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On The Existence Of Coincidence And Common Fixed Points Of Rational Type Contractions Via C-Class Functions In Branciari Distance Spaces*

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Abstract

The aim of this paper is to establish some coincidence point results for self-mappings satisfying rational type contractions in Branciari distance spaces. In this direction, we correct some false essential steps given in the papers [9], [40] and [44]. Our presented coincidence point theorems extend numerous existing theorems in the literature. We also provide an illustrated application.

1 Introduction

The Banach contraction principle [15] has been generalized and extended in many directions, see [1, 13, 20, 29, 33, 35, 37, 38, 39, 41, 46]. In 1973, Dass and Gupta [25] defined the following rational type contraction which is more general than the Banach contraction condition:

$$d(Ax, Ay) \leq ad(x, y) + \frac{bd(y, Ay)(d(x, Ax) + 1)}{1 + d(x, y)} \tag{1}$$

for all $x, y \in X$ and $a, b \geq 0$ with $a + b < 1$, where $A : X \to X$ is a mapping from a metric space X into itself. There are many generalizations of this principle (see [20], [29], [41], [46]). Later, Almeida, Roldan-Lopez-de-Hierro and Sadarangani [5] introduced an extension of the condition (1) of Dass and Gupta [25] as follows

$$d(Ax, Ay) \leq \phi(P(x, y)) + C \min\{d(x, Ax), d(y, Ay), d(x, Ay), d(y, Ax)\}, \tag{2}$$

for all $x, y \in X$ with $C \geq 0$, where $\phi : [0, \infty) \to [0, \infty)$ is a non-decreasing upper semi-continuous function with $\phi(t) < t$ for all $t > 0$, and $P(x, y)$ is defined by

$$P(x, y) = \max\left\{d(x, y), \frac{d(x, Ax)(d(y, Ay) + 1)}{1 + d(x, y)}, \frac{d(y, Ay)(d(x, Ax) + 1)}{1 + d(x, y)}\right\}.$$

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It is worth to mention that the use of triangle inequality in a metric space (X, d) is of extreme importance since it implies that d is continuous, each open ball is an open set, a sequence may converge to a unique point and every convergent sequence is Cauchy. In 2000, Branciari [18] introduced a new concept of a generalized distance space by replacing the triangle inequality by a so-called quadrilateral inequality. Since then, various works have dealt with fixed point results in such spaces (see [3, 4, 7, 9, 10, 11, 12, 14, 16, 17, 22, 23, 24, 26, 27, 28, 31, 32, 43, 44]). Following the paper of Suzuki [45], these spaces are called Branciari distance spaces (B.D.S, for short).

The following definitions and results will be needed in the sequel.

DEFINITION 1 ([18]). Suppose that X be a nonempty set and $d : X \times X \to [0, \infty)$ be a distance function such that for all $w, x \in X$ and all distinct points $y, z \in X$, each distinct from w and x:

(i) $d(w, x) = 0 \Leftrightarrow w = x$;

(ii) $d(w, x) = d(x, w)$;

(iii) $d(w, x) \leq d(x, y) + d(y, z) + d(z, w)$ (quadrilateral inequality).

Then (X, d) is called a B.D.S.

EXAMPLE 1 ([44]). Suppose that $X = \{\frac{5}{6}, \frac{2}{3}, \frac{7}{12}, \frac{8}{15}\}$. Define d on $X \times X$ as follows

$$d(\frac{5}{6}, \frac{2}{3}) = d(\frac{7}{12}, \frac{8}{15}) = \frac{4}{9} \quad , d(\frac{5}{6}, \frac{8}{12}) = d(\frac{2}{3}, \frac{7}{12}) = \frac{1}{3},$$

$$d(\frac{5}{6}, \frac{7}{12}) = d(\frac{2}{3}, \frac{8}{12}) = \frac{8}{9} \quad , d(x, x) = 0,\ d(x, y) = d(y, x).$$

Then (X, d) is a B.D.S. Note that (X, d) is not a metric space.

