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# Existence And Uniqueness Of Solution For Hadamard Fractional Differential Equations On An Infinite Interval With Integral Boundary Value Conditions *

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**Abstract**

In this paper, we consider Hadamard fractional differential equations on an infinite interval with the nonlinearity depending on fractional derivatives of lower order. We establish a new compactness criterion in a special space. We employ Schauder fixed point theorem and Banach contraction principle to show the existence and uniqueness of solutions for the proposed equation. Finally, we provide an example to illustrate our results.

## 1 Introduction

In recent years, fractional differential equations have drawn much attention due to its applications in a number of fields such as physics, mechanics, chemistry, biology, economics, biophysics, control theory, signal and image processing, etc [14, 15, 20].

In contrast to integer-order differential and integral operators, fractional order differential operators are nonlocal in nature and thus provide the possibility to look into hereditary properties of several materials and processes. The monographs [8, 10, 12, 16, 18, 21] are commonly cited of the theory of fractional derivatives and integrals and their applications. However, it has been noticed that most of the works on this topic is concerned with either Riemann-Liouville or Caputo type fractional differential equations. Besides these fractional derivatives, another kind of fractional derivatives found in the literature is the Hadamard fractional derivative [16, 19] which differs from the aforementioned derivatives in the sense that the kernel of the integral contains logarithmic function of arbitrary exponent. It should be mensioned that most of the results on fractional calculus are devoted to the solvability of fractional differential equations on finite interval.

Recently, there have been few papers concerning the fractional differential equations with various boundary conditions on infinite interval [6, 11, 17, 22, 24]. Boundary value problem on infinite intervals appear often in applied mathematics and physics, such as in unsteady flow of gas through a semi-infinite porous medium, the theory of drain flows, etc.

Zhao and Ge [24] studied the existence of unbounded solutions for the following boundary value problem on the infinite interval

$$\begin{cases} D^{\alpha}u(t) + f(t, u(t)) = 0, \quad t \in (0, +\infty), \\ u(0) = 0, \\ \lim\limits_{t\to+\infty} D^{\alpha}u(t) = \beta u(\xi), \end{cases}$$

where $1 < \alpha < 2$, $D^{\alpha}$ is the Riemann-Liouville fractional derivative and $0 < \beta, \xi < \infty$. By means of fixed point theorems, sufficient conditions were obtained that guarantee the existence of solutions to the boundary value problem.

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Thiramanus et al. [17] investigated the existence of nonnegative multiple solutions for nonlinear Hadamard fractional differential equations, with nonlocal fractional integral boundary conditions on an unbounded domain

$$\begin{cases} D^{\alpha}u(t) + a(t)f(u(t)) = 0, \quad t \in (1, +\infty), \\ u(1) = 0, \\ D^{\alpha-1}u(+\infty) = \sum\limits_{i=1}^{m} \lambda_i I^{\beta_i} u(\eta), \end{cases}$$

where $1 < \alpha \leqslant 2$, $D^{\alpha}$ is the Hadamard fractional derivative, $\eta \in (1, \infty)$ and $I^{\beta_i}$ is the Hadamard fractional integral of order $\beta_i > 0, i = 1, 2, ..., m$ and $\lambda_i \geq 0, i = 1, 2, ..., m$, are given constants. The authors applyied Leggett-Williams fixed point theorem to obtain the existence of at least three distinct nonnegative solutions under some conditions. Then, to prove the existence of at least one positive solution they used Guo-Krasnoselskii fixed point theorem.

Wang, Pei and Baleanu [21] considered the following Hadamard fractional integro-differential equations with Hadamard fractional integral boundary conditions on an infinite interval

$$\begin{cases} D^{\alpha}u(t) + f(t, u(t), I^{\beta}u(t)) = 0, \quad t \in (1, +\infty), \\ u(1) = u'(1) = 0, \\ D^{\alpha-1}u(+\infty) = \sum\limits_{i=1}^{m} \lambda_i I^{\beta_i} u(\eta), , \end{cases}$$

where $2 < \alpha \leqslant 3$, $\beta > 0$, $\eta \geq 1, f \in C((1, +\infty) \times \mathbb{R}^2, \mathbb{R})$, $D^{\alpha}$ denote the Hadamard fractional derivative, $I^{\beta}$ is the Hadamard fractional integral and $\lambda_i \geq 0$, $i = 1, 2, ..., m$, are given constants. The authors investigates the existence of the unique solution with the monotone iterative technique. Inspired by the aforementioned works, in this paper, we investigate the boundary value problem (BVP) on an infinite interval

$$\begin{cases} D^{\alpha}u(t) + f(t, u(t), D^{\alpha-2}u(t), D^{\alpha-1}u(t)) = 0, \quad t \in J = [1, +\infty), \\ u(1) = u'(1) = 0, \\ D^{\alpha-1}u(+\infty) = \xi I^{\beta} u(\eta), \end{cases} \tag{1}$$

where $2 < \alpha \leqslant 3$, $\beta > 0$, $\xi \geq 0$, $\eta \geq 1$, $\Gamma(\alpha + \beta) \neq \xi \left(\log \eta\right)^{\alpha+\beta-1}$, $f \in C(J \times \mathbb{R}^3, \mathbb{R})$, $D^{\alpha}$, $D^{\alpha-1}$ and $D^{\alpha-2}$ denote the Hadamard fractional derivatives, $I^{\beta}$ is the Hadamard fractional integral.

Under suitable growth conditions on the nonlinear term $f$, we show the existence and uniqueness results of solutions for problem (1) by using Schauder's fixed point theorem and Banach contraction principle.

The paper is organized as follows. In Section 2, we begin with some definitions and lemmas that will be used to prove our main result. Section 3 is devoted to existence and uniqueness of a solution to boundary value problem (1) and an example illustrating our results is presented. In Section 4, brief conclusion is given.

## 2 Preliminaries

In this section, we introduce some notations and definitions of fractional calculus and present preliminary results needed in our proofs later.

**Definition 1 ([13])** *The Hadamard derivative of fractional order $q$ for a $C^{n-1}$ function $g : [1, \infty) \to \mathbb{R}$ is defined by*

$$D^q g(t) = \frac{1}{\Gamma(n-q)} \Big(t \frac{d}{dt}\Big)^n \int_1^t \left(\log \frac{t}{s}\right)^{n-q-1} \frac{g(s)}{s} ds \qquad n - 1 < q < n, \; n = [q] + 1.$$

**Definition 2 ([13])** *The Hadamard fractional integral of order $q$ for a continuous function $g$ is defined as a function*

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_1^t \left(\log \frac{t}{s}\right)^{q-1} \frac{g(s)}{s} ds, \quad q > 0,$$

*provided the integral exists.*

**Lemma 1 ([13])** *If $a, \alpha, \beta > 0$, then*

$$\left( D_a^{\alpha} \left( \log \frac{t}{a} \right)^{\beta-1} \right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left( \log \frac{x}{a} \right)^{\beta-\alpha-1},$$

