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Computation Of Exact Solutions Of Abel Type Differential Equations*

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Abstract

In this note, we present fractional polynomials, fractional rational functions and find explicit new fractional polynomial or rational solutions of Abel differential equations.

1 Introduction

In 2006, Behloul and Cheng [1] gave an algorithm to compute all rational solutions of differential equations of the form

$$Q(z)\frac{dy}{dz} = P_m(z)y^m + P_{m-1}(z)y^{m-1} + ... + P_1(z)y + P_0(z), z \in C, m \in \mathbb{N}, m \geq 3$$

where Q, P_i for $i = 0, 1, 2, ..., m$ are polynomials in z and Q, P_m are not trivial.

By putting $y(z) = g(z)/P_m(z)$ the authors are able to obtain a new equation such that each of its rational solutions is a polynomial. It is also shown that the polynomial solutions of this new equation have degrees bounded above, and that they can be computed in automatic manners.

In 2011, Behloul and Cheng [2] gave another algorithm to look for the rational solutions of differential equations of the following more general form

$$\frac{dy}{dz} = \frac{A_n(z)y^n + A_{n-1}(z)y^{n-1} + \cdots + A_0(z)}{B_m(z)y^m + B_{m-1}(z)y^{m-1} + \cdots + B_0(z)},\ z \in C,\ m, n \in \mathbb{N},$$

where $A_0, A_1, ..., A_n$ and $B_0, B_1, ..., B_m$ are polynomials in z such that A_n and B_m are not trivial.

In this note, our purpose is to characterize all polynomials with nonnegative rational powers (fractional-polynomials) and quotient of such polynomials which are solutions of Abel differential equations of the first kind with fractional-polynomial coefficients.

Fractional-polynomials and fractional-rational functions are generalizations of polynomials and rational functions that may occur in the characteristic equations of functional differential and/or difference equations [4], or as interpolating functions that behave better than the usual interpolating polynomials and rational functions [3].

Exact solutions of nonlinear differential equations, on the other hand, are needed in the theory of limit cycles [5], and elsewhere as illustrated by Llibre and Valls [6], Thieu Vo, Grasegger and Winkler [8] who used the results in [1, 2] to determine respectively "maximum number of polynomial solutions" and "rational general solutions" of some nonlinear differential equations.

We therefore believe that exact generalized polynomial and rational solutions of differential equations will be of use in similar research areas.

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2 Main Results

Let $\mathbb{N}$, $\mathbb{C}$ and $\mathbb{Q}$ be respectively the sets of nonnegative integers, complex numbers and rational numbers. First of all, a fractional-polynomial is a function of the form

$$f(x) = a_n x^{\gamma_n} + a_{n-1} x^{\gamma_{n-1}} + \cdots + a_1 x^{\gamma_1} + a_0 \tag{1}$$

where $n \in \mathbb{N}$, $a_i \in \mathbb{C}$, $\gamma_i \in \mathbb{Q}$, and $\gamma_n > \gamma_{n-1} > \cdots > \gamma_1 > 0$ and $x \in [0, +\infty)$. If $n \geq 1$ and $a_n \neq 0$ then we call γ_n the fractional degree of $f(x)$. If $n = 0$ then the fractional degree of $f(x)$ is its usual degree as a constant polynomial. The index of a fractional-polynomial of the form (1) with irreducible $\gamma_1, ..., \gamma_n$ is the integer r which is defined as the least common multiple of the denominators of γ_i. The index of a constant fractional-polynomial is equal to 1. A fractional-rational function is a function of the form

$$R(x) = \frac{a_n x^{\gamma_n} + a_{n-1} x^{\gamma_{n-1}} + \cdots + a_1 x^{\gamma_1} + a_0}{b_m x^{s_m} + b_{m-1} x^{s_{m-1}} + \cdots + b_1 x^{s_1} + b_0} \tag{2}$$

where $n, m \in \mathbb{N}$, $a_i, b_i \in \mathbb{C}$, $\gamma_i, s_i \in \mathbb{Q}$, γ_i, s_i irreducible, $\gamma_n > \cdots > \gamma_1 > 0$, $s_m > \cdots > s_1 > 0$, $b_m \neq 0$, and $x \in [0, +\infty) \backslash \Omega$ where

$$\Omega = \{x \in [0, +\infty) : b_m x^{s_m} + b_{m-1} x^{s_{m-1}} + \cdots + b_1 x^{s_1} + b_0 = 0 \}.$$

