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Repdigits As Sums Of Two Lucas Numbers*

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Abstract

Let (L_n) be the Lucas sequence defined by $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$ with initial conditions $L_0 = 2$ and $L_1 = 1$. A repdigit is a nonnegative integer whose digits are all equal. In this paper, we show that if $L_n + L_m$ is a repdigit, then $L_n + L_m = 2, 3, 4, 5, 6, 7, 8, 9, 11, 22, 33, 77, 333$.

1 Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence satisfying the recurrence relation $F_{n+2} = F_{n+1} + F_n$ with initial conditions $F_0 = 0$ and $F_1 = 1$. Let $(L_n)_{n\geq 0}$ be the Lucas sequence following the same recursive pattern as the Fibonacci sequence, but with initial conditions $L_0 = 2$ and $L_1 = 1$. F_n and L_n are called n^{th} Fibonacci number and n^{th} Lucas number, respectively. It is well known that

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n, \tag{1}$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1-\sqrt{5}}{2},$$

which are the roots of the characteristic equation $x^2 - x - 1 = 0$. Also, the following relation between n^{th} Lucas number L_n and α is well known:

$$\alpha^{n-1} \leq L_n \leq 2\alpha^n \tag{2}$$

for $n \geq 0$. The inequality (2) can be proved by induction.

A repdigit is a nonnegative integer whose digits are all equal. Recently, some mathematicians have investigated the repdigits which are sums or products of any number of Fibonacci numbers, Lucas numbers, and Pell numbers. In [2], Luca determined that the largest repdigits in Fibonacci and Lucas sequences are $F_{10} = 55$ and $L_5 = 11$. Then, in [1], the authors have found all repdigits in the Pell and Pell-Lucas sequences. Here, they showed that the largest repdigits in these sequences are $P_3 = 5$ and $Q_2 = 6$. After that, in [3], Luca proved that all nonnegative integer solutions (m_1, m_2, m_3, n) of the equation

$$N = F_{m_1} + F_{m_2} + F_{m_3} = d\left(\frac{10^n - 1}{9}\right) \text{ with } d \in \{1, 2, ..., 9\}$$

have

$$N \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 22, 44, 55, 66, 77, 99, 111, 555, 666, 11111\}.$$

Later, in [4], the authors studied the similar problem for sums of four Pell numbers. They found all repdigits, which are sums of four Pell numbers. Moreover, in [8], Marques and Togbe studied on repdigits as products of consecutive Fibonacci numbers. They proved that the Diophantine equation

$$F_n \cdots F_{n+(k-1)} = d\left(\frac{10^m - 1}{9}\right) \text{ with } d \in \{1, 2, ..., 9\}$$

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in positive integers n, m, k such that $m > 1$ has only the solution $(n, k, m, d) = (10, 1, 2, 5)$. In [6], Irmak and Togbe handled the above problem with $m \geq 1$ for Lucas numbers, and found only the solution $(n, k, m, d) = (4, 2, 2, 7)$. In [10], the authors found all repdigits which are sums of four Fibonacci or Lucas numbers. Later, in [11], they found all repdigits which are sums of three Lucas Numbers. In order to solve the above mentioned problems, some authors have used only elementary methods, and some have used very technical methods such as linear forms in logarithms. In this paper, we will find all repdigits which are sums of two Lucas numbers. That is, we deal with the Diophantine equation

$$N = L_{m_1} + L_{m_2} = d\left(\frac{10^k - 1}{9}\right) \text{ with } d \in \{1, 2, ..., 9\}. \tag{3}$$

Our main result, which is proved in the third section, is the following.

Theorem 1 *All nonnegative integer solutions (m_1, m_2, k, N) of the equation (3) with $0 \leq m_2 \leq m_1$ are given by*

$$(m_1, m_2, k, N) \in \left\{ \begin{array}{c} (1,1,1,2), (1,0,1,3), (0,0,1,4), (2,1,1,4), (2,0,1,5), \\ (3,1,1,5), (2,2,1,6), (3,0,1,6), (3,2,1,7), \\ (3,3,1,8), (4,1,1,8), (4,0,1,9), (4,3,2,11), (5,5,2,22), \\ (6,3,2,22), (7,3,2,33), (9,1,2,77), (12,5,3,333) \end{array} \right\}.$$

2 Auxiliary Results

Lately, in many articles, to solve Diophantine equations such as the equation (3), the authors have used Baker's theory lower bounds for a nonzero linear form in logarithms of algebraic numbers. Since such bounds are of crucial importance in effectively solving of Diophantine equations, we start with recalling some basic notions from algebraic number theory.