*and*

$$\left( I_a^{\alpha} \left( \log \frac{t}{a} \right)^{\beta-1} \right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} \left( \log \frac{x}{a} \right)^{\beta+\alpha-1}.$$

**Lemma 2 ([13])** *Let $q > 0$ and $u \in C[1,+\infty) \cap L^1[1,+\infty)$. Then the Hadamard fractional differential equation*

$$D^q u(t) = 0,$$

*has a solution*

$$u(t) = \sum_{k=1}^{n} c_k (\log t)^{\alpha-k},$$

*and the following formula holds*

$$I^q D^q u(t) = u(t) - \sum_{k=1}^{n} c_k (\log t)^{\alpha-k},$$

*where $c_k \in \mathbb{R}$, $k = 1, 2, ..., n$ and $n-1 < q < n$.*

Now, we give the exact expression of the Green's function associated to the fractional order differential equation with nonlocal boundary value conditions

$$\begin{cases} D^{\alpha} u(t) + h(t) = 0, \quad t \in J, \\ u(1) = u'(1) = 0, \\ D^{\alpha-1} u(+\infty) = \xi I^{\beta} u(\eta). \end{cases} \tag{2}$$

**Lemma 3** *Let $2 < \alpha \leq 3$, $h \in C[1,+\infty)$, $0 < \int\limits_1^{\infty} h(s) \frac{ds}{s} < +\infty$ and*

$$\Omega = \Gamma(\alpha) - \frac{\xi \Gamma(\alpha)}{\Gamma(\alpha+\beta)} (\log \eta)^{\alpha+\beta-1}. \tag{3}$$

*Then problem (2) has a solution $u$ given by the integral equation*

$$u(t) = \int_1^{\infty} G(t,s) h(s) \, \frac{ds}{s}, \tag{4}$$

*where*

$$G(t,s) = g(t,s) + \frac{\xi (\log t)^{\alpha-1}}{\Omega \Gamma(\alpha+\beta)} g(\eta, s), \tag{5}$$

*with*

$$g(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (\log t)^{\alpha-1} - \left(\log \frac{t}{s}\right)^{\alpha-1} & \text{for } 1 \leq s \leq t < \infty, \\ (\log t)^{\alpha-1} & \text{for } 1 \leq t \leq s < \infty, \end{cases}$$

*and*

$$g(\eta,s) = \begin{cases} (\log \eta)^{\alpha+\beta-1} - \left(\log \frac{\eta}{s}\right)^{\alpha+\beta-1} & \text{for } 1 \leq s \leq \eta < \infty, \\ (\log \eta)^{\alpha-1} & \text{for } 1 \leq \eta \leq s < \infty. \end{cases}$$

**Proof.** Applying the result of Lemma 2, we get the general solution of (2)

$$u(t)=c_1(\log t)^{\alpha-1}+c_2(\log t)^{\alpha-2}+c_3(\log t)^{\alpha-3}-\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds, \tag{6}$$

where $c_1,c_2,c_3\in\mathbb{R}$. Using the first boundary conditions of the problem (2) we obtain $c_2=c_3=0$. Therefore,

$$u(t)=c_1(\log t)^{\alpha-1}-\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds.$$

In accordance with Lemma 1, we have

$$D^{\alpha-1}u(t)=c_1\Gamma(\alpha)-\int_1^t\frac{h(s)}{s}ds.$$

By the second condition of (2), we get

$$c_1=\frac{1}{\Omega}\left(\int_1^\infty\frac{h(s)}{s}ds-\frac{\xi}{\Gamma(\alpha+\beta)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\alpha+\beta-1}\frac{h(s)}{s}ds\right),$$

where $\Omega$ is defined by (3).

Therefore, the unique solution of the fractional boundary value problem (2) is

$$\begin{aligned}
u(t) &= \frac{(\log t)^{\alpha-1}}{\Omega}\left[\int_1^\infty\frac{h(s)}{s}ds-\frac{\xi}{\Gamma(\alpha+\beta)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\alpha+\beta-1}\frac{h(s)}{s}ds\right]\\
&\quad -\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds\\
&= \frac{(\log t)^{\alpha-1}\Gamma(\alpha)}{\Gamma(\alpha)-\frac{\xi\Gamma(\alpha)}{\Gamma(\alpha+\beta)}(\log\eta)^{\alpha+\beta-1}}\int_1^\infty\frac{h(s)}{\Gamma(\alpha)}\frac{ds}{s}-\frac{\xi(\log t)^{\alpha-1}}{\Omega\Gamma(\alpha+\beta)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\alpha+\beta-1}\frac{h(s)}{s}ds\\
&\quad -\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds\\
&= \frac{(\log t)^{\alpha-1}\left(\Gamma(\alpha)-\xi\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)}(\log\eta)^{\alpha+\beta-1}+\xi\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)}(\log\eta)^{\alpha+\beta-1}\right)}{\Gamma(\alpha)-\xi\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)}(\log\eta)^{\alpha+\beta-1}}\int_1^\infty\frac{h(s)}{\Gamma(\alpha)}\frac{ds}{s}\\
&\quad -\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds-\frac{\xi(\log t)^{\alpha-1}}{\Omega\Gamma(\alpha+\beta)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\alpha+\beta-1}\frac{h(s)}{s}ds\\
&= \frac{1}{\Gamma(\alpha)}\int_1^t\left((\log t)^{\alpha-1}-\left(\log\frac{t}{s}\right)^{\alpha-1}\right)\frac{h(s)}{s}ds+\frac{1}{\Gamma(\alpha)}\int_t^\infty(\log t)^{\alpha-1}\frac{h(s)}{s}ds\\
&\quad +\frac{\xi(\log t)^{\alpha-1}}{\Omega\Gamma(\alpha+\beta)}\int_1^\eta\left((\log\eta)^{\alpha+\beta-1}-\left(\log\frac{\eta}{s}\right)^{\alpha+\beta-1}\right)\frac{h(s)}{s}ds\\
&\quad +\frac{\xi(\log t)^{\alpha-1}}{\Omega\Gamma(\alpha+\beta)}\int_\eta^\infty(\log\eta)^{\alpha+\beta-1}\frac{h(s)}{s}ds.\\
&= \int_1^\infty g(t,s)\frac{h(s)}{s}ds+\frac{\xi(\log t)^{\alpha-1}}{\Omega\Gamma(\alpha+\beta)}\int_1^\infty g(\eta,s)\frac{h(s)}{s}ds=\int_1^\infty G(t,s)h(s)\,\frac{ds}{s}.
\end{aligned}$$

The proof is completed. ■

**Lemma 4** *The Green's function $G(t,s)$ defined by (5) satisfies the following conditions*

$(A_1)$ *$G(t,s)$ is a continuous function for $(t,s)\in[1,+\infty)\times[1,+\infty)$.*

$(A_2)$ $G(t,s) \geq 0$ *for all* $s,t \in [1,\infty)$.

