarbi *et al.* [8], Benchohra *et al.* [11], and the references therein.

The theory of functional differential equations has emerged as an important branch of nonlinear analysis. Differential delay equations, or functional differential equations, have been used in modeling scientific

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[†]Sciences and Technology Faculty, Mustapha Stambouli University of Mascara B.P. 763, 29000, Mascara, Algeria. Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbés, B.P. 89, 22000, Sidi Bel-Abbés, Algeria

phenomena for many years. Often, it has been assumed that the delay is either a fixed constant or is given as an integral in which case it is called a distributed delay; see for instance the books by Hale and Verduyn Lunel [15], Hino *et al.* [18], Kolmanovskii and Myshkis [20], Lakshmikantham *et al.* [22], and Wu [28], and the papers [12, 13].

Motivated by the previous papers, in this paper, we consider the existence of solutions for the problems (1)–(3). Our main result for this problem is based a nonlinear alternative for the sum of a completely continuous operator and a contraction in Fréchet spaces due to Avramescu [6] and a fractional version of Gronwall's inequality. This is the main motivation to look for sufficient conditions ensuring existence of solutions for each of our problems. The present results extend those considered with finite and/or infinite constant delay on bounded domains in [1, 9]. To our Knowledge, there are very few papers devoted to fractional differential equations with delay on Fréchet spaces. The aim of this paper is to continue this study.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $p \in \mathbb{N}$ and $J_0 = [0,p] \times [0,p]$. By $C(J_0,\mathbb{R})$ we denote the Banach space of all continuous functions from J_0 into $\mathbb{R}^n$ with the norm

$$\|w\|_\infty = \sup_{(t,x)\in J_0} \|w(t,x)\|,$$

where $\|.\|$ denotes a suitable complete norm on $\mathbb{R}^n$. As usual, by $AC(J_0,\mathbb{R})$ we denote the space of absolutely continuous functions from J_0 into $\mathbb{R}^n$ and $L^1(J_0,\mathbb{R})$ is the space of Lebesgue-integrable functions $w : J_0 \to \mathbb{R}^n$ with the norm

$$\|w\|_{L^1} = \int_0^p \int_0^p \|w(t,x)\| dt dx.$$

Definition 1 ([27]) *Let $r = (r_1, r_2) \in (0,\infty)\times(0,\infty), \theta = (0,0)$ and $u \in L^1(J_0,\mathbb{R}^n)$. The left-sided mixed Riemann-Liouville integral of order r of u is defined by*

$$(I_\theta^r u)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} u(s,\tau) d\tau ds.$$

In particular,

$$(I_\theta^\theta u)(t,x) = u(t,x), \;\; (I_\theta^\sigma u)(t,x) = \int_0^t \int_0^x u(s,\tau) d\tau ds; \;\; \textit{for almost all } (t,x)\in J_0,$$

where $\sigma = (1,1)$.

For instance, $I_\theta^r u$ exists for all $r_1, r_2 \in (0,\infty)\times(0,\infty)$, when $u \in L^1(J_0,\mathbb{R}^n)$. Note also that when $u \in C(J_0,\mathbb{R}^n)$, then $(I_\theta^r u) \in C(J_0,\mathbb{R}^n)$, moreover

$$(I_\theta^r u)(t,0) = (I_\theta^r u)(0,x) = 0; \;\; (t,x)\in J_0.$$

Example 1 *Let $\lambda, \omega \in (-1,\infty)$ and $r = (r_1,r_2) \in (0,\infty)\times(0,\infty)$. Then*

$$I_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} t^{\lambda+r_1} x^{\omega+r_2}, \;\; \textit{for almost all } (t,x)\in J_0.$$

By $1-r$ we mean $(1-r_1, 1-r_2) \in (0,1]\times(0,1]$. Denote by $D_{tx}^2 := \frac{\partial^2}{\partial t \partial x}$, the mixed second order partial derivative.

