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Exact Moments And Characterizations Of The Weibull-Rayleigh Distribution Based On Generalized Upper Record Statistics*

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Abstract

A new distribution, called the Weibull-Rayleigh distribution, was recently introduced by Ganji, *et al.* (2016). In this paper, we consider the generalized upper record values (or k-th upper record values) from this distribution and obtained exact explicit expressions as well as several recurrence relations satisfied by single and product moments. The results include as particular cases the above relations for moments of upper record statistics. In addition, conditional expectation and recurrence relations for single moments are used to characterize this distribution and some computational works are also carried out.

1 Introduction

A random variable X is said to have a Weibull-Rayleigh distribution (Ganji *et al.* (2016)), if its probability density function (*pdf*) is of the form

$$f(x) = \frac{\beta x}{\theta\alpha^2}\left(\frac{x^2}{2\theta\alpha^2}\right)^{\beta-1} e^{-(x^2/2\theta\alpha^2)^\beta}, \quad x \geq 0, \quad \alpha,\ \beta,\ \theta > 0 \tag{1}$$

with corresponding distribution function (*df*)

$$\bar{F}(x) = e^{-(x^2/2\theta\alpha^2)^\beta}, \quad x \geq 0, \quad \alpha,\ \beta,\ \theta > 0. \tag{2}$$

In view of (1) and (2), it is easy to see that

$$f(x) = \frac{\beta}{2^{\beta-1}(\theta\alpha^2)^\beta} x^{2\beta-1}\bar{F}(x). \tag{3}$$

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Ganji *et al.* (2016) pointed out Weibull-Rayleigh distribution is quite effective to provide the best fits for real data sets. Since the results, on real life data compared with other known distributions like Beta-Pareto, Three parameters Weibull, Generalized exponential and Pareto, revealed that this distribution provides a better fit for modelling real life data.

The statistical study of record values in a sequence of independently and identically distributed (*iid*) continuous random variables was first formulated by Chandler (1952). For a survey on important results developed in this area one may refer to Arnold *et al.* (1998), Ahsanullah (1995) and Ahsanullah and Nevzorov (2015). Dziubdziela and Kopociński (1976) have generalized the concept of record values of Chandler (1952) by random variables of a more generalized nature and called them the k-th record values. Later, Minimol and Thomas (2013) called the record values defined by Dziubdziela and Kopociński (1976) also as the generalized record values, since the r-th member of the sequence of the ordinary record values is also known as the r-th record value. Setting $k = 1$, we obtain ordinary record statistics.

For applications of generalized upper record values or k-th record values one may refer to Kamps (1995) and Danielak and Raqab (2004).

Let $\{X_n,\ n \geq 1\}$ be a sequence of *iid* random variables with *df* $F(x)$ and *pdf* $f(x)$. The j-th order statistic of a sample $X_1, X_2, \ldots, X_n$ is denoted by $X_{j:n}$. For a fixed positive integer k, we define the sequence $\{U_n^{(k)},\ n \geq 1\}$ of k-th upper record times of $\{X_n,\ n \geq 1\}$ as follows:

$$\begin{aligned} U_1^{(k)} &= 1, \\ U_{n+1}^{(k)} &= \min\left\{ j > U_n^{(k)} : X_{j:j+k-1} > X_{U_n^{(k)}:U_n^{(k)}+k-1} \right\}. \end{aligned}$$

The sequence $\{Y_n^{(k)},\ n \geq 1\}$, where $Y_n^{(k)} = X_{U_n^{(k)}}$ is called the sequence of generalized upper record values or k-th upper record values of $\{X_n,\ n \geq 1\}$. Note that for $k = 1$, we have $Y_n^{(1)} = X_{U_n},\ n \geq 1$, which are the record values of $\{X_n,\ n \geq 1\}$ as defined in Ahsanullah (1995). Moreover, we see that $Y_0^{(k)} = 0$ and $Y_1^{(k)} = \min(X_1, X_2, ..., X_k) = X_{1:k}$. The *pdf* of $Y_n^{(k)}$ and the joint *pdf* of $Y_m^{(k)}$ and $Y_n^{(k)}$ are given by (Dziubdziela and Kopociński (1976), Grudzień (1982))