$(A_3)$ $\frac{G(t,s)}{1+(\log t)^{\alpha-1}} \leq \frac{1}{\Gamma(\alpha)} + \frac{\xi g(\eta,s)}{\Omega\Gamma(\alpha+\beta)}$ *for* $t,s \in [1,+\infty)$.

$(A_4)$ $\min\limits_{\eta\leq t\leq k\eta} \frac{G(t,s)}{1+(\log t)^{\alpha-1}} \geq \dfrac{\xi(\log\eta)^{\alpha-1}g(\eta,s)}{\Omega\Gamma(\alpha+\beta)\left(1+(\log\eta)^{\alpha-1}\right)}$ *for* $k>1$ *and* $s \in [1,+\infty)$.

**Proof.** It is easy to check that $(A_1)$ and $(A_2)$ hold, we only prove $(A_3)$ and $(A_4)$. To prove $(A_3)$, we have, for $s,t \in [1,\infty)$,

$$\begin{aligned}
\frac{G(t,s)}{1+(\log t)^{\alpha-1}} &= \frac{g(t,s)}{1+(\log t)^{\alpha-1}} + \frac{\xi(\log t)^{\alpha-1}g(\eta,s)}{\Omega\Gamma(\alpha+\beta)\left(1+(\log\eta)^{\alpha-1}\right)} \\
&\leq \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)\left(1+(\log t)^{\alpha-1}\right)} + \frac{\xi g(\eta,s)}{\Omega\Gamma(\alpha+\beta)}.\frac{(\log t)^{\alpha-1}}{1+(\log t)^{\alpha-1}} \\
&\leq \frac{1}{\Gamma(\alpha)} + \frac{\xi g(\eta,s)}{\Omega\Gamma(\alpha+\beta)}.
\end{aligned}$$

To prove $(A_4)$, from $g(t,s)\geq 0$ and $g(\eta,s)\geq 0$, for all $s,t \in [1,\infty)$, for $k>1$, we have

$$\begin{aligned}
\min_{\eta\leq t\leq k\eta}\frac{G(t,s)}{1+(\log t)^{\alpha-1}} &= \min_{\eta\leq t\leq k\eta}\left[\frac{g(t,s)}{1+(\log t)^{\alpha-1}} + \frac{\xi(\log t)^{\alpha-1}g(\eta,s)}{\Omega\Gamma(\alpha+\beta)\left(1+(\log\eta)^{\alpha-1}\right)}\right] \\
&\geq \min_{\eta\leq t\leq k\eta}\left(\frac{g(t,s)}{1+(\log t)^{\alpha-1}}\right) + \min_{\eta\leq t\leq k\eta}\left(\frac{\xi(\log t)^{\alpha-1}g(\eta,s)}{\Omega\Gamma(\alpha+\beta)\left(1+(\log\eta)^{\alpha-1}\right)}\right) \\
&\geq \min_{\eta\leq t\leq k\eta}\left(\frac{\xi(\log t)^{\alpha-1}g(\eta,s)}{\Omega\Gamma(\alpha+\beta)\left(1+(\log\eta)^{\alpha-1}\right)}\right) \\
&\geq \frac{\xi(\log\eta)^{\alpha-1}g(\eta,s)}{\Omega\Gamma(\alpha+\beta)\left(1+(\log\eta)^{\alpha-1}\right)},\ for\ s\in[1,\infty).
\end{aligned}$$

The proof is completed. ∎

In this paper, we will use the Banach space $E$ defined by

$$E = \left\{u \in C(J,\mathbb{R}),\ \sup_{t\in J}\left(\frac{|u(t)|}{1+(\log t)^{\alpha-1}}\right) < \infty\right\},$$

equipped with the norm

$$\|u\|_E = \sup_{t\in J}\left(\frac{|u(t)|}{1+(\log t)^{\alpha-1}}\right).$$

Set

$$F = \left\{u(t)\in E, D^{\alpha-1}u(t),\ D^{\alpha-2}u(t)\in C(J,\mathbb{R}), \sup_{t\in J}\frac{\left|D^{\alpha-2}u(t)\right|}{1+\log t} < +\infty, \sup_{t\in J}\left|D^{\alpha-1}u(t)\right| < +\infty\right\},$$

and

$$\|u\|_F = \max\left\{\sup_{t\in J}\frac{|u(t)|}{1+(\log t)^{\alpha-1}},\ \sup_{t\in J}\frac{\left|D^{\alpha-2}u(t)\right|}{1+\log t},\ \sup_{t\in J}\left|D^{\alpha-1}u(t)\right|\right\}.$$

**Lemma 5** $(F,\|.\|_F)$ *is a Banach space.*

**Proof.** Let $(u_n)$ be a cauchy sequence in the space $F$, so $(u_n)$ is a Cauchy sequence in the space $E$. Moreover, $(D^{\alpha-1}u_n)$ and $(D^{\alpha-2}u_n)$ converge uniformly to some $v,w\in C(J,\mathbb{R})$ where

$$\sup_{t\in J}|v(t)| < +\infty,\ \sup_{t\in J}|w(t)| < +\infty.$$

Define

$$\frac{M_0}{2} = \sup_{t\in J}\frac{|u(t)|}{1+(\log t)^{\alpha-1}}.$$

Then for $\frac{M_0}{2} > 0,\ \exists N > 0$ such that

$$\left|\frac{u_n(t)}{1+(\log t)^{\alpha-1}} - \frac{u(t)}{1+(\log t)^{\alpha-1}}\right| < M_0,\ \forall t \in J.$$

Setting

$$M_i = \sup_{t\in J}\frac{|u_i(t)|}{1+(\log t)^{\alpha-1}},\ i = 1, ..., N$$

and

$$M = \max\{M_i,\ i = 0, 1, ..., N\},$$

we can arrive at

$$\frac{|u_n(t)|}{1+(\log t)^{\alpha-1}} \le M, \quad n = 1, 2, ....$$

Therefore, for any $t \in J$ and $2 < \alpha \le 3$, we get

$$\begin{aligned}
&\left|\int_1^t (\log\frac{t}{s})^{2-\alpha}[1+(\log s)^{\alpha-1}](\frac{u_n(s)}{1+(\log s)^{\alpha-1}})\frac{ds}{s}\right| \\
\le\ & M\int_1^t (\log\frac{t}{s})^{2-\alpha}[1+(\log s)^{\alpha-1}]\frac{ds}{s} \\
\le\ & M\int_1^t (\log t - \log s)^{2-\alpha}[1+(\log s)^{\alpha-1}]\frac{ds}{s} \\
\le\ & M\frac{(\log t)^{3-\alpha}}{3-\alpha} + M(\log t)^2 B(\alpha, 3-\alpha),
\end{aligned}$$

where $B(\alpha, 3-\alpha)$ is the beta-function. According to the uniform convergence of $D^{\alpha-1}u_n(t)$ and Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}
w(t) &= \lim_{n\to+\infty} D^{\alpha-2}u_n(t) \\
&= \lim_{n\to+\infty}\frac{1}{\Gamma(3-\alpha)}\left(t\frac{d}{dt}\right)\int_1^t\left(\log\frac{t}{s}\right)^{2-\alpha}[1+(\log s)^{\alpha-1}]\left(\frac{u_n(s)}{1+(\log s)^{\alpha-1}}\right)\frac{ds}{s} \\
&= \left(t\frac{d}{dt}\right)\frac{1}{\Gamma(3-\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{2-\alpha}[1+(\log s)^{\alpha-1}]\left(\frac{u(s)}{1+(\log s)^{\alpha-1}}\right)\frac{ds}{s} \\
&= D^{\alpha-2}u(t).
\end{aligned}$$