REMARK 1. Condition (iii) in Definition 1 does not ensure that d is continuous on its domain, see [18].

DEFINITION 2 ([18, 40]). Let (X, d) be a B.D.S. Let $\{x_n\}$ be a sequence in X. Then

(i) $\{x_n\}$ converges to $x \in X$ iff $\lim_{n\to\infty} d(x_n, x) = 0$;

(ii) $\{x_n\}$ is a Cauchy iff $\forall \epsilon > 0, \exists K(\epsilon) > 0$ such that $d(x_r, x_s) < \epsilon$ for all $r > s \geq K(\epsilon)$;

(iii) (X, d) is called a complete B.D.S if every Cauchy sequence in X converges to a point in X.

In 2009, Sarma et al. [42] introduced the following example illustrating Remark 1.

EXAMPLE 2 ([42]). Suppose that $X = D \cup E$, where $D = \{0, 2\}$ and $E = \{\frac{1}{n} : n \in \mathbb{N}$ (the set of all natural numbers)$\}$. Define $d : X \times X \to [0, \infty)$ as

$$d(u,v) = \begin{cases} 0, & u = v \\ 1, & u \neq v \ \& \ \{u,v\} \subset D \text{ or } \{u,v\} \subset E, \end{cases}$$

and $d(u,v) = d(v,u) = u$ if $u \in D$ and $v \in E$.
Then (X, d) is a complete B.D.S. Moreover, one can see that

(i) $d(\frac{1}{n}, 0) = 0$ and $d(\frac{1}{n}, 2) = 2 \Rightarrow \{\frac{1}{n}\}$ is not a Cauchy sequence.

(ii) $d(\frac{1}{n}, \frac{1}{2}) \neq d(\frac{1}{2}, 0) \Rightarrow d$ is not continuous.

DEFINITION 3 ([40]). Let $A, B : X \to X$ and $\beta : X \times X \to [0, \infty)$. The mapping A is $B-\beta-$admissible if, for all $x, y \in X$ such that $\beta(Bx, By) > 1$, we have $\beta(Ax, Ay) > 1$. If B is the identity mapping, then A is called $\beta-$admissible.

DEFINITION 4 ([40]). Let (X, d) be a B.D.S and $\beta : X \times X \to [0, \infty)$. X is $\beta-$regular if for each sequence $\{x_n\}$ in X such that $\beta(x_n, x_{n+1}) > 1$ for all $n \in \mathbb{N}$ and $x_n \to x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\beta(x_{n_k}, x) > 1$ $\forall k \in N$.

LEMMA 1 ([30]). Let (X, d) be a B.D.S and let $\{x_n\}$ be a sequence in X with distinct elements ($x_n \neq x_m$ for all $n \neq m$). Suppose that $d(x_n, x_{n+1})$ and $d(x_n, x_{n+2})$ tend to 0 as $n \to \infty$ and that $\{x_n\}$ is not a Cauchy sequence. Then there exist $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and the following four sequences

$$d(x_{m_k}, x_{n_k}), \quad d(x_{m_k}, x_{n_{k+1}}), \quad d(x_{m_{k-1}}, x_{n_k}), \quad d(x_{m_{k-1}}, x_{n_{k+1}}) \tag{3}$$

tend to ϵ as $k \to \infty$.

In 2014, the concept of C-class functions was introduced by Ansari in [6].

DEFINITION 5 ([6]). A mapping $F : [0, \infty)^2 \to \mathbb{R}$ is called a C-*class* function if it is continuous and satisfies the following axioms:

(1) $F(s, t) \leq s$ for all $s, t \in [0, \infty)$;

(2) $F(s, t) = s$ implies that either $s = 0$ or $t = 0$.

We denote $\mathcal{C}$ as the set of C-class functions.