Let r be the least common multiple of the denominators of γ_i and s_i. Put $x = t^r$, $t \geq 0$, then (2) becomes

$$R(x) = R(t^r) = \frac{a_n t^{r\gamma_n} + a_{n-1} t^{r\gamma_{n-1}} + \cdots + a_1 t^{r\gamma_1} + a_0}{b_m t^{rs_m} + b_{m-1} t^{rs_{m-1}} + \cdots + b_1 t^{rs_1} + b_0} \equiv F(t).$$

It is clear that $F(t)$ is a rational function. Let $g(t) = \gcd(a_n t^{r\gamma_n} + a_{n-1} t^{r\gamma_{n-1}} + \cdots + a_0,\ b_m t^{rs_m} + b_{m-1} t^{rs_{m-1}} + \cdots + b_0)$. Then

$$F(t) = \frac{g(t)h(t)}{g(t)k(t)} = \frac{h(t)}{k(t)},$$

where $h(t), k(t)$ are polynomials such that $\gcd(h(t), k(t)) = 1$. But since $t = x^{\frac{1}{r}}$, we see that $F(t)$ becomes

$$F(t) = F(x^{\frac{1}{r}}) = R(x) = \frac{h(x^{\frac{1}{r}})}{k(x^{\frac{1}{r}})}.$$

Let r_1 be the index of $h(x^{\frac{1}{r}})$ and r_2 be the index of $k(x^{\frac{1}{r}})$. Then the index of a fractional-rational function of the form (2) is the least common multiple of r_1 and r_2.

Example 1 *Let* $R(x) = \dfrac{x^{\frac{3}{2}} + x}{x^{\frac{5}{6}} + x^{\frac{1}{3}} + x^{\frac{1}{2}} + 1}$, $x \in [0, +\infty) \backslash \Omega$. *The least common multiple of the denominators of* γ_i *and* s_i*, in this case, is* $r = 6$. *Put* $x = t^6$, $t \geq 0$. *Then we obtain*

$$R(x) = R(t^6) = \frac{t^9 + t^6}{t^5 + t^2 + t^3 + 1}.$$

But $\gcd(t^9 + t^6, t^5 + t^2 + t^3 + 1) = t^3 + 1$. *Thus,*

$$\frac{t^9 + t^6}{t^5 + t^2 + t^3 + 1} = \frac{(t^3 + 1)t^6}{(t^3 + 1)(t^2 + 1)} = \frac{t^6}{t^2 + 1} = \frac{x}{x^{\frac{1}{3}} + 1} = R(x).$$

We conclude that index of the fractional-rational function $R(x)$ *is equal to* 3.

2.1 Abel Differential Equations of the First Kind

Abel equations of the second kind take the form

$$(g_0 + g_1 y)\, y' = f_0 + f_1 y + f_2 y^2 + f_3 y^3 + \cdots,$$

where $g_0, g_1, f_0, f_1, f_2, \ldots$ are functions in x and y is the unknown function in x to be sought. These equations appear early in mathematics and some of them can be explicitly solved (see the Handbook of Exact Solutions for Ordinary Differential Equations by Polyanin and Zaitsev [7]). When g_0, f_0, f_1, f_2, f_3 are polynomials, and $g_1 = f_4 = f_5 = \cdots \equiv 0$, it has been shown that the number of its rational solutions is finite and they can be computed in a systematic manner [1, 2].