On the other hand, if $\alpha = 3$, then

$$w(t) = \lim_{n\to+\infty} u_n'(t) = \frac{d}{dt}\lim_{n\to+\infty} u_n(t) = u'(t).$$

Similary, we obtain

$$v(t) = D^{\alpha-1}u_n(t) \to D^{\alpha-1}u(t).$$

We conclude that $(F, \|.\|_F)$ is a Banach space. ■

**Remark 1** *Note that to applied the Arzela-Ascoli theorem in basic space $F$, we need to establish the following modified compactness criterion.*

**Lemma 6** *Let $U \subseteq F$ be a bounded set. Then, $U$ is relatively compact in $F$ if the following conditions hold*

**(i)** *for any* $u(t) \in U$, $\frac{|u(t)|}{1+(\log t)^{\alpha-1}}$, $\frac{\left|D^{\alpha-2}u(t)\right|}{1+\log t}$, $\left|D^{\alpha-1}u(t)\right|$ *are equicontinuous on any compacts interval of* $J$,

**(ii)** *for any* $\varepsilon > 0$, *there exists a constant* $T = T(\varepsilon) > 0$ *such that*

$$\left|\frac{u(t_1)}{1+(\log t_1)^{\alpha-1}} - \frac{u(t_2)}{1+(\log t_2)^{\alpha-1}}\right| < \varepsilon,$$

$$\left|\frac{D^{\alpha-2}u(t_1)}{1+\log t_1} - \frac{D^{\alpha-2}u(t_2)}{1+\log t_2}\right| < \varepsilon,$$

$$\left|D^{\alpha-1}u(t_1) - D^{\alpha-1}u(t_2)\right| < \varepsilon,$$

*for any* $t_1, t_2 < T$ *and* $u(t) \in U$.

**Proof.** Obviously, it is sufficient to show that $U$ is totally bounded.

**Step 1** The case $t \in [1, T]$. We define the set

$$U_{[1,T]} = \{u(t),\ u(t) \in U;\ t \in [1, T]\},$$

which is a Banach space with the norm

$$\|u\|_{[1,T]} = \sup_{t\in[1,\infty)} \left|\frac{u(t)}{1+(\log t)^{\alpha-1}}\right|.$$

It follows from condition (i) and Arzela-Ascoli theorem that $U_{[1,T]}$ is relatively compact. Thus, for any $\epsilon > 0$, there exist finite balls $\{B_\epsilon(u_i)\}$; $i = 1, ..., n$ such that

$$U_{[1,T]} \subset \bigcup_{i=1}^{n} B_\epsilon(u_i),$$

where

$$B_\epsilon(u_i) = \left\{u(t) \in U_{[1,T]},\ \|u-u_i\|_{[1,T]} = \sup_{t\in[1,\infty)} \left|\frac{u(t)}{1+(\log t)^{\alpha-1}} - \frac{u_i(t)}{1+(\log t)^{\alpha-1}}\right| < \epsilon\right\}.$$

Similarly,

$$U^{\alpha-2}_{[1,T]} = \{D^{\alpha-2}u(t),\ u(t) \in U_{[1,T]},\ t \in [1, T]\},$$

is a Banach space with the norm

$$\left\|D^{\alpha-2}u\right\|_{[1,T]} = \sup_{t\in[1,\infty)} \left|\frac{D^{\alpha-2}u(t)}{1+\log t}\right|.$$

The set $U^{\alpha-2}_{[1,T]}$ is totally bounded, thus, for any $\epsilon > 0$, there exist balls $\{B_\epsilon(D^{\alpha-2}v_j)\}$, $j = 1, ..., m$ such that $U^{\alpha-2}_{[1,T]} \subset \bigcup_{j=1}^{m} B_\epsilon(D^{\alpha-2}v_j)$ where

$$B_\epsilon(D^{\alpha-2}v_j) = \left\{D^{\alpha-2}u(t) \in U^{\alpha-2}_{[1,T]}, \left\|D^{\alpha-2}u - D^{\alpha-2}v_j\right\|_{[1,T]} = \sup_{t\in[1,\infty)} \left|\frac{D^{\alpha-2}u(t)}{1+\log t} - \frac{D^{\alpha-2}v_j(t)}{1+\log t}\right| < \epsilon\right\}.$$

In a similar manner, the space

$$U^{\alpha-1}_{[1,T]} = \{D^{\alpha-1}u(t),\ u(t) \in U_{[1,T]},\ t \in [1, T]\},$$

is a Banach space with the norm

$$\left\|D^{\alpha-1}u\right\|_{[1,T]} = \sup_{t\in[1,\infty)} \left|D^{\alpha-1}u(t)\right|.$$

The set $U^{\alpha-1}_{[1,T]}$ is totally bounded, thus, for any $\epsilon>0$, there exist balls $\{B_\epsilon(D^{\alpha-1}w_p)\}$, $p=1,...,k$ such that $U^{\alpha-1}_{[1,T]}\subset\bigcup\limits_{p=1}^{k}B_\epsilon(D^{\alpha-1}w_p)$ where

$$B_\epsilon(D^{\alpha-1}w_p)=\left\{D^{\alpha-1}u(t)\in U^{\alpha-1}_{[1,T]},\left\|D^{\alpha-1}u-D^{\alpha-1}w_p\right\|_{[1,T]}=\sup_{t\in[1,\infty)}\left|D^{\alpha-1}u(t)-D^{\alpha-1}w_p(t)\right|\ <\epsilon\right\}.$$

**Step 2** We define

$$U_{ijp}=\left\{u(t)\in U,\ u_{[1,T]}\in B_\epsilon(u_i),\ D^{\alpha-2}u_{[1,T]}\in B_\epsilon(D^{\alpha-2}v_j),\ D^{\alpha-1}u_{[1,T]}\in B_\epsilon(D^{\alpha-1}w_p)\right\}.$$

Clearly,

$$U_{[1,T]}\subset\bigcup_{\substack{1\le i\le n,\\1\le j\le m,\\1\le p\le k}}U_{ijp_{[1,T]}}\Longrightarrow\begin{cases}U\subset\bigcup\limits_{\substack{1\le i\le n,\\1\le j\le m,\\1\le p\le k}}B_{4\epsilon}(u_{ijp}),\ u_{ijp}\in U_{ijp}\\ \\ B_{4\epsilon}(u_{ijp})=\left\{u(t)\in U,\ \left\|u-u_{ijp}\right\|_F\le4\epsilon\right\}\end{cases}$$