EXAMPLE 3 ([6]). The following functions $F : [0, \infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$, for all $s, t \in [0, \infty)$:

(1) $F(s, t) = s - t$;

(2) $F(s, t) = ms$ where $0 < m < 1$;

(3) $F(s,t) = s\beta(s)$ where $\beta : [0,\infty) \to [0,1)$ is continuous;

(4) $F(s,t) = s - \varphi(s)$ where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$;

(5) $F(s,t) = \phi(s)$ where $\phi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$.

DEFINITION 6 ([34]). A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is non-decreasing and continuous;

(ii) $\psi(t) = 0$ if and only if $t = 0$.

We denote Ψ the set of altering distance functions.

DEFINITION 7. For $\psi, \varphi \in \Psi$ and $F \in \mathcal{C}$, the tripled (ψ, φ, F) is said to be monotone if for any $x, y \in [0,\infty)$

$$x \leqslant y \Longrightarrow F(\psi(x), \varphi(x)) \leqslant F(\psi(y), \varphi(y)).$$

EXAMPLE 4. Let $F(s,t) = s - t$, $\phi(x) = \sqrt{x}$ and

$$\psi(x) = \begin{cases} \sqrt{x} & \text{if } 0 \leq x \leq 1, \\ x^2 & \text{if } x > 1, \end{cases}$$

then (ψ, ϕ, F) is monotone.

In this paper, we state some coincidence point and common fixed point results involving rational type contractive self-mappings using C-class functions in a complete B.D.S. Mention that the proof of Theorem 10 in [40] is false (same remark for the proof of Theorem 5 in [9]). To be more clear, the case $z_n = z_m$ (for $n \neq m$) is not treated and the end of equation (17) is not correct in [40]. Also, in [9] there is a gap in the proof of $\lim_{n\to\infty} d(y_n, y_{n+2}) = 0$ (the same remark for the proof of Theorem 10 in [44]). Indeed, the authors in [9] take the limit $n \to \infty$ in inequalities (19) and (20), which only hold for some integer n. Here, we provide a correct proof which goes as well for these mentioned papers. Our corrections are given within step 2 and step 3 in the proof of Theorem 2 (next section).

2 Main Results

We present some coincidence point theorems for (α, ψ, ϕ)-contraction self-mappings of a rational type using C-class functions in the setting of B.D.S.

THEOREM 2. Let (X, d) be a B.D.S and let $A, B : X \to X$ be two self-mappings satisfy the following:

$$\psi(\beta(Bx, By)d(Ax, Ay)) \leq F(\psi(M(x, y)), \phi(M(x, y))) \; \forall x, y \in X, \tag{4}$$

where $\psi, \phi \in \Psi$, $F \in C$, $AX \subset BX$, (BX, d) is a complete B.D.S. and

$$M(x, y) = \max\left\{d(Bx, By), \frac{d(Bx, Ax)(d(By, Ay) + 1)}{1 + d(Bx, By)}, \frac{d(By, Ay)(d(Bx, Ax) + 1)}{1 + d(Bx, By)}\right\}.$$

Assume also that

(i) there exists $x_0 \in X$ such that $\beta(Ax_0, Bx_0) \geq 1$;

(ii) A is $B - \beta-$admissible;

(iii) X is $\beta-$regular and $\beta(x_m, x_n) \geq 1$ for each $x_n \in X$ and $\forall m, n \in N, m \neq n$;

(iv) either $\beta(Bx, By) \geq 1$ or $\beta(By, Bx) \geq 1$, whenever $Bx = Ax$ and $By = Ay$;

(v) (ψ, ϕ, F) is monotone;

(vi) B is one to one.

Then A and B have a unique point of coincidence in X. Moreover, if A and B are weakly compatible, then A and B have a unique common fixed point.

PROOF. Let $x_0 \in X$ be arbitrary. Consider the sequences $\{x_n\}$ and $\{z_n\}$ in X defined by

$$z_n = Bx_{n+1} = Ax_n.$$

Suppose also that $\beta(Bx_0, Ax_0) \geq 1$. If for some n, $z_n = z_{n+1}$, then z_n is a point of coincidence of A and B. This completes the proof.