Definition 2 ([27]) *Let $r \in (0,1] \times (0,1]$ and $u \in L^1(J_0, \mathbb{R}^n)$. The mixed fractional Riemann-Liouville derivative of order r of u is defined by the expression*

$$D_\theta^r u(t,x) = (D_{tx}^2 I_\theta^{1-r} u)(t,x)$$

and the Caputo fractional-order derivative of order r of u is defined by the expression

$$({}^cD_0^r u)(t,x) = (I_\theta^{1-r} \frac{\partial^2}{\partial t \partial x} u)(t,x).$$

The case $\sigma = (1,1)$ is included and we have

$$(D_\theta^\sigma u)(t,x) = ({}^cD_\theta^\sigma u)(t,x) = (D_{tx}^2 u)(t,x), \text{ for almost all } (t,x) \in J_0.$$

Example 2 *Let $\lambda, \omega \in (-1, \infty)$ and $r = (r_1, r_2) \in (0,1] \times (0,1]$. Then*

$$D_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} t^{\lambda-r_1} x^{\omega-r_2}, \text{ for almost all } (t,x) \in J_0.$$

In the sequel we will make use of the following generalization of Gronwall's lemma for two independent variables and singular kernel.

Lemma 1 ([16]) *Let $\upsilon : J \to [0,\infty)$ be a real function and $\omega(.,.)$ be a nonnegative, locally integrable function on J. If there are constants $c > 0$ and $0 < r_1, r_2 < 1$ such that*

$$\upsilon(t,x) \le \omega(t,x) + c \int_0^t \int_0^x \frac{\upsilon(s,\tau)}{(t-s)^{r_1}(x-\tau)^{r_2}} d\tau ds,$$

then there exists a constant $\delta = \delta(r_1, r_2)$ such that

$$\upsilon(t,x) \le \omega(t,x) + \delta c \int_0^t \int_0^x \frac{\omega(s,\tau)}{(t-s)^{r_1}(x-\tau)^{r_2}} d\tau ds,$$

for every $(t,x) \in J$.

3 Some Properties in Fréchet Spaces

Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies :

$$\|u\|_1 \le \|u\|_2 \le \|u\|_3 \le ... \text{ for every } u \in X.$$

Let $Y \subset X$, we say that Y is bounded if for every $n \in \mathbb{N}$, there exists $\overline{M}_n > 0$ such that

$$\|v\|_n \le \overline{M}_n \quad \text{for all } v \in Y.$$

To X we associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ as follows : For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by : $u \sim_n v$ if and only if $\|u - v\|_n = 0$ for $u, v \in X$. We denote $X^n = (X|_{\sim_n}, \|\cdot\|_n)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y \subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows : For every $u \in X$, we denote $[u]_n$ the equivalence class of u of subset X^n and we define $Y^n = \{[u]_n : u \in Y\}$. We denote $\overline{Y^n}$, $int_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|_n$ in X^n. For more information about this subject see [14].

Definition 3 *Let X be a Fréchet space. A function $N : X \longrightarrow X$ is said to be a contraction if for each $n \in \mathbb{N}$ there exists $k_n \in (0,1)$ such that*

$$\|N(u) - N(v)\|_n \le k_n \|u - v\|_n \text{ for all } u, v \in X.$$

Theorem 2 (Nonlinear Alternative of Avramescu [6]) *Let $(X, |.|_n)$ be a Fréchet space and let $A, B : X \to X$ two operators. Suppose that the following hypothesis are fulfilled:*

(i) *A is a compact operator.*

(ii) *B is a contraction operator with respect to a family of seminorms $||.||_n$ equivalent with the family $|.|_n$.*

(iii) *the set $\mathcal{E} = \{u \in X : u = \lambda A(u) + \lambda B(\frac{u}{\lambda})$ for some $\lambda \in (0,1)\}$ is bounded.*

Then there is $u \in X$ such that $u = Au + Bu$.

4 Existence of Solutions

In this section, we give our main existence result for problem (1)–(3).

For each $p \in \mathbb{N}$ we consider the following set

$$C_p = C([-\alpha, p] \times [-\beta, p], \mathbb{R}^n)$$

and we define in $C_0 := C([-\alpha, \infty) \times [-\beta, \infty), \mathbb{R}^n)$ the semi-norms by:

$$\|u\|_p = \{\sup \|u(t,x)\| : -\alpha \le t \le p, -\beta \le x \le p\}.$$

Then C_0 is a Fréchet space with the family of semi-norms $\{\| \cdot \|_p\}$.

Before starting and proving this result, we give what we mean by a solution of the problem (1)–(3).

