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Mapping Properties Of Certain Subclasses Of Analytic Functions Associated With Generalized Distribution Series*

Saurabh Porwal†

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Abstract

The purpose of the present paper is to obtain some necessary and sufficient conditions for generalized distribution series to be in certain special subclasses.

1 Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions of the form (1) and univalent in $\mathbb{U}$. As usual, we denote by $\mathcal{T}$ [22] the subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad n = 2, 3, \ldots. \tag{2}$$

A function $f(z)$ of the form (1) is said to be starlike of order α $(0 \leq \alpha < 1)$, if it satisfies the following condition

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \qquad z \in \mathbb{U}.$$

A function $f(z)$ of the form (1) is said to be convex of order α $(0 \leq \alpha < 1)$, if it satisfies the following condition

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \qquad z \in \mathbb{U}.$$

The classes of all starlike and convex functions of order α are denoted by $S^*(\alpha)$ and $K(\alpha)$ were introduced and studied by Robertson [19] and Silverman [22]. We also write $S^*(0) = S^*$ and $K(0) = K$ are the well-known classes of starlike and convex functions.

A function $f(z)$ of the form (1) is said to be in the class $\mathcal{G}(\lambda, \alpha)$, if it satisfies the following condition

$$\Re\left\{\frac{zf'(z) + \lambda z^2 f''(z)}{f(z)}\right\} > \alpha, \qquad z \in \mathbb{U},$$

where $0 \leq \alpha < 1$ and $0 \leq \lambda < 1$.

A function $f(z)$ of the form (1) is said to be in the class $\mathcal{K}(\lambda, \alpha)$, if it satisfies the following condition

$$\Re\left\{\frac{z\left[zf'(z) + \lambda z^2 f''(z)\right]'}{zf'(z)}\right\} > \alpha, \qquad z \in \mathbb{U},$$

*Mathematics Subject Classifications: 30C45.

†Lecturer Mathematics, Sri Radhey Lal Arya Inter College, Aihan-204101, Hathras, (U.P.) India

where $0 \leq \alpha < 1$ and $0 \leq \lambda < 1$.

Also we write $\mathcal{TG}(\lambda, \alpha) \equiv \mathcal{G}(\lambda, \alpha) \cap \mathcal{T}$ and $\mathcal{TK}(\lambda, \alpha) \equiv \mathcal{K}(\lambda, \alpha) \cap \mathcal{T}$.

Remark 1 *It is easy to verify that for $\lambda = 0$, we have $\mathcal{G}(\lambda, \alpha) \equiv \mathcal{S}^*(\alpha)$ and $\mathcal{K}(\lambda, \alpha) \equiv \mathcal{K}(\alpha)$.*

A function $f \in \mathcal{A}$ is said to be in the class $f \in \Re^\tau(A, B)$ ($\tau \in \mathbb{C}\backslash\{0\}$, $-1 \leq B < A \leq 1$), if it satisfies the inequality

$$\left| \frac{f'(z) - 1}{(A - B)\tau - B[f'(z) - 1]} \right| < 1, \quad (z \in \mathbb{U}).$$

The class $\Re^\tau(A, B)$ was introduced earlier by Dixit and Pal [6].

The applications of hypergeometric function ([5], [9], [12], [23], [24]), confluent hypergeometric functions ([4], [7]), Wright's function [18], generalized Bessel functions ([3], [8], [15]) are interesting topics of research in Geometric Function Theory. In 2014, Porwal [13] introduced Poisson distribution series and give a nice application on analytic univalent functions and co-relates probability density function with univalent function. After the appearance of this paper several researchers introduced hypergeometric distribution series [1], hypergeometric distribution type series [16], confluent hypergeometric distribution series [17], Binomial distribution series [11], Mittag-Leffler type distribution series [2] and obtain sufficient conditions and inclusion relations for various classes of univalent functions. Very recently, Porwal [14] introduced generalized distribution and studied its geometric properties associated with univalent functions. Now we recall the definition of generalized distribution. The probability mass function of the generalized distribution is given as

$$p(n) = \frac{t_n}{S}, \quad n = 0, 1, 2, \ldots,$$

where $t_n \geq 0$ and the series $\sum_{n=0}^{\infty} t_n$ is convergent and

$$S = \sum_{n=0}^{\infty} t_n. \tag{3}$$

Further, we introduce the series

$$\phi(x) = \sum_{n=0}^{\infty} t_n x^n. \tag{4}$$

From (3) it is easy to see that the series given by (4) is convergent for $|x| < 1$ and for $x = 1$, it is also convergent. Porwal [14] introduce generalized distribution series as

$$K_\phi(z) = z + \sum_{n=2}^{\infty} \frac{t_{n-1}}{S} z^n. \tag{5}$$

Now, we define

$$TK_\phi(z) = 2z - K_\phi(z) = z - \sum_{n=2}^{\infty} \frac{t_{n-1}}{S} z^n. \tag{6}$$

We define the convolution (or Hadamard product) of two functions $f \in \mathcal{A}$ given by (1) and $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad (z \in \mathbb{U}). \tag{7}$$