Consider

$$P_4(x)y' = P_3(x)y^3 + P_2(x)y^2 + P_1(x)y + P_0(x), \; x > 0 \tag{3}$$

where P_0, P_1, P_2, P_3, P_4 are fractional-polynomials such that P_4 and P_3 are not trivial.

The aim of this note is to establish the following main result.

Theorem 1 *Equation (3) has a finite number of fractional-rational solutions and they can be determined in a systematic manner.*

Let r_i be the index of each fractional polynomial P_i in (3). Let $s = \operatorname{lcm}(r_0, r_1, r_2, r_3, r_4)$. If we put

$$x = t^s \text{ and } w(t) = y(x), \; t > 0,$$

then Eq. (3) becomes

$$P_4(t^s)w' = st^{s-1}\left(P_3(t^s)w^3 + P_2(t^s)w^2 + P_1(t^s)w + P_0(t^s)\right), \; t > 0$$

where $P_4(t^s)$, $st^{s-1}P_3(t^s)$, $st^{s-1}P_2(t^s)$, $st^{s-1}P_1(t^s)$ and $st^{s-1}P_0(t^s)$ are all polynomials. We can then assume that P_0, P_1, P_2, P_3, P_4 are polynomials in equation (3).

2.2 Fractional-Rational Solutions

Let us now seek fractional-rational solutions of (3).

Proposition 1 *Every rational-fractional solution $y(x)$ of (3) is of the form*

$$y(x) = \frac{v(x)}{xP_3(x)}$$

where $v(x)$ is a fractional-polynomial.

To prove this, let $y(x)$ be such a fractional-rational solution with index r. Then we can write it as

$$y(x) = \frac{a_n x^{\frac{n}{r}} + a_{n-1}x^{\frac{n-1}{r}} + \cdots + a_1 x^{\frac{1}{r}} + a_0}{b_m x^{\frac{m}{r}} + b_{m-1}x^{\frac{m-1}{r}} + \cdots + b_1 x^{\frac{1}{r}} + b_0}.$$

If we put

$$x = t^r \text{ and } w(t) = y(x),$$

then

$$w(t) = \frac{a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0}{b_m t^m + b_{m-1}t^{m-1} + \cdots + b_1 t + b_0}$$

which is a rational function. Equation (3) becomes

$$P_4(t^r)w' = rt^{r-1}\left(P_3(t^r)w^3 + P_2(t^r)w^2 + P_1(t^r)w + P_0(t^r)\right), \; t > 0 \tag{4}$$

According to [1, Section 4], we know that

$$w(t) = \frac{f(t)}{rt^{r-1}P_3(t^r)} \tag{5}$$

where $f(t)$ is a polynomial.

Finally, from (5), we have $w(t) = \frac{\frac{tf(t)}{r}}{t^r P_3(t^r)}$ and then $y(x) = \frac{v(x)}{xP_3(x)}$ where $v(x)$ is a fractional-polynomial.

Now, if we put $y(x) = \frac{v(x)}{xP_3(x)}$ in equation (3), we obtain

$$Q(x)v' = v^3 + Q_2(x)v^2 + Q_1(x)v + Q_0(x),\ x > 0, \tag{6}$$

where Q, Q_i are polynomials. Therefore we may now consider

$$P_3(x)y' = y^3 + P_2(x)y^2 + P_1(x)y + P_0(x),\ x > 0 \tag{7}$$

and look for its fractional-polynomials solutions where P_i are polynomials.

Let us start with a particular case.