So $U$ can be covered by the balls $B_{4\epsilon}(u_{ijp})$. Hence, $U$ is totally bounded. ■

# 3 Main Results

We define the operator $T:F\to F$ by

$$Tu(t)=\int_1^\infty G(t,s)f(t,u(t),D^{\alpha-2}u(t),D^{\alpha-1}u(t))\frac{ds}{s},\ t\in J.$$

By Lemma 3, if $u$ is a fixed point of operator $T$, then $u$ is a solution of BVP (1). Direct computations show that

$$\begin{aligned}D^{\alpha-2}Tu(t)&=\int_1^\infty G_2(t,s)f(s,u(s),D^{\alpha-2}u(s),D^{\alpha-1}u(s))\frac{ds}{s},\ t\in J,\\D^{\alpha-1}Tu(t)&=\int_1^\infty G_1(t,s)f(s,u(s),D^{\alpha-2}u(s),D^{\alpha-1}u(s)\frac{ds}{s},\ t\in J,\end{aligned}$$

where

$$G_1(t,s)=\begin{cases}\frac{\zeta\Gamma(\alpha)g(\eta,s)}{\Omega\Gamma(\alpha+\beta)}, & 1\le s\le t,\\ 1+\frac{\zeta\Gamma(\alpha)g(\eta,s)}{\Omega\Gamma(\alpha+\beta)}, & 1\le t\le s,\end{cases}$$

and

$$G_2(t,s)=\frac{1}{\Gamma(2)}\begin{cases}\log s+\frac{\zeta g(\eta,s)}{\Omega\Gamma(\alpha+\beta)}(\log t), & 1\le s\le t,\\ \left(1+\frac{\zeta g(\eta,s)}{\Omega\Gamma(\alpha+\beta)}\right)\log t. & 1\le t\le s.\end{cases}$$

Throughout this paper, we assume that the following conditions hold

$(H_1)$ There exist nonnegative functions $a(t),b(t),c(t),p(t)\in L^1(J)$ with $(\log t)^{\alpha-1}a(t)$, $(\log t)\,b(t)\in L^1(J)$ such that:

$$|f(t,u,v,w)|\le a(t)\,|u|+b(t)\,|v|+c(t)\,|w|+p(t),\ \ (t,u,v,w)\in J\times\mathbb{R}^3.$$

$(H_2)$ There exist nonnegative functions $a(t),b(t),c(t)\in L^1(J)$ with $(\log t)^{\alpha-1}a(t)$, $(\log t)\,b(t)\in L^1(J)$ such that for $t\in J$ and $u_1,v_1,w_1,u_2,v_2,w_2\in\mathbb{R}$, we have

$$|f(t,u_1,v_1,w_1)-f(t,u_2,v_2,w_2)|\le a(t)\,|u_1-u_2|+b(t)\,|v_1-v_2|+c(t)\,|w_1-w_2|\,.$$

Denote

$$a^{*}=\int_{1}^{+\infty}\left(1+(\log t)^{\alpha-1}\right)a(t)dt,\ b^{*}=\int_{1}^{+\infty}(1+\log t)\,b(t)dt,\ c^{*}=\int_{1}^{+\infty}c(t)dt,\ p^{*}=\int_{1}^{+\infty}p(t)dt.$$

**Lemma 7** *Assume that $(H_1)$ holds. Then*

$$\int_{1}^{+\infty}\left|f(s,u(s),D^{\alpha-2}u(s),D^{\alpha-1}u(s)\right|ds\leq(a^{*}+b^{*}+c^{*})\left\|u\right\|_{F}+p^{*},\quad\forall u\in F.$$

**Proof.** By $(H_1)$, for any $u\in F$, we have

$$\begin{aligned}
&\int_{1}^{+\infty}\left|f(s,u(s),D^{\alpha-2}u(s),D^{\alpha-1}u(s)\right|ds\\
\leq&\int_{1}^{+\infty}\left[a(s)\left|u(s)\right|+b(s)\left|D^{\alpha-2}u(s)\right|+c(s)\left|D^{\alpha-1}u(s)\right|+p(s)\right]ds\\
=&\int_{1}^{+\infty}\left[a(s)\left(1+(\log s)^{\alpha-1}\right)\frac{\left|u(s)\right|}{1+(\log s)^{\alpha-1}}+b(s)\left(1+\log s\right)\frac{\left|D^{\alpha-2}u(s)\right|}{(1+\log s)}+c(s)\left|D^{\alpha-1}u(s)\right|+p(s)\right]ds\\
\leq&\left(\int_{1}^{+\infty}a(s)\left(1+(\log s)^{\alpha-1}\right)ds\right)\left\|u\right\|_{F}+\left(\int_{1}^{+\infty}b(s)\left(1+\log s\right)ds\right)\left\|u\right\|_{F}\\
&+\left(\int_{1}^{+\infty}c(s)ds\right)\left\|u\right\|_{F}+\int_{1}^{+\infty}p(s)ds=(a^{*}+b^{*}+c^{*})\left\|u\right\|_{F}+p^{*}.
\end{aligned}$$

■

**Lemma 8** *Assume that $(H_1)$ holds. Then the operator $T:F\rightarrow F$ is completely continuous.*

**Proof.** We divide this proof into the following four steps.

**Step 1.** We prove that $T:F\rightarrow F$ is continuous.
Let $U$ be bounded subset of $F$. Then there exists constant $C>0$ such that $||u||_F\leq C,\forall u\in U$. Let $(u_n)\in U$ and $\lim\limits_{n\rightarrow+\infty}\left\|u_n-u\right\|_F=0$. We have

$$\left\|u_n-u\right\|_F=\max\left\{\sup_{t\in J}\frac{\left|u_n(t)-u(t)\right|}{1+(\log t)^{\alpha-1}},\sup_{t\in J}\frac{\left|D^{\alpha-2}u_n(t)-D^{\alpha-2}u(t)\right|}{1+\log t},\sup_{t\in J}\left|D^{\alpha-1}u_n(t)-D^{\alpha-1}u(t)\right|\right\}.$$

Then

$$\lim_{n\rightarrow+\infty}\left|u_n(t)-u(t)\right|=0,\quad\lim_{n\rightarrow+\infty}\left|D^{\alpha-2}u_n(t)-D^{\alpha-2}u(t)\right|=0\ \text{ and }\lim_{n\rightarrow+\infty}\left|D^{\alpha-1}u_n(t)-D^{\alpha-1}u(t)\right|=0.$$