Let η be an algebraic number of degree d with minimal polynomial

$$a_0 x^d + a_1 x^{d-1} + ... + a_d = a_0 \prod_{i=1}^{d} \left(X - \eta^{(i)}\right) \in \mathbb{Z}[x],$$

where the a_i's are relatively prime integers with $a_0 > 0$ and $\eta^{(i)}$'s are conjugates of η. Then

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\left\{|\eta^{(i)}|, 1\right\}\right)\right)$$

is called the logarithmic height of η. In particular, if $\eta = a/b$ is a rational number with $\gcd(a, b) = 1$ and $b > 1$, then $h(\eta) = \log(\max\{|a|, b\})$.

The following properties of logarithmic height are found in many works stated in the references:

$$h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2, \tag{4}$$

$$h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma), \tag{5}$$

$$h(\eta^m) = |m| h(\eta). \tag{6}$$

The following theorem, which is deduced from Corollary 2.3 of Matveev [9], provides a large upper bound for the subscript m_1 in the equation (3) (also see Theorem 9.4 in [5]).

Theorem 2 *Assume that $\gamma_1, \gamma_2, \ldots, \gamma_t$ are positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D, $b_1, b_2, \ldots, b_t$ are rational integers, and*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1$$

is not zero. Then

$$|\Lambda| > \exp\left(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2(1+\log D)(1+\log B)A_1A_2\cdots A_t\right),$$

where

$$B \geq \max\{|b_1|, \ldots, |b_t|\},$$

and $A_i \geq \max\{Dh(\gamma_i), |\log\gamma_i|, 0.16\}$ for all $i = 1, \ldots, t$.

The following lemma can be found in [4]. And this lemma will be used to reduce the upper bound for the subscript m_1 in the equation (3). In this lemma, the function $||\cdot||$ denotes the distance from x to the nearest integer. That is, $||x|| = \min\{|x-n| : n \in \mathbb{Z}\}$ for a real number x.

Lemma 1 *Let $\Lambda = \epsilon + x_1v_1 + x_2v_2$ such that $\epsilon v_1 v_2 \neq 0$ and $x_1, x_2 \in \mathbb{Z}$. Let X_0, c, and δ be positive integers such that $\max\{|x_1|, |x_2|\} \leq X_0$ and*

$$|\Lambda| < c\exp(-\delta X).$$

Put $v = -v_1/v_2$ and $\Psi = \epsilon/v_2$. Let p/q be a convergent of v with $q > X_0$. Assume that $||q\Psi|| > \frac{2X_0}{q}$. Then $X < \frac{1}{\delta}\log\left(\dfrac{cq^2}{|v_2X_0|}\right)$.

The following lemma is given in [7].

Lemma 2 *Let $n \in \mathbb{N} \cup \{0\}$ and $k, m \in \mathbb{Z}$. Then*

$$L_{2mn+k} \equiv (-1)^{(m+1)n} L_k \,(mod L_m), \tag{7}$$

$$L_{2mn+k} \equiv (-1)^{mn} L_k \,(mod F_m). \tag{8}$$

3 Proof of Theorem 1

We assume that the equation (3) holds with $0 \leq m_2 \leq m_1$. If we run a program with *Mathematica* in the range $0 \leq m_2 \leq m_1 \leq 200$, we obtain only the solutions stated in theorem. So, from now on, we can assume that $m_1 \geq 201$. Thus, we have

$$L_{201} \leq L_{m_1} + L_{m_2} = d\left(\frac{10^k - 1}{9}\right) < 10^k - 1.$$

This shows that

$$42 \leq \frac{\log(L_{201}+1)}{\log 10} < k.$$

On the other hand, using (2), we see that

$$10^{k-1} \leq d\left(\frac{10^k-1}{9}\right) = L_{m_1} + L_{m_2} \leq 2L_{m_1} \leq 4\alpha^{m_1} < \alpha^{m_1+4}.$$

Taking the logarithm both sides of the last inequality gives

$$(k-1)\left(\frac{\log 10}{\log\alpha}\right) \leq (m_1+4).$$

This inequality shows that $4.7 \times k - 8.7 < m_1$. This implies that $42 < k < m_1$.