Example 2 *Let us look for fractional-polynomial solutions of an arbitrary index of*

$$2x^2y' = y^3 - (3x-1)\,y^2 + \left(3x^2 - 2x\right)y - x^3 + 3x^2 - x,\ x > 0. \tag{8}$$

For $r = k,\ k \in \mathbb{N}\backslash\{0\}$, put $x = t^k,\ t > 0$ and $w(t) = y(x)$. Equation (8) becomes

$$\frac{2t^{k+1}}{k}w' = w^3 - \left(3t^k - 1\right)w^2 + \left(3t^{2k} - 2t^k\right)w - t^{3k} + 3t^{2k} - t^k,\ t > 0. \tag{9}$$

Let $n = \deg w$. Then $\deg(\frac{2t^{k+1}}{k}w') = n+k,\ \deg(w^3) = 3n,\ \deg(\left(3t^k-1\right)w^2) = 2n+k, \deg(\left(3t^{2k}-2t^k\right)w) = n+2k$ and $\deg(-t^{3k}+3t^{2k}-t^k) = 3k$. Now, if $n > k$ then $3n > k+n,\ 2n+k,\ n+2k, 3k$, and then there is no polynomial solution of (9). If $k > n$ then $3k > k+n, 3n,\ 2n+k,\ n+2k$, and again there is no polynomial solution of (9).

We conclude that $k = n$, in this case, we see that

$$\begin{aligned}\deg(w^3) &= \deg((3t^n - 1)\,w^2) = \deg(\left(3t^{2n} - 2t^n\right)w)\\ &= \deg(-t^{3n} + 3t^{2n} - t^n) > \deg(\frac{2t^{n+1}}{n}w').\end{aligned}$$

If $w(t) = a_nt^n + ... + a_1t + a_0$ such that $n \geq 1$ and $a_n \neq 0$, then after substituting $w(t)$ in equation (9) we get : $a_n^3 - 3a_n^2 + 3a_n - 1 = 0$, then $a_n = 1$. Put $z(t) = w(t) - t^n$, ($\deg(z) \leq n-1$), in equation (9), then we get

$$\frac{2t^{n+1}}{n}(z+t^n)' = (z+t^n)^3 - (3t^n-1)(z+t^n)^2 + \left(3t^{2n} - 2t^n\right)(z+t^n) - t^{3n} + 3t^{2n} - t^n,$$

thus,

$$\frac{2t^{n+1}}{n}z' = z^3 + z^2 - t^n. \tag{10}$$

Let $a = \deg(z) > 0$. Then $\deg(\frac{2t^{n+1}}{n}z') = a+n,\ \deg(z^3) = 3a, \deg(z^2) = 2a$, and $\deg(t^n) = n$. We have $a < n$. Then $a+n > 2a, n$, it is necessary that $a+n = 3a$, i.e. $a = \frac{n}{2}$. We can write equation (10) as $(\frac{2t}{n}z'+1)t^n = (z+1)z^2$. Then t^n divides $(z+1)z^2$ but $\gcd(z+1, z^2) = 1$, (because $z^2 - (z-1)(z+1) = 1$), and t^n does not divide $z+1$, then t^n divides z^2. We conclude that $z(t) = bt^{\frac{n}{2}}$. Now, substituting $z(t)$ by $bt^{\frac{n}{2}}$

in (10), we get $bt^{\frac{3n}{2}} = b^3t^{\frac{3n}{2}} + b^2t^n - t^n$ *and then* $b = 1$ *or* $b = -1$*. Hence,* $w(t) = t^n + t^{\frac{n}{2}}$ *and* $w(t) = t^n - t^{\frac{n}{2}}$ *are all polynomial solutions of (9).*

Finally, we conclude that $y_1(x) = x + x^{\frac{1}{2}}$ *and* $y_2(x) = x - x^{\frac{1}{2}}$ *are all fractional-polynomial solutions of equation (8) and their index is equal to* 2 *and there is no other fractional-polynomial solution of index different from* 2*.*

Let us now analyze indices of (7) in the general case. Before solving the general case, we recall a classical identity.