$$f_{Y_n^{(k)}}(x) = \frac{k^n}{(n-1)!} \, [-ln\bar{F}(x)]^{n-1} \, [\bar{F}(x)]^{k-1} \, f(x), \quad n \geq 1, \tag{4}$$

and

$$\begin{aligned} f_{Y_m^{(k)},Y_n^{(k)}}(x,y) &= \frac{k^n}{(m-1)!(n-m-1)!}[-ln\bar{F}(x)]^{m-1}\frac{f(x)}{\bar{F}(x)} \\ &\quad \times[ln\bar{F}(x) - ln\bar{F}(y)]^{n-m-1}[\bar{F}(y)]^{k-1} \, f(y) \end{aligned} \tag{5}$$

for $x < y,\ 1 \leq m < n,\ n \geq 2$ where $\bar{F}(x) = 1 - F(x)$.

For some recent developments on record values and generalized upper record values (k-th upper record values) with special reference to those arising from generalized Rayleigh, Kumaraswamy-Burr III, exponentiated Pareto type I, exponential, Gumble, Pareto, generalized Pareto, Burr, Weibull, Gompertz, Makeham, exponential-Weibull, additive Weibull and modified Weibull distributions, see Kumar (2015), Kumar *et al.* (2017), Kumar and Kumar (2018), Grudzień and Szynal (1997), Pawlas and Szynal (1998, 1999, 2000), Khan and Zia (2009), Minimol and Thomas (2013, 2014), Khan and Khan (2016) and Khan *et al.* (2017), respectively. In this work we mainly focus on the study of generalized upper record values arising from the Weibull-Rayleigh distribution.

2 Single Moments and Relations

In this section we will derive the exact explicit expressions and some recurrence relations for single moments of generalized upper record values from the Weibull-Rayleigh distribution.

THEOREM 1. For the Weibull-Rayleigh distribution given in (2) and $1 \leq k \leq n$, $j = 0, 1, ...$

$$E(Y_n^{(k)})^j = \frac{(2\theta\alpha^2)^{j/2}\Gamma(n + (j/2\beta))}{k^{j/2\beta}\Gamma(n)} \tag{6}$$

PROOF. From (2) and (4), we have

$$E(Y_n^{(k)})^j = \frac{(2\theta\alpha^2)^{j/2}k^n}{\Gamma(n)} \int_0^\infty [-ln\bar{F}(x)]^{n+(j/2\beta)-1}[\bar{F}(x)]^{k-1} f(x)dx. \tag{7}$$

By setting $t = \bar{F}(x)$ in (7), we get

$$E(Y_n^{(k)})^j = \frac{(2\theta\alpha^2)^{j/2}k^n}{\Gamma(n)} \int_0^1 [-lnt]^{n+(j/2\beta)-1} t^{k-1} dt. \tag{8}$$

In view of Gradshetyn and Ryzhik (2007, p-557), note that

$$\int_0^1 [-lnx]^{\mu-1} x^{\nu-1} dx = \frac{\Gamma(\mu)}{\nu^\mu}. \tag{9}$$

On substituting (9) in (8), we obtained the result given in (6).

REMARK 1. Setting $\alpha = 1/\sqrt{2}$ and $\beta = 1$ in (6), we get the exact expression for single moments of generalized upper record values from the Rayleigh distribution in the form

$$E(Y_n^{(k)})^j = \frac{\theta^{j/2}\Gamma(n + (j/2))}{k^{j/2}\Gamma(n)}.$$

COROLLARY 1. The exact expression for single moments of upper record values from the Weibull-Rayleigh distribution has the form

$$E(X_{U_n})^j = \frac{(2\theta\alpha^2)^{j/2}\Gamma(n + (j/2\beta))}{\Gamma(n)}. \tag{10}$$

REMARK 2. Putting $\alpha = 1/\sqrt{2}$ and $\beta = 1$ in (10), the result for single moments of upper record values is deduced for the Rayleigh distribution as

$$E(Y_n^{(k)})^j = \frac{\theta^{j/2}\Gamma(n+(j/2))}{\Gamma(n)}.$$

THEOREM 2. For the distribution given in (2) and $1 \leq k \leq n$, $j = 0, 1, ...,$

$$E(Y_n^{(k)})^{j+2\beta} = E(Y_{n-1}^{(k)})^{j+2\beta} + \frac{2^{\beta-1}(\theta\alpha^2)^{\beta}(j+2\beta)}{\beta k} E(Y_n^{(k)})^j. \tag{11}$$