Firstly, assume that $d = 9$. Then we have $L_{m_1} + L_{m_2} = 10^k - 1 = 9(1 + 10 + 10^2 + ... + 10^{k-1})$. This implies that $9|L_{m_1} + L_{m_2}$. Writing $m_1 = 60q_1 + r_1$ and $m_2 = 60q_2 + r_2$ with $0 \leq r_1, r_2 \leq 59$, we get $L_{m_1} + L_{m_2} \equiv L_{r_1} + L_{r_2} (mod L_6)$ by (7). Since $9|L_6$, it follows that $9|L_{r_1} + L_{r_2}$. Also, since $k > 42$ and $F_6 = 8$, it is obvious that $L_{m_1} + L_{m_2} = 10^k - 1 \equiv 7(mod 8)$, which implies $L_{m_1} + L_{m_2} \equiv L_{r_1} + L_{r_2} \equiv 7(mod 8)$ by (8). Furthermore, we can see that $L_{m_1} + L_{m_2} \equiv L_{r_1} + L_{r_2} (mod F_5)$ by (8). Since $L_{m_1} + L_{m_2} = 10^k - 1 \equiv 4(mod 5)$ and $5|F_5$, it follows that $L_{r_1} + L_{r_2} \equiv 4(mod 5)$. A search with *Mathematica* gives us that there is no pairs (r_1, r_2) satisfying congruences $L_{r_1} + L_{r_2} \equiv 0(mod 9), L_{r_1} + L_{r_2} \equiv 7(mod 8)$, and $L_{r_1} + L_{r_2} \equiv 4(mod 5)$. Therefore, from now on, we assume that $1 \leq d \leq 8$.

Now, if we arrange the equation (3) as

$$\alpha^{m_1} + \alpha^{m_2} - d\frac{10^k}{9} = -\left(\beta^{m_1} + \beta^{m_2} + \frac{d}{9}\right),$$

we get the inequality

$$\left|\alpha^{m_1}(1 + \alpha^{m_2 - m_1}) - d\frac{10^k}{9}\right| \leq |\beta|^{m_1} + |\beta|^{m_2} + \frac{d}{9} \leq 3.$$

Dividing this inequality by $\alpha^{m_1}(1 + \alpha^{m_2 - m_1})$, we obtain

$$\left|1 - 10^k \alpha^{-m_2} \frac{d}{9(1 + \alpha^{m_1 - m_2})}\right| \leq \frac{3}{\alpha^{m_1}} < \alpha^{2.3 - m_1}. \tag{9}$$

Let

$$\Gamma_1 = 1 - 10^k \alpha^{-m_2} \frac{d}{9(1 + \alpha^{m_1 - m_2})}.$$

If $\Gamma_1 = 0$, then we have $\alpha^{m_1} + \alpha^{m_2} = d\frac{10^k}{9}$, which is impossible since $\alpha^{m_1} + \alpha^{m_2}$ is irrational. Therefore $\Gamma_1 \neq 0$. Now we put

$$\gamma_1 = \alpha, \gamma_2 = 10, \gamma_3 = \frac{d}{9(1 + \alpha^{m_1 - m_2})}$$

and

$$b_1 = -m_2, b_2 = k, b_3 = 1.$$

Then, using (4), (5), and (6), we obtain $h(\gamma_1) = \frac{\log \alpha}{2} = \frac{0.4812}{2}$, $h(\gamma_2) = \log 10$ and

$$\begin{aligned} h(\gamma_3) &\leq h(d) + h(9) + h(\alpha^{m_1 - m_2}) + \log 2 \\ &\leq \log 9 + \log 9 + (m_1 - m_2)\frac{\log \alpha}{2} + \log(2) \\ &< 5.1 + (m_1 - m_2)\frac{\log \alpha}{2}. \end{aligned}$$

It is clear that the degree of $\mathbb{Q}(\sqrt{5})$ is 2. Since $1 \leq |\log \alpha| \leq 2\ h(\alpha)$, $|\log 10| \leq 2h(10)$, and $\left|\log \frac{d}{9(1+\alpha^{n-m})}\right| \leq 2h(\gamma_3)$, we can take $A_1 := 1$, $A_2 := 4.61$ and $A_3 := 10.2 + (m_1 - m_2)\log \alpha$. Also, $B = \max\{m_2, k, 1\} \leq m_1$. Thus applying Theorem 2 to the inequality (9), we get

$$m_1 \log \alpha - 2.3 \log \alpha < 4.5 \cdot 10^{12}(1 + \log m_1)\left(10.2 + (m_1 - m_2)\log \alpha\right). \tag{10}$$