Definition 4 *A function $u \in C_0$ is said to be a solution of (1)–(3) if u satisfies equations (1) and (3) on J and the condition (2) on $\tilde{J}$.*

For the existence of solutions for the problem (1)–(3), we need the following lemma:

Lemma 3 *A function $u \in C_0$ is a solution of problem (1)–(3) if and only if u satisfies the equation*

$$\begin{aligned} u(t,x) &= z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau,u_{(s,\tau)})d\tau ds \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} g(s,\tau,u_{(s,\tau)})d\tau ds, \end{aligned}$$

for all $(t,x) \in J$ and the condition (2) on $\tilde{J}$, where

$$z(t,x) = \varphi(t) + \psi(x) - \varphi(0).$$

Our main existence result in this section is based on the nonlinear alternative of Avramescu. We will need to introduce the following hypothesis:

Theorem 4 *Assume the following conditions hold:*

$(H1)$ *The functions $f, g : J \times C([-\alpha, 0] \times [-\beta, 0], \mathbb{R}^n) \to \mathbb{R}^n$ are continuous.*

$(H2)$ *For each $p \in \mathbb{N}$, there exists $\ell_p \in C(J_0, \mathbb{R}^n)$ such that for each $(t,x) \in J_0$*

$$\|g(t,x,u) - g(t,x,v)\| \le \ell_p(t,x)\|u - v\|_C, \text{ for each } u, v \in C.$$

$(H3)$ *There exist $p, q \in C(J, \mathbb{R}_+)$ such that*

$$\|f(t,x,u)\| \le p(t,x) + q(t,x)\|u\|_C, \text{ for } (t,x) \in J_0 \text{ and each } u \in C.$$

If

$$\frac{\ell_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} < 1, \tag{4}$$

where

$$\ell_p^* = \sup_{(t,x)\in J_0} \ell_p(t,x),$$

then there exists a unique solution for IVP (1)–(3) on $[-\alpha,\infty)\times[-\beta,\infty)$.

Proof. Transform the problem (1)–(3) into a fixed point problem. Consider the operators $F, G : C_0 \to C_0$ defined by

$$F(u)(t,x) = \begin{cases} \phi(t,x), & (t,x)\in \tilde{J}, \\ z(t,x) + \dfrac{1}{\Gamma(r_1)\Gamma(r_2)}\displaystyle\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} & \\ f(s,\tau,u_{(s,\tau)})d\tau ds, & (t,x)\in J. \end{cases}$$

and

$$G(u)(t,x) = \begin{cases} 0, & (t,x)\in \tilde{J}, \\ \dfrac{1}{\Gamma(r_1)\Gamma(r_2)}\displaystyle\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} & \\ g(s,\tau,u_{(s,\tau)})d\tau ds, & (t,x)\in J. \end{cases}$$

The problem of finding the solutions of the IVP (1)–(3) is reduced to finding the solutions of the operator equation $(Fu)(t,x)+(Gu)(t,x)=u(t,x)$, $(t,x)\in J$. We shall show that the operators F and G satisfies all the conditions of Theorem 2.

For better readability, we break the proof into a sequence of steps.

Step 1. F is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in C_0. Then

$$\begin{aligned} \|(Fu_n)(t,x)-(Fu)(t,x)\| &\le \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \\ &\quad \times\|f(s,\tau,u_{n(s,\tau)})-f(s,\tau,u_{(s,\tau)})\|d\tau ds. \end{aligned}$$

Since f is a continuous function, we have

$$\|(Fu_n)-(Fu)\|_p \le \frac{p^{r_1+r_2}\|f(.,.,u_{n_{(.,.)}})-f(.,.,u_{(.,.)})\|_p}{\Gamma(r_1+1)\Gamma(r_2+1)} \to 0 \text{ as } n\to\infty.$$

Thus F is continuous.

Step 2. F maps bounded sets into bounded sets in C_0. Indeed, it is enough show that, for any $\eta>0$, there exists a positive constant $\widetilde{\ell}$ such that, for each $u\in B_\eta=\{u\in C_0 : \|u\|_p\le\eta\}$, we have $\|F(u)\|_p\le\widetilde{\ell}$. Let $u\in B_\eta$. By $(H3)$, we have for each $(t,x)\in J_0$,

$$\begin{aligned} \|(Fu)(t,x)\| &\le \|z(t,x)\| + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \\ &\quad \times\|f(s,\tau,u_{(s,\tau)})\|d\tau ds \\ &\le \|z(t,x)\| + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}p(s,\tau)d\tau ds \\ &\quad +\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}q(s,\tau) \\ &\quad \times\|u_{(s,\tau)}\|_C d\tau ds \end{aligned}$$

$$
\begin{aligned}
&\le \|z(t,x)\| + \frac{\|p\|_p}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}d\tau ds\\
&\quad +\frac{\|q\|_p\eta}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}d\tau ds\\
&\le \|z(t,x)\| + \frac{\|p\|_p+\|q\|_p\eta}{\Gamma(r_1+1)\Gamma(r_2+1)}p^{r_1+r_2}.
\end{aligned}
$$