PROOF. From (3) and (4), we have

$$E(Y_n^{(k)})^j = \frac{\beta k^n}{2^{\beta-1}(\theta\alpha^2)^{\beta}\Gamma(n)} \int_0^{\infty} x^{j+2\beta-1}[-ln\bar{F}(x)]^{n-1}[\bar{F}(x)]^k dx.$$

Now, (11) can be seen in view of Khan *et al.* (2017) by noting that

$$E(Y_n^{(k)})^j - E(Y_{n-1}^{(k)})^j = \frac{jk^{n-1}}{\Gamma(n)} \int_0^{\infty} x^{j-1}[-ln\bar{F}(x)]^{n-1}[\bar{F}(x)]^k dx.$$

REMARK 3. Putting $\alpha = 1/\sqrt{2}$ and $\beta = 1$ in (11), we deduce the recurrence relation for single moments of generalized upper record values from the Rayleigh distribution with parameter θ as established by Khan *et al.* (2015).

COROLLARY 2. The recurrence relation for single moments of upper record values from the Weibull-Rayleigh distribution has the form

$$E(X_{U_n})^{j+2\beta} = E(X_{U_{n-1}})^{j+2\beta} + \frac{2^{\beta-1}(\theta\alpha^2)^{\beta}(j+2\beta)}{\beta} E(X_{U_n})^j. \tag{12}$$

REMARK 4. Letting $\alpha = 1/\sqrt{2}$ and $\beta = 1$ in (12), the recurrence relation for single moments of upper record values from the Rayleigh distribution is deduced as established by Khan *et al.* (2015).

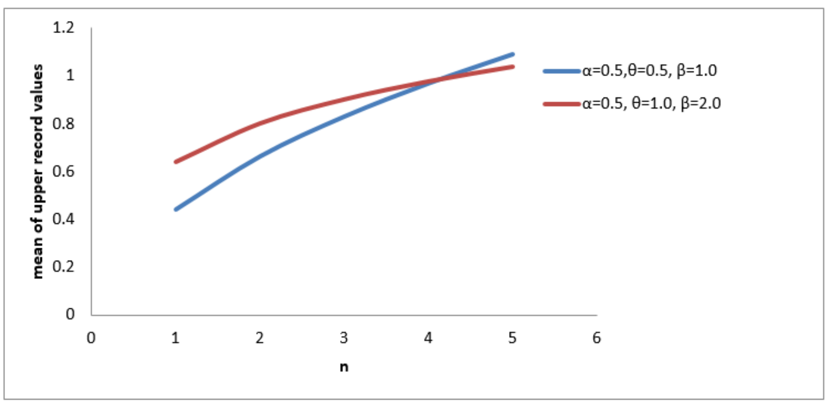
Numerical computations for the first four moments of upper record values from the Weibull-Rayleigh distribution for arbitrary choosen values of α, β, θ, and various sample size $n = 1, 2, \ldots, 5$ are given in Table 1.

Trend of the first four moments of upper record values from the Weibull-Rayleigh distribution for $n = 1, 2, ..., 5$ and different values of parameters are presented in following figures.

Table 1. First four moments of upper record values

n	$\alpha = 0.5, \theta = 0.5$				$\alpha = 0.5, \theta = 1$			
	$\beta = 1$				$\beta = 2$			
	$E(X)$	$E(X^2)$	$E(X^3)$	$E(X^4)$	$E(X)$	$E(X^2)$	$E(X^3)$	$E(X^4)$
1	0.4431	0.2500	0.1661	0.125	0.6409	0.4431	0.3249	0.2500
2	0.6646	0.5000	0.4154	0.375	0.8012	0.6646	0.5686	0.5000
3	0.8308	0.7500	0.7269	0.750	0.9013	0.8308	0.7818	0.7500
4	0.9693	1.0000	1.090	1.250	0.9764	0.9693	0.9773	1.0000
5	1.0904	1.2500	1.4994	1.875	1.0374	1.0904	1.1606	1.2500