Rearranging the equation (3) as $\alpha^{m_1} - d\frac{10^k}{9} = -\left(\beta^{m_1} + \beta^{m_2} + \frac{d}{9} + \alpha^{m_2}\right)$ and taking absolute value, we obtain

$$\left|\alpha^{m_1} - d\frac{10^k}{9}\right| \leq |\beta|^{m_1} + |\beta|^{m_2} + \frac{d}{9} + \alpha^{m_2} \leq \alpha^{m_2} + 3 < \alpha^{m_2 + 2.9}.$$

This leads to

$$\left|1 - \alpha^{-m_1} 10^k \frac{d}{9}\right| < \alpha^{m_2 - m_1 + 2.9}. \tag{11}$$

We now put $\gamma_1 = \alpha, \gamma_2 = 10, \gamma_3 = \dfrac{d}{9}$ and $b_1 = -m_1, b_2 = k, b_3 = 1$. A similar argument to the above gives that $A_1 := 1,\ A_2 := 4.61, A_3 := 8.8$, and $B = m_1$. Let $\Gamma_2 = \alpha^{-m_1}10^k\frac{d}{9}$. Similarly, one can justify that $\Gamma_2 \neq 0$. Thus, again applying Theorem 2, we get

$$(m_1 - m_2)\log\alpha - 2.9\log\alpha < 3.94\cdot 10^{13}(1+\log m_1). \tag{12}$$

Substituting the inequality (12) into (10), a computer search with *Mathematica* gives us that $m_1 < 1.85\cdot 10^{30}$. Put $X_0 = 1.85\cdot 10^{30}$.

Let $\Lambda_1 = \log\left(\frac{d}{9}\right) - m_1\log\alpha + k\log 10$. From (3), we see that

$$\alpha^{m_1} - d\frac{10^k}{9} = -\frac{d}{9} - \beta^{m_1} - L_{m_2} \leq -\frac{d}{9} - \beta^{m_1} - 1 < 0.$$

A simple computation shows that $\Lambda_1 > 0$. By (11), it is seen that

$$0 < \Lambda_1 < e^{\Lambda_1} - 1 < \alpha^{m_2 - m_1 + 2.9},$$

which leads to

$$|\Lambda_1| < \alpha^{2.9}\alpha^{m_2-m_1} < \alpha^3\exp(-0.48(m_1-m_2)).$$

We now put

$$c = \alpha^3, X = m_1 - m_2, \delta = 0.48, x_1 = m_1, x_2 = k, \epsilon = \log\left(\frac{d}{9}\right),\ \Psi = \frac{\log\left(\frac{d}{9}\right)}{\log 10}$$

and $v = \frac{\log\alpha}{\log 10}$. Also we have

$$\frac{\Lambda_1}{\log 10} = \frac{\log\left(\frac{d}{9}\right)}{\log 10} - m_1\frac{\log\alpha}{\log 10} + k.$$

It is clear that $\max\{|x_1|, |x_2|\} = m_1 \leq X_0$. We found that q_{63}, the denominator of the 63^{th} convergent of v satisfies the hypothesis of Lemma 1. Thus we get $X = m_1 - m_2 < 165$.

Now take $m_1 - m_2 < 165$ and say

$$\Lambda_2 = \log\left(\frac{d}{9(1+\alpha^{m_1-m_2})}\right) - m_2\log\alpha + k\log 10.$$

It can be easily seen that $\Lambda_2 > 0$. Then, it follows that $0 < \Lambda_2 < e^{\Lambda_2} - 1 < \alpha^{2.3-m_1}$ by (9). This yields to

$$|\Lambda_2| < \alpha^{2.3}\alpha^{-m_1} < \alpha^{2.3}\exp(-0.48m_1).$$

Put

$$\begin{aligned} c &= \alpha^{2.3}, X = m_1, \delta = 0.48, x_1 = m_2, x_2 = k, \epsilon = \log\left(\frac{d}{9(1+\alpha^{m_1-m_2})}\right), \\ \Psi &= \frac{\log\left(\frac{d}{9(1+\alpha^{m_1-m_2})}\right)}{\log 10}, v = \frac{\log\alpha}{\log 10}. \end{aligned}$$

We found that q_{69}, the denominator of the 69^{th} convergent of v satisfies the hypothesis of Lemma 1. Applying Lemma 1, we get $m_1 < 188$. This contradicts the assumption that $m_1 \geq 201$. This completes the proof.

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