From now on, we assume that $z_n \neq z_{n+1}$ for all $n \in \mathbb{N}$.

Step 1: We shall prove that

$$\lim_{n\to\infty} d(z_n, z_{n+1}) = 0. \tag{5}$$

From (i), $\beta(Bx_0, Ax_0) = \beta(Bx_0, Bx_1) \geq 1$. Applying (ii), we have that $\beta(Ax_0, Ax_1) = \beta(Bx_1, Bx_2) \geq 1$ and $\beta(Ax_1, Ax_2) = \beta(Bx_2, Bx_3) \geq 1$. Continuing in this process, we get that $\beta(Bx_n, Bx_{n+1}) \geq 1$.

We shall prove that

$$d(z_n, z_{n+1}) \leq d(z_{n-1}, z_n) \quad \text{for all } n \geq 1. \tag{6}$$

Suppose that $d(z_n, z_{n+1}) > d(z_{n-1}, z_n)$ for some $n \geq 1$. By using (4), we have

$$\begin{aligned} \psi(d(z_n, z_{n+1})) = \psi(d(Ax_n, Ax_{n+1})) &\leq \psi(\beta(Bx_n, Bx_{n+1})d(Ax_n, Ax_{n+1})) \\ &\leq F(\psi(M(x_n, x_{n+1})), \phi(M(x_n, x_{n+1}))) \end{aligned} \tag{7}$$

where

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\left\{d(Bx_n, Bx_{n+1}), \frac{d(Bx_n, Ax_n)(d(Bx_{n+1}, Ax_{n+1})+1)}{1+d(Bx_n, Bx_{n+1})},\right. \\ &\left.\frac{d(Bx_{n+1}, Ax_{n+1})(d(Bx_n, Ax_n)+1)}{1+d(Bx_n, Bx_{n+1})}\right\} \\ &= \max\{d(z_{n-1}, z_n), \frac{d(z_{n-1}, z_n)(1+d(z_n, z_{n+1}))}{1+d(z_{n-1}, z_n)}, d(z_n, z_{n+1})\} \\ &= d(z_n, z_{n+1}). \end{aligned}$$

Then

$$\psi(d(z_n, z_{n+1})) \le F(\psi(d(z_n, z_{n+1})), \phi(d(z_n, z_{n+1}))),$$

which implies that $\psi\,(d(z_n, z_{n+1})) = 0$ or $\phi(d(z_n, z_{n+1})) = 0$. That is $d(z_n, z_{n+1}) = 0$. This is a contradiction. So (6) holds. Finally, (7) becomes

$$\psi(d(z_n, z_{n+1})) \le F(\psi(d(z_{n-1}, z_n)), \phi(d(z_{n-1}, z_n)) \quad \forall n \ge 1. \tag{8}$$

From (6), the positive real sequence $\{d(z_n, z_{n+1})\}$ is decreasing, so it converges to a nonnegative number $s \ge 0$. Letting $n \to +\infty$ in (8), we obtain

$$\psi(s) \le F(\psi(s), \phi(s)).$$

Thus, $\psi\,(s) = 0$ or $\phi(s) = 0$. Hence $s = 0$ and hence (5) holds.

Step 2: We shall prove that

$$z_n \ne z_m \quad \text{for all } n \ne m. \tag{9}$$

We argue by contradiction. Suppose that $z_n = z_m$ for some $m, n \in \mathbb{N}$ with $m \ne n$. Since $d(z_p, z_{p+1}) > 0$ for each $p \in \mathbb{N}$, without loss of generality, we may assume that $m \ge n+1$.