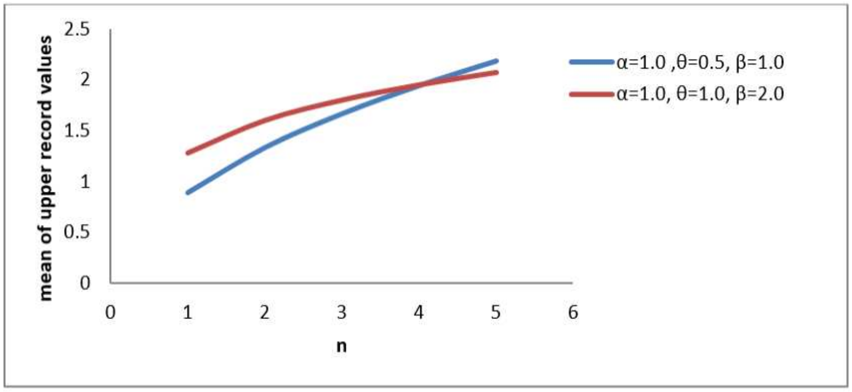
n	$\alpha = 1.0, \theta = 0.5$				$\alpha = 1.0, \theta = 1$			
	$\beta = 1$				$\beta = 2$			
	$E(X)$	$E(X^2)$	$E(X^3)$	$E(X^4)$	$E(X)$	$E(X^2)$	$E(X^3)$	$E(X^4)$
1	0.8862	1.0000	1.3293	2.0000	1.2818	1.7724	2.5995	4.0000
2	1.3293	2.0000	3.3233	6.0000	1.6023	2.6586	4.5491	8.0000
3	1.6616	3.0000	5.8158	12.000	1.8025	3.3233	6.2551	12.000
4	1.9386	4.0000	8.7237	20.000	1.9528	3.8772	7.8188	16.000
5	2.1809	5.0000	11.995	30.000	2.0748	4.3618	9.2848	20.000



3 Product Moments and Relations

This section contains the explicit expressions and recurrence relations for product moments of generalized upper record values from the Weibull-Rayleigh distribution. We shall first establish the explicit expression for the product moments of generalized upper record values.

THEOREM 3. For the distribution as given in (2). Fix a positive integer $k \geq 1$,



for $1 \leq m \leq n-1$ and $i, j = 0, 1, ...$

$$E[(Y_m^{(k)})^i (Y_n^{(k)})^j] = \frac{(2\theta\alpha^2)^{(i+j)/2}}{k^{(i+j)/2\beta}\Gamma(m)\Gamma(n-m)} \sum_{u=0}^{n-m-1} (-1)^{n-m-u-1} \binom{n-m-1}{u} \times \frac{\Gamma(n+(i+j)/2\beta)}{[n+(i/2\beta)-u-1]}. \quad (13)$$

PROOF. From (5), for $1 \leq m \leq n-1$ and $i, j = 0, 1, ...$, we have

$$E[(Y_m^{(k)})^i (Y_n^{(k)})^j] = \frac{k^n}{\Gamma(m)\Gamma(n-m)} \int_0^\infty \int_0^y x^i y^j [-ln\bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)} \times [ln\bar{F}(x) - ln\bar{F}(y)]^{n-m-1} [\bar{F}(y)]^{k-1} f(y) dx dy \quad (14)$$

On expanding $[ln\bar{F}(x) - ln\bar{F}(y)]^{n-m-1}$ in (14), we have

$$E[(Y_m^{(k)})^i (Y_n^{(k)})^j] = \frac{k^n}{\Gamma(m)\Gamma(n-m)} \sum_{u=0}^{n-m-1} (-1)^{n-m-u-1} \binom{n-m-1}{u} \times \int_0^\infty y^j [-ln\bar{F}(y)]^u [\bar{F}(y)]^{k-1} f(y) I(y) dy, \quad (15)$$

where

$$I(y) = \int_0^y x^i [-ln\bar{F}(x)]^{n-u-2} \frac{f(x)}{\bar{F}(x)} dx. \quad (16)$$

By setting $t = -ln\bar{F}(x)$ in (16), we find that

$$I(y) = (2\theta\alpha^2)^{i/2} \int_0^{-ln\bar{F}(y)} t^{n+(i/2\beta)-u-2} dt$$
$$= \frac{(2\theta\alpha^2)^{i/2} [-ln\bar{F}(y)]^{n+(i/2\beta)-u-1}}{[n+(i/2\beta)-u-1]}.$$