Thus

$$
\|(Fu)\|_p \le \|z\|_p + \frac{\|p\|_p+\|q\|_p\eta}{\Gamma(r_1+1)\Gamma(r_2+1)}p^{r_1+r_2} := \tilde{\ell}.
$$

Step 3. F maps bounded sets into equicontinuous sets in C_0. Let $(t_1,x_1),(t_2,x_2)\in J$, $t_1<t_2$, $x_1<x_2$, B_η be a bounded set as in step 2, and let $u\in B_\eta$. Then

$$
\begin{aligned}
&\|(Fu)(t_2,x_2)-(Fu)(t_1,x_1)\|\\
\le\ & \|z(t_1,x_1)-z(t_2,x_2)\| + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{t_1}\int_0^{x_1}[(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}\\
&-(t_1-s)^{r_1-1}(x_1-\tau)^{r_2-1}]||f(s,\tau,u_{(s,\tau)})||d\tau ds\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_{x_1}^{x_2}(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}||f(s,\tau,u_{(s,\tau)})||d\tau ds\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{0}^{t_1}\int_{x_1}^{x_2}(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}||f(s,\tau,u_{(s,\tau)})||d\tau ds\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_{0}^{x_1}(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}||f(s,\tau,u_{(s,\tau)})||d\tau ds\\
\le\ & \|z(t_1,x_1)-z(t_2,x_2)\| + \frac{\|p\|_p+\|q\|_p\eta}{\Gamma(r_1)\Gamma(r_2)}\\
&\times\int_0^{t_1}\int_0^{x_1}[(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}-(t_1-s)^{r_1-1}(x_1-\tau)^{r_2-1}]d\tau ds\\
&+\frac{\|p\|_p+\|q\|_p\eta}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_{x_1}^{x_2}(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}d\tau ds\\
&+\frac{\|p\|_p+\|q\|_p\eta}{\Gamma(r_1)\Gamma(r_2)}\int_{0}^{t_1}\int_{x_1}^{x_2}(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}d\tau ds\\
&+\frac{\|p\|_p+\|q\|_p\eta}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_{0}^{x_1}(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}d\tau ds\\
\le\ & \|z(t_1,x_1)-z(t_2,x_2)\| + \frac{\|p\|_p+\|q\|_p\eta}{\Gamma(r_1+1)\Gamma(r_2+1)}[x_2^{r_2}(t_2-t_1)^{r_1}\\
&+t_2^{r_1}(x_2-x_1)^{r_2}-(t_2-t_1)^{r_1}(x_2-x_1)^{r_2}+t_1^{r_1}x_1^{r_2}-t_2^{r_1}x_2^{r_2}]\\
&+\frac{\|p\|_p+\|q\|_p\eta}{\Gamma(r_1+1)\Gamma(r_2+1)}(t_2-t_1)^{r_1}(x_2-x_1)^{r_2}\\
&+\frac{\|p\|_p+\|q\|_p\eta}{\Gamma(r_1+1)\Gamma(r_2+1)}[t_2^{r_1}-(t_2-t_1)^{r_1}](x_2-x_1)^{r_2}\\
&+\frac{\|p\|_p+\|q\|_p\eta}{\Gamma(r_1+1)\Gamma(r_2+1)}(t_2-t_1)^{r_1}[x_2^{r_2}-(x_2-x_1)^{r_2-1}]\\
\le\ & \|z(t_1,x_1)-z(t_2,x_2)\| + \frac{\|p\|_p+\|q\|_p\eta}{\Gamma(r_1+1)\Gamma(r_2+1)}[2x_2^{r_2}(t_2-t_1)^{r_1}\\
&+2t_2^{r_1}(x_2-x_1)^{r_2}+t_1^{r_1}x_1^{r_2}-t_2^{r_1}x_2^{r_2}-2(t_2-t_1)^{r_1}(x_2-x_1)^{r_2}].
\end{aligned}
$$

The right-hand side of the above inequality tends to zero as $t_1 \to t_2,\ x_1 \to x_2$. The equicontinuity for the cases $t_1 < t_2 < 0,\ x_1 < x_2 < 0$ and $t_1 \leq 0 \leq t_2,\ x_1 \leq 0 \leq x_2$ is obvious.