Since B is one to one and as $z_n = z_m$, we get $z_{n+1} = z_{m+1}$. Then by (4) and (6), we have

$$\begin{aligned} \psi(d(z_n, z_{n+1})) = \psi(d(z_m, z_{m+1})) &\le \psi(\beta(d(Bx_m, Bx_{m+1})d(Ax_m, Ax_{m+1})) \\ &\le F(\psi(M(x_m, x_{m+1})), \phi(M(x_m, x_{m+1}))) \\ &\le \psi(M(x_m, x_{m+1})), \end{aligned}$$

where

$$\begin{aligned} M(x_m, x_{m+1}) &= \max\left\{d(Bx_m, Bx_{m+1}), \frac{d(Bx_m, Ax_m)(d(Bx_{m+1}, Ax_{m+1})+1)}{1+d(Bx_m, Bx_{m+1})},\right. \\ &\left.\frac{d(Bx_{m+1}, Ax_{m+1})(d(Bx_m, Ax_m)+1)}{1+d(Bx_m, Bx_{m+1})}\right\} \\ &= \max\left\{d(z_{m-1}, z_m), \frac{d(z_{m-1}, z_m)(1+d(z_m, z_{m+1}))}{1+d(z_{m-1}, z_m)}, d(z_m, z_{m+1})\right\} \\ &= d(z_{m-1}, z_m). \end{aligned}$$

As (ψ,ϕ,F) is monotone, we obtain

$$\begin{aligned}\psi(d(z_n,z_{n+1})) &\leq F(\psi(d(z_{m-1},z_m)),\phi(d(z_{m-1},z_m))\\ &\leq F(\psi(d(z_{m-2},z_{m-1})),\phi(d(z_{m-2},z_{m-1}))\\ &\cdots\\ &\leq F(\psi(d(z_n,z_{n+1})),\phi(d(z_n,z_{n+1})),\end{aligned}$$

which implies that $\psi(d(z_n,z_{n+1}))=0$ or $\phi(d(z_n,z_{n+1}))=0$, i.e., $d(z_n,z_{n+1})=0$. This is a contradiction. So (9) holds.

Step 3: We shall show that

$$\lim_{n\to\infty} d(z_n,z_{n+2})=0. \tag{10}$$

By using (4), we have

$$\begin{aligned}\psi(d(z_n,z_{n+2})) &\leq \psi(\beta(Bx_n,Bx_{n+2})d(Ax_n,Ax_{n+2})) \\ &\leq F(\psi(M(x_n,x_{n+2})),\phi(M(x_n,x_{n+2})))\end{aligned} \tag{11}$$

where

$$\begin{aligned}M(x_n,x_{n+2}) &= \max\left\{d(Bx_n,Bx_{n+2}),\frac{d(Bx_n,Ax_n)(d(Bx_{n+2},Ax_{n+2})+1)}{1+d(Bx_n,Bx_{n+2})},\right.\\ &\left.\frac{d(Bx_{n+2},Ax_{n+2})(d(Bx_n,Ax_n)+1)}{1+d(Bx_n,Bx_{n+2})}\right\}\\ &= \max\left\{d(z_{n-1},z_{n+1}),\frac{d(z_{n-1},z_n)(1+d(z_{n+1},z_{n+2}))}{1+d(z_{n-1},z_{n+1})},\right.\\ &\left.\frac{d(z_{n+1},z_{n+2})(1+d(z_{n-1},z_n))}{1+d(z_{n-1},z_{n+1})}\right\}.\end{aligned}$$

Let

$$I=\{n\in\mathbb{N}:M(x_n,x_{n+2})=d(z_{n-1},z_{n+1})\}.$$

We distinguish the two following cases:

Case 1: Assume that $|I|<\infty$. In this case

$$M(x_n,x_{n+2})=\max\left\{\frac{d(z_{n-1},z_n)(1+d(z_{n+1},z_{n+2}))}{1+d(z_{n-1},z_n)},\frac{d(z_{n+1},z_{n+2})(1+d(z_{n-1},z_n))}{1+d(z_{n-1},z_{n+1})}\right\},$$

for n large enough. From (5),

$$\lim_{n\to\infty} M(x_n,x_{n+2})=0.$$

Using the properties of F and ψ, we get

$$\lim_{n\to\infty} d(z_n,z_{n+2})=0.$$

Case 2: Assume that $|I|=\infty$. In this case

$$M(x_n,x_{n+2})=d(z_{n-1},z_{n+1}),$$

for n large enough. It follows that the real positive sequence $\{d(z_n, z_{n+2})\}$ is non-increasing. Similarly, we have

$$\lim_{n\to\infty} d(z_n, z_{n+2}) = 0.$$

Step 4: We shall prove that $\{z_n\}$ is Cauchy.

Suppose that $\{z_n\}$ is not a Cauchy sequence. By Lemma 1, there exist $\varepsilon > 0$ and two subsequences $\{z_{m(k)}\}$ and $\{z_{n(k)}\}$ of $\{z_n\}$ with $m(k) > n(k) > k$ such that $d(z_{m(k)}, z_{n(k)}) \geq \varepsilon$, $d(z_{m(k)}, z_{2n(k)-2}) < \varepsilon$ and

$$\begin{aligned}\lim_{k\to\infty} d\left(z_{n(k)}, z_{m(k)}\right) &= \lim_{k\to\infty} d\left(z_{n(k)+1}, z_{m(k)}\right) = \lim_{k\to\infty} d\left(z_{n(k)}, z_{m(k)-1}\right) \\ &= \lim_{k\to\infty} d\left(z_{n(k)+1}, z_{m(k)+1}\right) = \varepsilon.\end{aligned}$$

Applying (4) with $x = x_{n_k}$ and $y = x_{m_k}$, we obtain

$$\begin{aligned}\psi(d(Ax_{m_k}, Ax_{n_k})) &\leq \psi(\beta(d(Bx_{m_k}, Bx_{n_k}))d(Ax_{m_k}, Ax_{n_k})) \\ &\leq F(\psi(M(x_{m_k}, x_{n_k})), \phi(M(x_{m_k}, x_{n_k})))\end{aligned}$$

where

$$\begin{aligned}M(x_{m_k}, x_{n_k}) &= \max\left\{d(Bx_{m_k}, Bx_{n_k}), \frac{d(Bx_{m_k}, Ax_{m_k})(d(Bx_{n_k}, Ax_{n_k}) + 1}{1 + d(Bx_{m_k}, Bx_{n_k})},\right. \\ &\quad \left.\frac{d(Bx_{n_k}, Ax_{n_k})(d(Bx_{m_k}, Ax_{m_k}) + 1)}{1 + d(Bx_{m_k}, Bx_{n_k})}\right\} \\ &= \max\left\{d(z_{m_k-1}, z_{n_k-1}), \frac{d(z_{m_k-1}, z_{m_k})(d(z_{n_k-1}, z_{n_k}) + 1)}{1 + d(z_{m_k-1}, z_{n_k-1})},\right. \\ &\quad \left.\frac{d(z_{n_k-1}, z_{n_k})(d(z_{m_k-1}, z_{m_k}) + 1)}{1 + d(z_{m_k-1}, z_{n_k-1})}\right\}.\end{aligned}$$

Using the continuity of ϕ, F, ψ and letting $k \to +\infty$,

$$\psi(\epsilon) \leq F(\psi(\epsilon), \phi(\epsilon)).$$

So $\psi(\epsilon) = 0$ or $\phi(\epsilon) = 0$. Hence $\epsilon = 0$, which is a contradiction. Thus $\{z_n\}$ is a Cauchy sequence. Since (BX, d) is complete, there exists $z \in BX$ such that $\lim_{n\to\infty} z_n = z$. Let $w \in X$ be such that $Bu = z$. Applying (4) by taking $x = x_{n_k}$,

$$\psi(d(Au, Ax_{n_k})) \leq F(\psi(M(u, x_{n_k})), \phi(M(u, x_{n_k}))), \tag{12}$$