Proposition 2 *For all positive integer* n *and complex numbers* $a_1, a_2, ..., a_n$ *we have*

$$\left(\sum_{i=1}^{n} a_i\right)^3 = \sum_{i} a_i^3 + 3\sum_{i\neq j} a_i a_j^2 + 6\sum_{i<j<k} a_i a_j a_k \text{ where } i, j, k \in \{1, 2, ..., n\}.$$

Proposition 3 *The possible values of an index of a fractional-polynomial solution of (7) are* 1, 2, 3 *or* 6*.*

Proof. Let $y = a_n x^{\gamma_n} + a_{n-1}x^{\gamma_{n-1}} + \cdots + a_1 x^{\gamma_1} + a_0$ be a fractional-polynomial solution of (7). We have $\gamma_i = \dfrac{\alpha_i}{\beta_i}$ where α_i, β_i are positive integers with $\gcd(\alpha_i, \beta_j) = 1$ for $i = 1, ..., n$. Assume that there exists i such that $\gcd(3, \beta_i) = 1,\ \beta_i > 3$. Let

$$\beta = \max_{1\leq i\leq n}\{\beta_i : \gcd(3, \beta_i) = 1\} \text{ and } \gamma_j = \max_{1\leq i\leq n}\{\gamma_i :\ \beta_i = \beta\}.$$

Then $a_j \neq 0$. After substituting y by $a_n x^{\gamma_n} + ... + a_j x^{\gamma_j} + \cdots + a_1 t^{\gamma_1} + a_0$ in (7) we see that the term $a_j^3 x^{3\gamma_j}$ given after expanding y^3 appears only for one time in (7). Then $a_j = 0$ which is a contradiction. We conclude that 3 divides β_i for all $i \in \{1, ..., n\}$ such that $\beta_i \neq 1, 2$, i.e. $\beta_i = 1$ or 2 or $3^{m_i} M_i$ where $m_i,\ M_i$ are positive integers such that $\gcd(3, M_i) = 1$. If there exists i such that $\beta_i \geq 3$, then let $m = \max m_i$ and $k = 3^m$. After putting $x = t^k$ and $w(t) = y(x)$ in equation (7), we have

$$P_3(t^k)w' = kt^{k-1}\left(w^3 + P_2(t^k)w^2 + P_1(t^k)w + P_0(t^k)\right),\ t > 0, \tag{11}$$

and

$$w(t) = a_n t^{k\gamma_n} + a_{n-1}t^{k\gamma_{n-1}} + \cdots + a_1{}^{k\gamma_1} + a_0$$

where

$$k\gamma_i = 3^m\alpha_i \text{ or } \frac{3^m\alpha_i}{2} \text{ or } \frac{3^{m-m_i}\alpha_i}{M_i} \text{ with } \gcd(3, M_i) = 1.$$

Assume that there exists i such that $M_i > 3$. Let $\gamma_j = \max\{\gamma_i \text{ where } M_i > 3\}$. Then $a_j \neq 0$. After substituting w by $a_n t^{k\gamma_n} + ... + a_j t^{k\gamma_j} + \cdots + a_1 t^{k\gamma_1} + a_0$ in (11), we see that the term $a_j^3 t^{k-1+3k\gamma_j}$ given after expanding $kt^{k-1}w^3$ appears only for one time in (11). Then $a_j = 0$ which is a contradiction. We conclude that $M_i \leq 2$ for all i, i.e. $\beta_i = 2^\varepsilon \times 3^{m_i}$ where $\varepsilon \in \{0, 1\}$ and m_i are nonnegative integers. After putting $z(x) = y(x) + \dfrac{P_2(x)}{3}$ in (7), we obtain

$$P_3(x)z' = z^3 + Q_1(x)z + Q_0(x),\ x > 0,$$

where Q_1, Q_0 are polynomials. In the following, we take $P_2 \equiv 0$ in (7) and $\gamma_0 = 0$. Assume that there exists i such that $m_i \geq 2$. Let $m = \max_{1\leq i\leq n} m_i$ and $k = 2\ \times 3^{m-1}$. After putting $x = t^k$ and $w(t) = y(x)$ in equation (7), we have