As a consequence of steps 1 to 3 together with Arzela-Ascoli theorem, we can conclude that $F : C_0 \to C_0$ is a compact operator.

Step 4. G is a contraction. Let $u, v \in C_0$. Then we have for each $(t,x) \in J_0$

$$\begin{aligned}
& \|(Gu)(t,x) - (Gv)(t,x)\| \\
\leq\ & \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \\
& \times \|g(s,\tau,u_{(s,\tau)}) - g(s,\tau,v_{(s,\tau)})\| d\tau ds \\
\leq\ & \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \ell_p(s,\tau) \|u_{(s,\tau)} - v_{(s,\tau)}\|_C \\
\leq\ & \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \ell_p(s,\tau) \\
& \times \sup_{(s,\tau)\in[0,T]\times[0,X]} \|u(s,\tau) - v(s,\tau)\| d\tau ds \\
\leq\ & \frac{\ell_p^*(s,\tau)}{\Gamma(r_1)\Gamma(r_2)} \int_0^p \int_0^p (t-s)^{r_1-1}(x-\tau)^{r_2-1} d\tau ds \|u-v\|_p.
\end{aligned}$$

Therefore

$$\|(Gu) - (Gv)\|_p \leq \frac{\ell_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \|u-v\|_p.$$

By (4), G is a contraction.

Step 5. (A priori bounds). Now it remains to show that the set

$$\mathcal{E} = \left\{ u \in C(J,\mathbb{R}) : u = \lambda F(u) + \lambda G(\frac{u}{\lambda}) \ \text{ for some } \lambda \in (0,1) \right\}$$

is bounded. Let $u \in \mathcal{E}$. Then and $u = \lambda F(u) + \lambda G(\frac{u}{\lambda})$ for some $0 < \lambda < 1$. Thus for each $(t,x) \in J_0$, we have

$$\begin{aligned}
u(t,x) &= \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau,u_{(s,\tau)}) d\tau ds \\
&\quad + \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} g(s,\tau,\frac{u_{(s,\tau)}}{\lambda}) d\tau ds.
\end{aligned}$$

This implies by $(H2)$ and $(H3)$ that, for each $(t,x) \in J_0$, we have

$$\begin{aligned}
\|u(t,x)\| &\leq ||z(t,x)|| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} [p(s,\tau) \\
&\quad + q(s,\tau)\|u_{(s,\tau)}\|_C] d\tau ds \\
&\quad + \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \\
&\quad \left| g(s,\tau,\frac{u_{(s,\tau)}}{\lambda}) - g(s,\tau,0) \right| d\tau ds \\
&\quad + \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} |g(s,\tau,0)| d\tau ds \\
&\leq ||z(t,x)|| + \frac{p^{r_1+r_2}\|p\|_p}{\Gamma(r_1+1)\Gamma(r_2+1)} + \frac{p^{r_1+r_2} g^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \\
&\quad + \frac{||q||_p}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \|u_{(s,\tau)}\|_C d\tau ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\ell_p(s,\tau)||u_{(s,\tau)}||_C d\tau ds\\
\leq\ & ||z(t,x)||+\frac{p^{r_1+r_2}(\|p\|_p+g^*)}{\Gamma(r_1+1)\Gamma(r_2+1)}\\
&+\frac{(||q||_p+\ell_p^*)}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\|u_{(s,\tau)}\|_C d\tau ds,
\end{aligned}
$$

where $g^*=\sup_{(s,\tau)\in J}|g(s,\tau,0)|$. Consider the function y defined by

$$
y(t,x)=\sup\{\|u(s,\tau)\|:-\alpha\leq s\leq t,\ -\beta\leq\tau\leq x\},\ 0\leq t\leq p,\ 0\leq x\leq p.
$$

Let $(t^*,x^*)\in[-\alpha,t]\times[-\beta,x]$ be such that $y(t,x)=\|u(t^*,x^*)\|$. If $(t^*,x^*)\in J_0$, then by the previous inequality, we have for $(t,x)\in J_0$,

$$
\begin{aligned}
y(t,x)\ \leq\ & ||z(t,x)||+\frac{p^{r_1+r_2}(\|p\|_p+g^*)}{\Gamma(r_1+1)\Gamma(r_2+1)}\\
&+\frac{||q||_p+\ell_p^*}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}y(s,\tau)d\tau ds.
\end{aligned}
$$