Therefore, for any $t\in J$ and $n>N$, we get

$$\begin{aligned}
&\left|\frac{Tu_n(t)-Tu(t)}{1+(\log t)^{\alpha-1}}\right|\\
=&\left|\frac{Tu_n(t)}{1+(\log t)^{\alpha-1}}-\frac{Tu(t)}{1+(\log t)^{\alpha-1}}\right|\\
\leq&\int_{1}^{+\infty}\frac{G(t,s)}{1+(\log t)^{\alpha-1}}\left|f(s,u_n(s),D^{\alpha-2}u_n(s),D^{\alpha-1}u_n(s))-f(s,u(s),D^{\alpha-2}u(s),D^{\alpha-1}u(s))\right|\frac{ds}{s}\\
\leq&\left(\frac{1}{\Gamma(\alpha)}+\frac{\zeta\left(\log\eta\right)^{\alpha+\beta-1}}{\Omega\Gamma\left(\alpha+\beta\right)}\right)
\end{aligned}$$

$$
\begin{aligned}
& \times \int_1^{+\infty} \left| f(s,u_n(s),D^{\alpha-2}u_n(s),D^{\alpha-1}u_n(s)) - f(s,u(s),D^{\alpha-2}u(s),D^{\alpha-1}u(s)) \right| \frac{ds}{s} \\
\leq\ & \frac{1}{\Omega}\int_1^{+\infty} \left| f(s,u_n(s),D^{\alpha-2}u_n(s),D^{\alpha-1}u_n(s)) \right| \frac{ds}{s} + \frac{1}{\Omega}\int_1^{+\infty} \left| f(s,u(s),D^{\alpha-2}u(s),D^{\alpha-1}u(s)) \right| \frac{ds}{s} \\
\leq\ & \frac{2}{\Omega}\left[ \left(a^*+b^*+c^*\right)C + p^* \right].
\end{aligned}
$$

Similarly, there exists $Q>0$ such that, for $n>Q$ and $t\in J$,

$$
\begin{aligned}
& \left| \frac{D^{\alpha-2}Tu_n(t) - D^{\alpha-2}Tu(t)}{1+\log t} \right| \\
=\ & \left| \frac{D^{\alpha-2}Tu_n(t)}{1+\log t} - \frac{D^{\alpha-2}Tu(t)}{1+\log t} \right| \\
\leq\ & \int_1^{+\infty} \frac{G_2(t,s)}{1+\log t} \left| f(s,u_n(s),D^{\alpha-2}u_n(s),D^{\alpha-1}u_n(s)) - f(s,u(s),D^{\alpha-2}u(s),D^{\alpha-1}u(s)) \right| \frac{ds}{s} \\
\leq\ & \left( 1 + \frac{\zeta\left(\log\eta\right)^{\alpha+\beta-1}}{\Omega\Gamma\left(\alpha+\beta\right)} \right) \\
& \times \int_1^{+\infty} \left| f(s,u_n(s),D^{\alpha-2}u_n(s),D^{\alpha-1}u_n(s)) - f(s,u(s),D^{\alpha-2}u(s),D^{\alpha-1}u(s)) \right| \frac{ds}{s} \\
\leq\ & 2\left( 1 + \frac{1}{\Omega} - \frac{1}{\Gamma(\alpha)} \right)\left[ \left(a^*+b^*+c^*\right)C + p^* \right],
\end{aligned}
$$

and

$$
\begin{aligned}
& \left| D^{\alpha-1}Tu_n(t) - D^{\alpha-1}Tu(t) \right| \\
\leq\ & \int_1^{+\infty} G_1(t,s) \left| f(s,u_n(s),D^{\alpha-2}u_n(s),D^{\alpha-1}u_n(s)) - f(s,u(s),D^{\alpha-2}u(s),D^{\alpha-1}u(s)) \right| \frac{ds}{s} \\
\leq\ & \left( 1 + \frac{\zeta\Gamma\left(\alpha\right)\left(\log\eta\right)^{\alpha+\beta-1}}{\Omega\Gamma\left(\alpha+\beta\right)} \right) \\
& \int_1^{+\infty} \left| f(s,u_n(s),D^{\alpha-2}u_n(s),D^{\alpha-1}u_n(s)) - f(s,u(s),D^{\alpha-2}u(s),D^{\alpha-1}u(s)) \right| \frac{ds}{s} \\
\leq\ & \frac{2\Gamma\left(\alpha\right)}{\Omega}\left[ \left(a^*+b^*+c^*\right)C + p^* \right].
\end{aligned}
$$

Then Lebesgue's dominated convergence theorem asserts that $\lim_{n\to+\infty}\|Tu_n(t)-Tu(t)\| = 0$. Hence, $T:F\to F$ is continuous.

**Step 2.** We show that $T:F\to F$ is uniformly bounded.
For any $u\in U$ and $t\in J$, we obtain

$$
\begin{aligned}
\left| \frac{Tu(t)}{1+(\log t)^{\alpha-1}} \right| &\leq \int_1^{+\infty} \frac{G(t,s)}{1+(\log t)^{\alpha-1}} \left| f(s,u(s),D^{\alpha-2}u(s),D^{\alpha-1}u(s)) \right| \frac{ds}{s} \\
&\leq \left( \frac{1}{\Gamma(\alpha)} + \frac{\zeta\left(\log\eta\right)^{\alpha+\beta-1}}{\Omega\Gamma\left(\alpha+\beta\right)} \right)\int_1^{+\infty} \left| f(s,u(s),D^{\alpha-2}u(s),D^{\alpha-1}u(s)) \right| \frac{ds}{s} \\
&\leq \frac{1}{\Omega}\left[ \left(a^*+b^*+c^*\right)R + p^* \right] = C_1.
\end{aligned}
$$

Similarly,

$$
\left| \frac{D^{\alpha-2}Tu(t)}{1+\log t} \right| \leq \left( 1 + \frac{1}{\Omega} - \frac{1}{\Gamma\left(\alpha\right)} \right)\left[ \left(a^*+b^*+c^*\right)R + p^* \right] = C_2,
$$

$$\left|D^{\alpha-1}Tu(t)\right| \leq \frac{\Gamma(\alpha)}{\Omega}\left[(a^*+b^*+c^*)R+p^*\right]=C_3.$$

Consequently, $T(U)$ is uniformly bounded.

**Step 3.** We show that $T(U)$ is equicontinuous on J.

Let $I \subseteq J$ be any compact interval. For any $t_1, t_2 \in I$, $t_1 < t_2$ and $u \in U$, we deduce