Substituting for $I(y)$ in (15) and simplifying the resulting expression, we get

$$E[(Y_m^{(k)})^i(Y_n^{(k)})^j] = A\int_0^\infty y^j[-ln\bar{F}(y)]^{n+(i/2\beta)-1}[\bar{F}(y)]^{k-1}f(y)dy, \tag{17}$$

where

$$A = \frac{(2\theta\alpha^2)^{i/2}k^n}{\Gamma(m)\Gamma(n-m)}\sum_{u=0}^{n-m-1}(-1)^{n-m-u-1}\binom{n-m-1}{u}\frac{1}{[n+(i/2\beta)-u-1]}.$$

Again by setting $z = -ln\bar{F}(y)$ in (17), we have

$$E[(Y_m^{(k)})^i(Y_n^{(k)})^j] = (2\theta\alpha^2)^{j/2}A\int_0^\infty z^{n+[(i+j)/2\beta]-1}e^{-kz}dz,$$

and hence the result given in (13).

REMARK 5. Setting $\alpha = 1/\sqrt{2}$ and $\beta = 1$ in (13), we get the exact expression for product moments of generalized upper record values from the Rayleigh distribution in the form

$$\begin{aligned} E[(Y_m^{(k)})^i(Y_n^{(k)})^j] &= \frac{\theta^{(i+j)/2}}{k^{(i+j)/2}\Gamma(m)\Gamma(n-m)}\sum_{u=0}^{n-m-1}(-1)^{n-m-u-1}\binom{n-m-1}{u} \\ &\times\frac{\Gamma(n+(i+j)/2)}{[n+(i/2)-u-1]}. \end{aligned}$$

IDENTITY 1. For $1 \le m < n$,

$$\sum_{u=0}^{n-m-1}(-1)^{n-m-u-1}\binom{n-m-1}{u}\frac{1}{(n-u-1)} = \frac{\Gamma(m)\Gamma(n-m)}{\Gamma(n)}. \tag{18}$$

PROOF. Putting $i = j = 0$ in (13), we get the required result.

REMARK 6. At $i = 0$ in (13), we have

$$\begin{aligned} E(Y_n^{(k)})^j &= \frac{(2\theta\alpha^2)^{j/2}}{k^{j/2\beta}\Gamma(m)\Gamma(n-m)}\sum_{u=0}^{n-m-1}(-1)^{n-m-u-1}\binom{n-m-1}{u} \\ &\times\frac{\Gamma(n+(j/2\beta))}{(n-u-1)}. \end{aligned} \tag{19}$$

On using (18) in (19), we find that

$$E(Y_n^{(k)})^j = \frac{(2\theta\alpha^2)^{j/2}\Gamma(n+(j/2\beta))}{k^{j/2\beta}\Gamma(n)},$$

which is the exact expression for single moments from the Weibull-Rayleigh distribution, as obtained in (6).

COROLLARY 3. The exact explicit expression for product moments of upper record values from the Weibull-Rayleigh distribution has the form

$$E[(X_{U_m})^i(X_{U_n})^j] = \frac{(2\theta\alpha^2)^{(i+j)/2}}{\Gamma(m)\Gamma(n-m)}\sum_{u=0}^{n-m-1}(-1)^{n-m-u-1}\binom{n-m-1}{u} \times\frac{\Gamma(n+(i+j)/2\beta)}{[n+(i/2\beta)-u-1]}. \tag{20}$$

REMARK 7. Putting $\alpha = 1/\sqrt{2}$ and $\beta = 1$ in (20), the result for product moments of upper record values is deduced for the Rayleigh distribution as

$$E[(X_{U_m})^i(X_{U_n})^j] = \frac{\theta^{(i+j)/2}}{\Gamma(m)\Gamma(n-m)}\sum_{u=0}^{n-m-1}(-1)^{n-m-u-1}\binom{n-m-1}{u} \times\frac{\Gamma(n+(i+j)/2)}{[n+(i/2)-u-1]}.$$