If $(t^*,x^*)\in\tilde J$, then $y(t,x)=\|\phi\|_C$ and the previous inequality holds. If $(t,x)\in J_0$, Lemma 1 implies that there exists $\delta=\delta(r_1,r_2)$ such that we have

$$
\begin{aligned}
y(t,x)\ \leq\ & \left[||z(t,x)||+\frac{p^{r_1+r_2}(\|p\|_p+g^*)}{\Gamma(r_1+1)\Gamma(r_2+1)}\right]\\
&\times\left[1+\frac{\delta(||q||_p+\ell_p^*)}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}d\tau ds\right]\\
\leq\ & \left[||z(t,x)||+\frac{p^{r_1+r_2}(\|p\|_p+g^*)}{\Gamma(r_1+1)\Gamma(r_2+1)}\right]\left[1+\frac{\delta p^{r_1+r_2}(||q||_p+\ell_p^*)}{\Gamma(r_1+1)\Gamma(r_2+1)}\right]:=M.
\end{aligned}
$$

Since for every $(t,x)\in J_0$, $\|u_{(t,x)}\|_C\leq y(t,x)$, we have

$$
\|u\|_p\leq\max(\|\phi\|_C,M):=M^*.
$$

This shows that the set $\mathcal{E}$ is bounded. As a consequence of Theorem 2 we deduce that $F+G$ has a fixed point u which is a solution of problem (1)–(3). ■

5 Application

As an application of our results we consider the following partial perturbed hyperbolic functional differential equations of the form

$$
({}^cD_0^r u)(t,x)=\frac{|u(t-1,x-2)|+2}{10c_pe^{t+x}(1+|u(t-1,x-2)|)},\quad\text{if } (t,x)\in J:=[0,\infty)\times[0,\infty), \tag{5}
$$

$$
u(t,0)=t,\ u(0,x)=x^2,\ (t,x)\in J, \tag{6}
$$

$$
u(t,x)=t+x^2,\ (t,x)\in\tilde J:=[-1,\infty)\times[-2,\infty)\backslash[0,\infty)\times[0,\infty), \tag{7}
$$

where

$$
f(t,x,u_{(t,x)})=\frac{|u(t-1,x-2)|}{(10c_pe^{t+x})(1+|u(t-1,x-2)|)},\ (t,x)\in J,
$$

$$g(t,x,u_{(t,x)}) = \frac{2}{(10c_p e^{t+x})(1+|u(t-1,x-2)|)}, \ (t,x) \in J,$$

and

$$c_p = \frac{3p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}.$$

For each $u, v \in C([-1,0]\times[-2,0],\mathbb{R})$ and $(t,x) \in J_0 = [0,p]\times[0,p]$, we have

$$|g(t,x,u) - g(t,x,v)| \leq \frac{1}{5c_p e^{t+x}} \|u-v\|_C.$$

Hence condition $(H2)$ is satisfied with $\ell_p e^{t+x} = \frac{1}{5c_p e^{t+x}}$. Since

$$\ell_p^* = \sup\left\{\frac{1}{5c_p e^{t+x}}, \quad (t,x)\in J_0\right\} \leq \frac{1}{5c}.$$

We shall show that condition (4) holds for each $(r_1, r_2) \in (0,1]\times(0,1]$ and all $p \in \mathbb{N}^*$. Indeed

$$\frac{\ell_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} = \frac{1}{15} < 1.$$

Also, the function f is continuous on $J_0 \times [0,\infty)$ and

$$|f(t,x,\varphi)| \leq |\varphi| \text{ for each } (t,x,\varphi) \in J_0 \times C([-1,0]\times[-2,0],\mathbb{R}).$$

Thus conditions $(H1)$ and $(H3)$ hold. Consequently Theorem 4 implies that problem (5)–(7) has at least one solution defined on $[-1,\infty)\times[-2,\infty)$.

6 Conclusion

Fractional calculus is a wide subject that requires extensive tools and various methods. In our consideration in this paper, we have presented a contribution to the study of different classes of Darboux problem for partial hyperbolic functional perturbed of fractional order involving the Caputo fractional derivative with finite delay in Fréchet spaces.

In most of this paper sufficient conditions were considered to get the existence and uniqueness results of solutions for our problem by reducing the research to the search of the existence and the uniqueness of fixed points of appropriate operators by applying a nonlinear alternative of Avramescu on unbounded interval. There are many directions in which we can extend the work done. We should observe the structure of the space and the properties of the operators to obtain existence results. Many other questions and issues can be investigeted regarding the existence in the space of weighted continuous functions, the uniqueness, the structure of the solutions set and also whether or not the condition satisfied by the operators are optimal.

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