$$\begin{aligned}
&\left|\frac{Tu(t_2)}{1+(\log t_2)^{\alpha-1}}-\frac{Tu(t_1)}{1+(\log t_1)^{\alpha-1}}\right| \\
\leq\ & \int_1^{+\infty}\left|\frac{G(t_2,s)}{1+(\log t_2)^{\alpha-1}}-\frac{G(t_1,s)}{1+(\log t_1)^{\alpha-1}}\right|\left|f(s,u(s),D^{\alpha-2}u(s),D^{\alpha-1}u(s))\right|\frac{ds}{s} \\
\leq\ & \int_1^{+\infty}\left(\frac{g(t_2,s)}{1+(\log t_2)^{\alpha-1}}-\frac{g(t_1,s)}{1+(\log t_1)^{\alpha-1}}\right)\left|f(s,u(s),D^{\alpha-2}u(s),D^{\alpha-1}u(s))\right|\frac{ds}{s} \\
&+\int_1^{+\infty}\left(\frac{(\log t_2)^{\alpha-1}}{1+(\log t_2)^{\alpha-1}}-\frac{(\log t_1)^{\alpha-1}}{1+(\log t_1)^{\alpha-1}}\right)\frac{\zeta g(\eta,s)}{\Omega\Gamma(\alpha+\beta)}\left|f(s,u(s),D^{\alpha-2}u(s),D^{\alpha-1}u(s))\right|\frac{ds}{s} \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_1^{t_1}\frac{(\log t_2)^{\alpha-1}-(\log t_1)^{\alpha-1}+(\log\frac{t_2}{s})^{\alpha-1}-(\log\frac{t_1}{s})^{\alpha-1}}{1+(\log t_2)^{\alpha-1}}f(s,u(s),D^{\alpha-2}u(s),D^{\alpha-1}u(s))\frac{ds}{s} \\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{(\log t_2)^{\alpha-1}-(\log t_1)^{\alpha-1}+(\log\frac{t_2}{s})^{\alpha-1}}{1+(\log t_2)^{\alpha-1}}f(s,u(s),D^{\alpha-2}u(s),D^{\alpha-1}u(s))\frac{ds}{s} \\
&+\frac{1}{\Gamma(\alpha)}\int_{t_2}^{+\infty}\frac{(\log t_2)^{\alpha-1}-(\log t_1)^{\alpha-1}}{1+(\log t_2)^{\alpha-1}}f(s,u(s),D^{\alpha-2}u(s),D^{\alpha-1}u(s))\frac{ds}{s}.
\end{aligned}$$

Then

$$\left|\frac{Tu(t_2)}{1+(\log t_2)^{\alpha-1}}-\frac{Tu(t_1)}{1+(\log t_1)^{\alpha-1}}\right| \to 0 \text{ uniformly as } t_1 \to t_2.$$

Similarly, we have

$$\left|\frac{D^{\alpha-2}Tu(t_2)}{1+\log t_2}-\frac{D^{\alpha-2}Tu(t_1)}{1+\log t_1}\right| \to 0 \text{ uniformly as } t_1 \to t_2,$$

and

$$\left|D^{\alpha-1}Tu(t_2)-D^{\alpha-1}Tu(t_1)\right| \to 0 \text{ uniformly as } t_1 \to t_2.$$

Thus, $T(U)$ is equicontinuous on $J$.

**Step 4.** We show that $T$ is equiconvergent at $\infty$.

$$\begin{aligned}
\lim_{t\to\infty}\left|\frac{Tu(t)}{1+(\log t)^{\alpha-1}}\right| &= \lim_{t\to\infty}\left|\frac{1}{1+(\log t)^{\alpha-1}}\int_1^{+\infty}G(t,s)f(s,u(s),D^{\alpha-2}u(s),D^{\alpha-1}u(s))\frac{ds}{s}\right| \\
&= \lim_{t\to\infty}\left|\frac{1}{1+(\log t)^{\alpha-1}}\int_1^{+\infty}\left(g(t,s)+(\log t)^{\alpha-1}\frac{\zeta g(\eta,s)}{\Omega\Gamma(\alpha+\beta)}\right)\right. \\
&\qquad \left. f(s,u(s),D^{\alpha-2}u(s),D^{\alpha-1}u(s))\frac{ds}{s}\right| \\
&\leq \left(\frac{1}{\Gamma(\alpha)}+\frac{\zeta g(\eta,s)}{\Omega\Gamma(\alpha+\beta)}\right)\int_1^{+\infty}\left|f(s,u(s),D^{\alpha-2}u(s),D^{\alpha-1}u(s))\right|\frac{ds}{s} \\
&\leq \left(\frac{1}{\Gamma(\alpha)}+\frac{\zeta g(\eta,s)}{\Omega\Gamma(\alpha+\beta)}\right)\left[(a^*+b^*+c^*)C+p^*\right] < +\infty.
\end{aligned}$$

Similarly, we have

$$\lim_{t\to\infty}\left|\frac{D^{\alpha-2}Tu(t)}{1+\log t}\right| = \lim_{t\to\infty}\left|\frac{1}{1+\log t}\int_1^{+\infty}G_2(t,s)f(s,u(s),D^{\alpha-2}u(s),D^{\alpha-1}u(s))\frac{ds}{s}\right|$$

$$\begin{aligned}
&\leq \left(1+\frac{\Gamma(\alpha)\zeta(\log\eta)^{\alpha+\beta-1}}{\Omega\Gamma(\alpha+\beta)}\right)\int_1^{+\infty}\left|f(s,u(s),D^{\alpha-2}u(s),D^{\alpha-1}u(s))\right|\frac{ds}{s}\\
&\leq \left(1+\frac{\Gamma(\alpha)\zeta(\log\eta)^{\alpha+\beta-1}}{\Omega\Gamma(\alpha+\beta)}\right)\left[(a^*+b^*+c^*)C+p^*\right]<+\infty.
\end{aligned}$$

and

$$\begin{aligned}
\lim_{t\to\infty}\left|D^{\alpha-1}Tu(t)\right| &= \lim_{t\to\infty}\left|\int_1^{+\infty}G_1(t,s)\ f(s,u(s),D^{\alpha-2}u(s),D^{\alpha-1}u(s))\frac{ds}{s}\right|\\
&\leq \left(1+\frac{\Gamma(\alpha)\zeta(\log\eta)^{\alpha+\beta-1}}{\Omega\Gamma(\alpha+\beta)}\right)\int_1^{+\infty}\left|f(s,u(s),D^{\alpha-2}u(s),D^{\alpha-1}u(s))\right|\frac{ds}{s}\\
&\leq \left(1+\frac{\Gamma(\alpha)\zeta(\log\eta)^{\alpha+\beta-1}}{\Omega\Gamma(\alpha+\beta)}\right)\left[(a^*+b^*+c^*)C+p^*\right]<+\infty.
\end{aligned}$$

Hence, $T(V)$ is equiconvergent at infinity. Thus, the operator $T$ is completely continuous. The proof is completed. ■

**Theorem 9** *Assume that* $(H_1)$ *holds. If* $\Omega>\Gamma(\alpha)(a^*+b^*+c^*)$, *then problem* (1) *has at least one solution.*