THEOREM 4. For $m \geq 1$, $m \geq k$ and $i, j = 0, 1, ...,$

$$E[(Y_m^{(k)})^i(Y_{m+1}^{(k)})^{j+2\beta}] = E[(Y_m^{(k)})^{i+j+2\beta} + \frac{2^{\beta-1}(\theta\alpha^2)^\beta(j+2\beta)}{\beta k}E[(Y_m^{(k)})^i(Y_{m+1}^{(k)})^j] \tag{21}$$

and for $1 \leq m \leq n-2,\ \ i, j = 0, 1, ...,$

$$E[(Y_m^{(k)})^i(Y_n^{(k)})^{j+2\beta}] = E[(Y_m^{(k)})^i(Y_{n-1}^{(k)})^{j+2\beta}] + \frac{2^{\beta-1}(\theta\alpha^2)^\beta(j+2\beta)}{\beta k}E[(Y_m^{(k)})^i(Y_n^{(k)})^j]. \tag{22}$$

PROOF. From (3) and (5), we have

$$E[(Y_m^{(k)})^i(Y_n^{(k)})^j] = \frac{\beta k^n}{2^{\beta-1}(\theta\alpha^2)^\beta\Gamma(m)\Gamma(n-m)}\int_0^\infty\int_x^\infty x^i y^{j+2\beta-1}[-ln\bar{F}(x)]^{m-1}\frac{f(x)}{\bar{F}(x)} \times[ln\bar{F}(x)-ln\bar{F}(y)]^{n-m-1}[\bar{F}(y)]^k dydx.$$

Now, (22) can be proved by noting that in view of Khan *et al.* (2017)

$$E[(Y_m^{(k)})^i(Y_n^{(k)})^j] - E[(Y_m^{(k)})^i(Y_{n-1}^{(k)})^j] = \frac{jk^{n-1}}{\Gamma(m)\Gamma(n-m)}\int_0^\infty\int_x^\infty x^i y^{j-1} \times[-ln\bar{F}(x)]^{m-1}\frac{f(x)}{\bar{F}(x)}[ln\bar{F}(x)-ln\bar{F}(y)]^{n-m-1}[\bar{F}(y)]^k dydx.$$

Now putting $n = m + 1$ in (22) and noting that $E[(Y_m^{(k)})^i (Y_m^{(k)})^j] = E(Y_m^{(k)})^{i+j}$, the recurrence relation given in (21) can easily be established.

REMARK 8. At $i = 0$ in (22), the recurrence relation for product moments reduces to relation for single moments as obtained in (11).

REMARK 9. Putting $\alpha = 1/\sqrt{2}$ and $\beta = 1$ in (22), we deduce the recurrence relation for product moments of generalized upper record values from the Rayleigh distribution with parameter θ, established by Khan *et al.* (2015).

COROLLARY 4. The recurrence relation for product moments of upper record values from the Weibull-Rayleigh distribution distribution has the form

$$\begin{aligned} E[(X_{U_m})^i (X_{U_n})^{j+2\beta}] &= E[(X_{U_m})^i (X_{U_{n-1}})^{j+2\beta}] \\ &\quad + \frac{2^{\beta-1}(\theta\alpha^2)^\beta (j+2\beta)}{\beta} E[(X_{U_m})^i (X_{U_n})^j]. \end{aligned} \tag{23}$$

REMARK 10. Letting $\alpha = 1/\sqrt{2}$ and $\beta = 1$ in (23), the recurrence relation for product moments of upper record values from the Rayleigh distribution is deduced, as established by Khan *et al.* (2015).

4 Characterizations

Following theorem contains characterization of the Weibull-Rayleigh distribution on using the recurrence relation for single moments of generalized upper record statistics.

THEOREM 5. Fix a positive integer $k \geq 1$ and let j be a non-negative integer, a necessary and sufficient condition for a random variable X to be distributed with *df* given by (2) is that

$$E(Y_n^{(k)})^{j+2\beta} = E(Y_{n-1}^{(k)})^{j+2\beta} + \frac{2^{\beta-1}(\theta\alpha^2)^\beta (j+2\beta)}{\beta k} E(Y_n^{(k)})^j \tag{24}$$

for $n = 1, 2, ..., n \geq k$.