Next, we introduce the convolution operator $K_\phi(f, z)$ for functions f of the form (1) as follows

$$K_\phi(f, z) = K_\phi(z) * f(z),$$

$$K_\phi(f,z)=z+\sum_{n=2}^{\infty}\frac{a_n t_{n-1}}{S}z^n. \tag{8}$$

Motivated by results on connections between various subclasses of analytic univalent functions by using generalized Bessel functions [8, 15], hypergeometric functions by [4, 5, 7, 9, 12, 20, 21, 23, 24], Wright's hypergeometric functions [18], Poisson distribution series [10], Hypergeometric distribution series [1, 16, 17], Binomial distribution series [11], we obtain necessary and sufficient condition for functions $T_\phi(z)$ in $\mathcal{TG}(\lambda,\alpha)$ and $\mathcal{TK}(\lambda,\alpha)$. Finally, we estimate certain inclusion relations between the classes $\Re^\tau(A,B)$ and $\mathcal{K}(\lambda,\alpha)$.

2 Main Results

To prove the main results, we need the following Lemmas.

Lemma 1 **([6])** *A function $f\in\Re^\tau(A,B)$ is of form (1), then*

$$|a_n|\le(A-B)\frac{|\tau|}{n},\quad n\in\mathbb{N}\backslash\{1\}. \tag{9}$$

The bound given in (9) is sharp.

Lemma 2 **([25])** *A function $f\in\mathcal{A}$ belongs to the class $\mathcal{G}(\lambda,\alpha)$ if*

$$\sum_{n=2}^{\infty}(n+\lambda n(n-1)-\alpha)|a_n|\le 1-\alpha. \tag{10}$$

Lemma 3 **([25])** *A function $f\in\mathcal{A}$ belongs to the class $\mathcal{K}(\lambda,\alpha)$ if*

$$\sum_{n=2}^{\infty}n(n+\lambda n(n-1)-\alpha)|a_n|\le 1-\alpha. \tag{11}$$

Further we can easily prove that the conditions are also necessary if $f\in\mathcal{T}$.

Lemma 4 *A function $f\in\mathcal{T}$ belongs to the class $\mathcal{TG}(\lambda,\alpha)$ if and only if (10) is satisfied.*

Lemma 5 *A function $f\in\mathcal{T}$ belongs to the class $\mathcal{TK}(\lambda,\alpha)$ if and only if (11) is satisfied.*

Theorem 1 *Let $TK_\phi(z)$ be of the form (6) is in the class $\mathcal{TG}(\lambda,\alpha)$ if and only if*

$$\lambda\phi''(1)+(1+2\lambda)\phi'(1)+(1-\alpha)\phi(1)\le(1-\alpha)(S+\phi(0)). \tag{12}$$

Proof. To prove that $TK_\phi(z)\in\mathcal{TG}(\lambda,\alpha)$, from Lemma 4 we have to show that

$$\sum_{n=2}^{\infty}(n+\lambda n(n-1)-\alpha)\frac{t_{n-1}}{S}\le 1-\alpha.$$

By the given hypothesis, we see that

$$\begin{aligned}
&\sum_{n=2}^{\infty}(n+\lambda n(n-1)-\alpha)\frac{t_{n-1}}{S} \\
=\ &\sum_{n=2}^{\infty}[\lambda(n-1)(n-2)+(1+2\lambda)(n-1)+(1-\alpha)]\times\frac{t_{n-1}}{S} \\
=\ &\frac{1}{S}\left[\sum_{n=2}^{\infty}[\lambda(n-1)(n-2)t_{n-1}+(1+2\lambda)(n-1)t_{n-1}+(1-\alpha)t_{n-1}]\right] \\
=\ &\frac{1}{S}\left[\sum_{n=1}^{\infty}[\lambda n(n-1)t_n+(1+2\lambda)nt_n+(1-\alpha)t_n]\right] \\
=\ &\frac{1}{S}\left[\lambda\phi''(1)+(1+2\lambda)\phi'(1)+(1-\alpha)(\phi(1)-\phi(0))\right] \\
\leq\ &1-\alpha.
\end{aligned}$$

This completes the proof of Theorem 1. ■

Theorem 2 *Let $TK_\phi(z)$ be of the form (6) is in the class $\mathcal{TK}(\lambda,\alpha)$, if and only if it satisfies the condition*

$$\lambda\phi'''(1)+(1+5\lambda)\phi''(1)+(3+4\lambda-\alpha)\phi'(1)+(1-\alpha)\phi(1)\leq(1-\alpha)(S+\phi(0)). \tag{13}$$