**Proof.** Let $U=\{u\in F,\ \|u\|_F\leq C\}$ with

$$C\geq\frac{\Gamma(\alpha)}{\Omega-\Gamma(\alpha)(a^*+b^*+c^*)}.$$

For any $u\in U$ and $t\in J$, we deduce

$$\begin{aligned}
\left|\frac{Tu(t)}{1+(\log t)^{\alpha-1}}\right| &\leq \int_1^{+\infty}\left|\frac{G(t,s)}{1+(\log t)^{\alpha-1}}\right|\left|f(s,u(s),D^{\alpha-2}u(s),D^{\alpha-1}u(s))\right|\frac{ds}{s}\\
&\leq \frac{1}{\Omega}\left[(a^*+b^*+c^*)R+p^*\right]\leq C,
\end{aligned}$$

$$\begin{aligned}
\left|\frac{D^{\alpha-2}Tu(t)}{1+\log t}\right| &\leq \int_1^{+\infty}\left|\frac{G_2(t,s)}{1+\log t}\right|\left|f(s,u(s),D^{\alpha-2}u(s),D^{\alpha-1}u(s))\right|\frac{ds}{s}\\
&\leq \left(1+\frac{1}{\Omega}-\frac{1}{\Gamma(\alpha)}\right)\left[(a^*+b^*+c^*)C+p^*\right]\leq C,
\end{aligned}$$

and

$$\begin{aligned}
\left|D^{\alpha-1}Tu(t)\right| &\leq \int_1^{+\infty}\left|G_1(t,s)\right|\left|f(s,u(s),D^{\alpha-2}u(s),D^{\alpha-1}u(s))\right|\frac{ds}{s}\\
&\leq \frac{\Gamma(\alpha)}{\Omega}\left[(a^*+b^*+c^*)C+p^*\right]\leq C.
\end{aligned}$$

Then $T(U)\subset U$ and $T:U\to U$ is completely continuous (from Lemma 8). Therefore, by the Schauder fixed point theorem, we conclude that BVP (1) has at least one solution in $U$. The proof is completed. ■

**Theorem 10** *Assume that* $(H_2)$ *holds. If* $\Omega>\Gamma(\alpha)(a^*+b^*+c^*)$, *then problem* (1) *has a unique solution in* $F$.

**Proof.** By $(H_2)$, we conclude that, for any $u,v\in F$ and $t\in J$

$$\left|\frac{Tu(t)-Tv(t)}{1+(\log t)^{\alpha-1}}\right|$$

$$
\begin{aligned}
&\leq \int_1^{+\infty} \left| \frac{G(t,s)}{1+(\log t)^{\alpha-1}} \right| \left| f(s,u(s),D^{\alpha-2}u(s),D^{\alpha-1}u(s)) - f(s,v(s),D^{\alpha-2}v(s),D^{\alpha-1}v(s)) \right| \frac{ds}{s} \\
&\leq \frac{\Gamma(\alpha)}{\Omega} \int_1^{+\infty} \left( a(s)\left|u(s)-v(s)\right| + b(s)\left|D^{\alpha-2}u(s)-D^{\alpha-2}v(s)\right| + c(s)\left|D^{\alpha-1}u(s)-D^{\alpha-1}v(s)\right| \right) \frac{ds}{s} \\
&= \frac{\Gamma(\alpha)}{\Omega} \int_1^{+\infty} (a(s)\left(1+(\log s)^{\alpha-1}\right) \frac{|u(s)-v(s)|}{1+(\log s)^{\alpha-1}} \\
&\quad +b(s)\left(1+\log s\right) \frac{\left|D^{\alpha-2}u(s)-D^{\alpha-2}v(s)\right|}{1+\log s} + c(s)\left|D^{\alpha-1}u(s)-D^{\alpha-1}v(s)\right|)\frac{ds}{s} \\
&\leq \frac{\Gamma(\alpha)}{\Omega}\left(a^*+b^*+c^*\right)\|u-v\|_F = k\|u-v\|_F ,
\end{aligned}
$$

with

$$k = \frac{\Gamma(\alpha)}{\Omega}\left(a^*+b^*+c^*\right) < 1.$$

Similarly, we obtain

$$\left|\frac{D^{\alpha-2}Tu(t)-D^{\alpha-2}Tv(t)}{1+\log t}\right| \leq k\|u-v\|_F , \quad \left|D^{\alpha-1}Tu(t)-D^{\alpha-1}Tv(t)\right| \leq k\|u-v\|_F .$$

Thus

$$\|Tu-Tv\|_F \leq k\|u-v\|_F .$$

It follows from the Banach contraction mapping theorem that $T$ has a unique fixed point in $F$. ■

**Example 1** *We consider the following Hadamard fractional boundary value problem*

$$\begin{cases} D^{\alpha}u(t)+f(t,u(t),D^{\alpha-2}u(t),D^{\alpha-1}u(t))=0 \quad t\in J=[1,+\infty), \\ u(1)=u'(1)=0, \\ D^{\alpha-1}u(+\infty)=\xi I^{\beta}u(\eta), \end{cases} \tag{7}$$

*with* $\alpha=\frac{5}{2}$, $\beta=\frac{1}{2}$, $\xi=\frac{1}{4}$, $\eta=3$ *and*

$$f(t,u,v,w)=\frac{\ln(1+|v|)}{(1+t^2)\,e^{5t}}+\frac{\sqrt{|uv|}}{3te^{2t}}+\frac{w}{30\,(1+t^2)}+\frac{t}{1+e^{t^2}}.$$

*Obviously,*

$$|f(t,u,v,w)| \leq \frac{1}{6te^{2t}}|u| + \left[\frac{1}{6te^{2t}}+\frac{1}{(1+t^2)\,e^{5t}}\right]|v| + \frac{|w|}{30\,(1+t^2)}+\frac{1}{1+e^{t^2}},$$

*and*

$$|f(t,u_1,v_1,w_1)-f(t,u_2,v_2,w_2)| \leq \frac{1}{6te^{2t}}|u_1-u_2| + \left[\frac{1}{6te^{2t}}+\frac{1}{(1+t^2)\,e^{5t}}\right]|v_1-v_2| + \frac{1}{30(1+t^2)}|w_1-w_2| .$$

*Then* $f$ *satisfies* $(H_1)$ *and* $(H_2)$*, with*

$$
\begin{gathered}
a(t)=\frac{1}{6te^{2t}} \Longrightarrow a^*=\frac{1}{12}, \\
b(t)=\frac{1}{6te^{2t}}+\frac{1}{(1+t^2)\,e^{5t}} \Longrightarrow b^*=\frac{17}{60}, \\
c(t)=\frac{1}{30\,(1+t^2)} \Longrightarrow c^*=\frac{\pi}{60}, \\
p(t)=\frac{1}{1+e^{t^2}} \Longrightarrow p^*=\frac{1}{2}.
\end{gathered}
$$

*Moreover, we find that*

$$k=\frac{\Gamma(\alpha)}{\Omega}\left(a^*+b^*+c^*\right)=0.488<1.$$

*Therefore, all conditions of Theorem 1 and 2 are satisfied. Thus, problem (7) has a unique solution.*

## 4 Conclusion

In this paper, we investigate the existence and uniqueness of solutions for Hadamard fractional equations on an infinite interval with integral boundary value conditions. We were able to overcome the main challenges by establishing a proper compactness criterion. Schauder fixed point theorem was the key of our analysis to establish existence of solutions of our problem by adding suitable conditions on the nonlinear term. We succeeded to obtain a unique solution by using Banach contraction principle. At the same time, we present an example to demonstrate the consistency of our the theoretical findings.

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