PROOF. The necessary part follows from (11). On the other hand if the recurrence relation (24) is satisfied, then on rearranging the terms in (24) and using (4), we have

$$\begin{aligned} & \frac{k^n}{\Gamma(n)} \int_0^\infty x^j [-ln\bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} f(x) dx \\ = \quad & \frac{\beta k^{n+1}}{2^{\beta-1}(\theta\alpha^2)^\beta (j+2\beta)\Gamma(n)} \int_0^\infty x^{j+2\beta} [-ln\bar{F}(x)]^{n-2} [\bar{F}(x)]^{k-1} f(x) \\ & \times \left(-ln\bar{F}(x) - \frac{n-1}{k} \right) dx. \end{aligned} \tag{25}$$

Let

$$h(x) = -\frac{1}{k}[-ln\bar{F}(x)]^{n-1}[\bar{F}(x)]^k. \tag{26}$$

Differentiating both the sides of (26), we get

$$h'(x) = [-ln\bar{F}(x)]^{n-2}[\bar{F}(x)]^{k-1}f(x)\Big\{-ln\bar{F}(x) - \frac{n-1}{k}\Big\}.$$

Thus

$$\begin{aligned} & \frac{k^n}{\Gamma(n)}\int_0^\infty x^j[-ln\bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}f(x)dx \\ = & \frac{\beta k^{n+1}}{2^{\beta-1}(\theta\alpha^2)^\beta(j+2\beta)\Gamma(n)}\int_0^\infty x^{j+2\beta}h'(x)dx. \end{aligned} \tag{27}$$

Integrating right hand side in (27) by parts and using the value of $h(x)$ from (26), we find that

$$\frac{k^n}{\Gamma(n)}\int_0^\infty x^j[-ln\bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}\Big\{f(x) - \frac{\beta x^{2\beta-1}}{2^{\beta-1}(\theta\alpha^2)^\beta}\bar{F}(x)\Big\} = 0. \tag{28}$$

Now applying a generalization of the Müntz-Szász Theorem (see for example Hwang and Lin (1984)) to (28), we obtain

$$f(x) = \frac{\beta x^{2\beta-1}}{2^{\beta-1}(\theta\alpha^2)^\beta}\bar{F}(x),$$

which proves that $f(x)$ has the form as in (3).

REMARK 11. If $k = 1$ in (24), we obtain the following characterization of the Weibull-Rayleigh distribution based on upper record values

$$E(X_{U_n})^{j+2\beta} = E(X_{U_{n-1}})^{j+2\beta} + \frac{2^{\beta-1}(\theta\alpha^2)^\beta(j+2\beta)}{\beta}E(X_{U_n})^j.$$

REMARK 12. If $\alpha = 1/\sqrt{2}$ and $\beta = 1$ in (24), the characterizing result based on generalized upper record values for the Rayleigh distribution is deduced as

$$E(Y_n^{(k)})^{j+2} = E(Y_{n-1}^{(k)})^{j+2} + \frac{\theta(j+2)}{2k}E(Y_n^{(k)})^j,$$

which verify the result obtained by Khan *et al.* (2015).

COROLLARY 5. Under the assumptions of Theorem 5 with $j = 0$, the following relation

$$E(Y_n^{(k)})^{2\beta} = E(Y_{n-1}^{(k)})^{2\beta} + \frac{2^\beta(\theta\alpha^2)^\beta}{k},$$

characterize the Weibull-Rayleigh distribution.

REMARK 13. If $k=1$, we obtain the following characterization of the Weibull-Rayleigh distribution.

$$E(X_{U_n})^{2\beta} = E(X_{U_{n-1}})^{2\beta} + 2^{\beta}(\theta\alpha^2)^{\beta}, \quad n = 1, 2, ...$$

Following theorem deal with the characterization of the Weibull-Rayleigh distribution through conditional expectation of function of generalized upper record statistics.

THEOREM 6. Let X be a non-negative random variable having an absolutely continuous *df* $F(x)$ with $F(0)=0$ and $0 \le F(x) \le 1$ for all $x>0$, then

$$E[\xi(Y_n^{(k)}) \mid (Y_l^{(k)}) = x] = \xi(x)\Big(\frac{k}{k+1}\Big)^{n-l}, \quad l = m, \quad m+1 \tag{29}$$

if and only if

$$\bar{F}(x) = e^{-(x^2/2\theta\alpha^2)^{\beta}}, \quad x \ge 0, \quad \alpha,\ \beta,\ \theta > 0,$$

where

$$\xi(y) = e^{-(y^2/2\theta\alpha^2)^{\beta}}.$$

PROOF. From (4) and (5), we have

$$E[\xi(Y_n^{(k)}) \quad | \quad (Y_m^{(k)}) = x] = \frac{k^{n-m}}{\Gamma(n-m)}\int_x^{\infty} e^{-(y^2/2\theta\alpha^2)^{\beta}}[ln\bar{F}(x) - ln\bar{F}(y)]^{n-m-1} \times\Big(\frac{\bar{F}(y)}{\bar{F}(x)}\Big)^{k-1}\frac{f(y)}{\bar{F}(x)}dy. \tag{30}$$