Proof. To prove that $TK_\phi(z)\in\mathcal{TK}(\lambda,\alpha)$, from Lemma 5 we have to show that

$$\sum_{n=2}^{\infty}n(n+\lambda n(n-1)-\alpha)\frac{t_{n-1}}{S}\leq 1-\alpha.$$

By the given hypothesis, we see that

$$\begin{aligned}
&\sum_{n=2}^{\infty}n(n+\lambda n(n-1)-\alpha)\frac{t_{n-1}}{S} \\
=\ &\sum_{n=2}^{\infty}\Big[\lambda(n-1)(n-2)(n-3)+(1+5\lambda)(n-1)(n-2) \\
&+(3+4\lambda-\alpha)(n-1)+(1-\alpha)\Big]\frac{t_{n-1}}{S} \\
=\ &\frac{1}{S}\sum_{n=2}^{\infty}\Big[\lambda(n-1)(n-2)(n-3)t_{n-1}+(1+5\lambda)(n-1)(n-2)t_{n-1} \\
&+(3+4\lambda-\alpha)(n-1)t_{n-1}+(1-\alpha)t_{n-1}\Big] \\
=\ &\frac{1}{S}\left[\sum_{n=1}^{\infty}[\lambda n(n-1)(n-2)t_n+(1+5\lambda)n(n-1)t_n+(3+4\lambda-\alpha)nt_n+(1-\alpha)t_n]\right] \\
=\ &\frac{1}{S}\left[\lambda\phi'''(1)+(1+5\lambda)\phi''(1)+(3+4\lambda-\alpha)\phi'(1)+(1-\alpha)(\phi(1)-\phi(0))\right] \\
\leq\ &1-\alpha.
\end{aligned}$$

This establishes the proof of Theorem 2. ■

Theorem 3 *If $f\in\Re^{\tau}(A,B)(\tau\in\mathbb{C}\backslash\{0\},-1\leq B<A\leq 1)$ and the inequality*

$$\frac{(A-B)|\tau|}{S}\left[\lambda\phi''(1)+(1+2\lambda)\phi'(1)+(1-\alpha)(\phi(1)-\phi(0))\right]\leq 1-\alpha.$$

is satisfied, then $\mathcal{K}_\phi(f,z)\in\mathcal{K}(\lambda,\alpha)$.

Proof. Let f be of the form (1) belong to the class $\Re^{\tau}(A,B)$. Then by virtue of Lemma 3, it suffices to show that

$$\sum_{n=2}^{\infty} n(n^2\lambda + n(1-\lambda) - \alpha)\frac{t_{n-1}}{S}|a_n| \leq 1-\alpha.$$

By Lemma 1 and given hypothesis, we see that

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(n+\lambda n(n-1)-\alpha)\frac{t_{n-1}}{S}|a_n| \\
&= \frac{(A-B)|\tau|}{S}\left[\sum_{n=2}^{\infty}\left[\lambda(n-1)(n-2)+(1+2\lambda)(n-1)+(1-\alpha)\right]t_{n-1}\right] \\
&= \frac{(A-B)|\tau|}{S}\left[\sum_{n=2}^{\infty}\left[\lambda(n-1)(n-2)t_{n-1}+(1+2\lambda)(n-1)t_{n-1}+(1-\alpha)t_{n-1}\right]\right] \\
&= \frac{(A-B)|\tau|}{S}\left[\sum_{n=1}^{\infty}\left[\lambda n(n-1)t_n+(1+2\lambda)nt_n+(1-\alpha)t_n\right]\right] \\
&= \frac{(A-B)|\tau|}{S}\left[\lambda\phi''(1)+(1+2\lambda)\phi'(1)+(1-\alpha)(\phi(1)-\phi(0))\right] \\
&\leq 1-\alpha.
\end{aligned}$$

This completes the proof of Theorem 3. ∎

Theorem 4 *Let*

$$TG_{\phi}(f,z) = \int_0^z \frac{TG_{\phi}(f,t)}{t})dt$$

is in $\mathcal{TK}(\lambda,\alpha)$ if and only if (12) is satisfied.

Proof. Since

$$TG_{\phi}(f,z) = z - \sum_{n=2}^{\infty}\frac{t_{n-1}}{S}\frac{z^n}{n}$$

by Lemma 5, we need only to show that

$$\sum_{n=2}^{\infty} n(n^2\lambda + n(1-\lambda) - \alpha)\frac{t_{n-1}}{nS} \leq 1-\alpha.$$

By the given hypothesis, we see that

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(n+\lambda n(n-1)-\alpha)\frac{t_{n-1}}{nS} \\
&= \sum_{n=2}^{\infty}\left[\lambda(n-1)(n-2)+(1+2\lambda)(n-1)+(1-\alpha)\right]\frac{t_{n-1}}{S} \\
&= \frac{1}{S}\left[\sum_{n=2}^{\infty}\left[\lambda(n-1)(n-2)t_{n-1}+(1+2\lambda)(n-1)t_{n-1}+(1-\alpha)t_{n-1}\right]\right] \\
&= \frac{1}{S}\left[\sum_{n=1}^{\infty}\left[\lambda n(n-1)t_n+(1+2\lambda)nt_n+(1-\alpha)t_n\right]\right] \\
&= \frac{1}{S}\left[\lambda\phi''(1)+(1+2\lambda)\phi'(1)+(1-\alpha)(\phi(1)-\phi(0))\right] \\
&\leq 1-\alpha.
\end{aligned}$$

This completes the proof of Theorem 4. ∎

Remark 2 *If we take* $t_n = \frac{m^n}{n!}$, *then we obtain the corresponding results of Murugusundaramoorthy et al. [10].*

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