$$P_3(t^k)w' = kt^{k-1}\left(w^3 + P_1(t^k)w + P_0(t^k)\right),\ t > 0, \tag{12}$$

and

$$w(t) = a_n t^{\gamma'_n} + a_{n-1}t^{\gamma'_{n-1}} + \cdots + a_1 t^{\gamma'_1} + a_0 t^{\gamma'_0}$$

where
$$\gamma_i' = k\gamma_i = \frac{2^{1-\varepsilon}\alpha_i 3^{m-m_i}}{3} \quad \text{and} \quad \varepsilon \in \{0,1\},$$
i.e. $\gamma_i' = \theta_i$ or $\dfrac{\delta_i}{3}$ where θ_i, δ_i are respectively nonnegative integers and positive integers such that $\gcd(3,\delta_i) = 1$. Let $\delta_j = \max \delta_i$. Then $a_j \neq 0$, $\gamma_j' = \dfrac{\delta_j}{3}$. Let i be an integer in $\{0,1,...,n\} - \{j\}$. After substituting w by $a_n t^{\gamma_n'} + ... + a_j t^{\gamma_j'} + \cdots + a_1 t^{\gamma_1'} + a_0 t^{\gamma_0'}$ in (12) we see that the term $3a_i a_j^2 t^{k-1+\gamma_i'+2\gamma_j'}$ given after expanding $kt^{k-1}w^3$ appears only for one time in (12). Then $a_i = 0$ for all $i \in \{0,1,...,n\} - \{j\}$. We conclude that $w(t) = a_j t^{\frac{\delta_i}{3}}$, i.e.
$$y(x) = a_j\ x^{\frac{\delta_i}{3k}} = a_j\ x^{\frac{\delta_i}{2^{\varepsilon}\ \times 3^m}}.$$

Now, after substituting y by $a_j\ x^{\frac{\delta_i}{2^{\varepsilon}\ \times 3^m}}$ in (7) we see that the term $a_j^3\ x^{\frac{\delta_i}{2^{\varepsilon}\ \times 3^{m-1}}}$ appears only for one time in (7). Then $a_j = 0$ which is a contradiction. We conclude that $m_i < 2$ for all $i \in \{1,...,n\}$.

Finally, we get $\beta_i = 2^{\varepsilon}\ \times 3^{m_i}$ where $\varepsilon, m_i \in \{0,1\}$ i.e. $\beta_i = 1$ or 2 or 3 or 6. ■

Corollary 1 *All fractional-polynomial solutions $y(x)$ of (7) are the polynomial solutions $w(t)$ of (11) for $k = 6$.*

Example 3 *Let*
$$3x^2y' = y^3 - 3xy^2 + \left(3x^2 + x\right)y - x^3 + 2x^2 - x,\ \ x > 0. \tag{13}$$
According to [1, 2], equation (13) has no polynomial solutions. After putting $x = t^6$ and $y(x) = y(t^6) \equiv w(t)$ in equation (13), we have
$$t^7 w' = 2(w^3 - 3t^6\ \ w^2 + (3t^{12}\ \ + t^6)w - t^{18}\ \ + 2t^{12} - t^6),\ \ t > 0. \tag{14}$$
According to [1, 2], all polynomial solutions of (14) are $w(t) = t^6 + t^2$, $t^6 + jt^2$ and $t^6 + j^2t^2$ where $j = \exp(\frac{2i\pi}{3})$. Then all fractional-polynomial solutions of (13) are $y(x) = x + x^{\frac{1}{3}}$, $x + jx^{\frac{1}{3}}$ and $x + j^2x^{\frac{1}{3}}$.

We have thus obtained in this short note the means to calculate all fractional rational solutions of the Abel differential equations. It is hoped that new exact solutions for other differential equations can be found in similar, if not identical, manners.

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