By setting $u = \frac{\bar{F}(y)}{\bar{F}(x)} = \frac{e^{-(y^2/2\theta\alpha^2)^{\beta}}}{e^{-(x^2/2\theta\alpha^2)^{\beta}}}$ from (2) in (30), we obtain

$$E[\xi(Y_n^{(k)}) \mid (Y_m^{(k)}) = x] = \frac{k^{n-m}}{\Gamma(n-m)}e^{-(x^2/2\theta\alpha^2)^{\beta}}\int_0^1(-lnu)^{n-m-1}u^k du. \tag{31}$$

On using (9) in (31), we derive relation given in (29). To prove sufficient part, we have

$$\frac{k^{n-m}}{\Gamma(n-m)}\int_x^{\infty} e^{-(y^2/2\theta\alpha^2)^{\beta}}[ln\bar{F}(x) - ln\bar{F}(y)]^{n-m-1}[\bar{F}(y)]^{k-1}f(y)dy = [\bar{F}(x)]^k g_{n|m}(x), \tag{32}$$

where

$$g_{n|m}(x) = e^{-(x^2/2\theta\alpha^2)^{\beta}}\Big(\frac{k}{k+1}\Big)^{n-m}.$$

Differentiating both sides of (32) with respect to x, we get

$$-\frac{k^{n-m}f(x)}{\bar{F}(x)\Gamma(n-m-1)}\int_x^{\infty} e^{-(y^2/2\theta\alpha^2)^{\beta}}[ln\bar{F}(x) - ln\bar{F}(y)]^{n-m-2} \times[\bar{F}(y)]^{k-1}f(y)dy = g'_{n|m}(x)[\bar{F}(x)]^k - k\ g_{n|m}(x)[\bar{F}(x)]^{k-1}f(x)$$

or

$$-k\ g_{n|m+1}(x)[\bar{F}(x)]^{k-1}f(x) = g'_{n|m}(x)[\bar{F}(x)]^{k} - k\ g_{n|m}(x)[\bar{F}(x)]^{k-1}f(x).$$

Therefore,

$$\frac{f(x)}{\bar{F}(x)} = -\frac{g'_{n|m}(x)}{k[g_{n|m+1}(x) - g_{n|m}(x)]} = \frac{\beta}{2^{\beta-1}(\theta\alpha^2)^{\beta}}x^{2\beta-1}, \tag{33}$$

where

$$g'_{n|m}(x) = -\frac{\beta}{2^{\beta-1}(\theta\alpha^2)^{\beta}}x^{2\beta-1}e^{-(x^2/2\theta\alpha^2)^{\beta}}\Big(\frac{k}{k+1}\Big)^{n-m},$$

$$g_{n|m+1}(x) - g_{n|m}(x) = \frac{1}{k}e^{-(x^2/2\theta\alpha^2)^{\beta}}\Big(\frac{k}{k+1}\Big)^{n-m}.$$

Integrating both sides of (33) with respect to x between $(0, y)$, the sufficiency part is proved.

REMARK 14. If $k = 1$ in (29), we obtain the following characterization of the Weibull-Rayleigh distribution based on upper record values

$$E[\xi(X_{U_n})|X_{U_l} = x] = e^{-(x^2/2\theta\alpha^2)^{\beta}}(1/2)^{n-l}, \quad l = m, \quad m+1.$$

REMARK 15. Putting $\alpha = 1/\sqrt{2}$ and $\beta = 1$ in (29), the characterizing result of generalized upper record values for the Rayleigh distribution with parameter θ, is deduced as

$$E[\xi(Y_n^{(k)}) \mid (Y_l^{(k)}) = x] = e^{-x^2/\theta}\Big(\frac{k}{k+1}\Big)^{n-l}, \quad l = m, \quad m+1$$

if and only if

$$\bar{F}(x) = e^{-(x^2/\theta)}, \quad x \geq 0, \quad \alpha,\ \beta,\ \theta > 0.$$

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