where

$$\begin{aligned}M(u, x_{n_k}) &= \max\left\{d(Bu, Bx_{n_k}), \frac{d(Bu, Au)(d(Bx_{n_k}, Ax_{n_k}) + 1)}{1 + d(Bu, Bx_{n_k})},\right. \\ &\quad \left.\frac{d(Bx_{n_k}, Ax_{n_k})(d(Bu, Au) + 1)}{1 + d(Bu, Bx_{n_k})}\right\} \\ &= \max\left\{d(z, z_{n_k-1}), \frac{d(Bu, Au)(d(z_{n_k-1}, z_{n_k}) + 1)}{1 + d(Bu, z_{n_k-1})},\right. \\ &\quad \left.\frac{d(z_{n_k-1}, z_{n_k})(d(Bu, Au) + 1)}{1 + d(Bu, z_{n_k-1})}\right\} \\ &\to d(Bu, Au) \text{ as } k \to \infty.\end{aligned}$$

By using (12), we have that

$$\begin{aligned}\psi(d(Bu,Au)) &\leq \limsup_{k\to\infty}[d(Bu,z_{n_k-1})+d(z_{n_k-1},z_{n_k})+d(Au,Ax_{n_k})] \quad (13)\\ &\leq \limsup_{k\to\infty}\psi(d(Au,Ax_{n_k}))\\ &= F(\psi(d(Bu,Au)),\phi(d(Bu,Au))).\end{aligned}$$

Again $\psi(d(Bu,Au))=0$ or $\phi(d(Bu,Au))=0$, that is $d(Bu,Au)=0$, i.e., $z=Bu=Au$ and so z is a coincidence point for A and B.

Finally, we prove that z is the unique coincidence point of A and B. Let x and y be two arbitrary coincidence points of A and B such that $x=Au=Bu$ and $y=Av=Bv$. Using (4), it follows that

$$\begin{aligned}&\psi(d(x,y))\\ =\ &\psi(d(Au,Av))\\ \leq\ &F\left(\psi(\max\left\{d(Bu,Bv),\frac{d(Bu,Au)(d(Bv,Av)+1)}{1+d(Bu,Bv)},\frac{d(Bv,Av)(d(Bu,Au)+1)}{1+d(Bu,Bv)}\right\})\right.\\ &\left.,\phi(\max\left\{d(Bu,Bv),\frac{d(Bu,Au)(d(Bv,Av)+1)}{1+d(Bu,Bv)},\frac{d(Bv,Av)(d(Bu,Au)+1)}{1+d(Bu,Bv)}\right\}\right)\\ =\ &F(\psi(d(Bu,Bv)),\phi(d(Bu,Bv)))\\ =\ &F(\psi(d(x,y)),\phi(d(x,y))).\end{aligned}$$

Similarly, $d(x,y)=0$. Thus A and B have a unique coincidence point.

Suppose that A and B are weakly compatible. We have

$$Az=ABu=BAu=Bz.$$

By (4),

$$\begin{aligned}&\psi(d(Az,z))\\ =\ &\psi(d(Az,Au))\\ \leq\ &F\left(\psi(\max\left\{d(Bz,Bu),\frac{d(Bz,Az)(d(Bu,Au)+1)}{1+d(Bz,Bu)},\frac{d(Bu,Au)(d(Bz,Az)+1)}{1+d(Bz,Bu)}\right\})\right.\\ &\left.,\phi(\max\left\{d(Bz,Bu),\frac{d(Bz,Az)(d(Bu,Au)+1)}{1+d(Bz,Bu)},\frac{d(Bu,Au)(d(Bz,Az)+1)}{1+d(Bz,Bu)}\right\}\right)\\ =\ &F(\psi(d(z,Bz)),\phi(d(z,Bz)))\\ =\ &F(\psi(d(z,Az)),\phi(d(z,Az))),\end{aligned}$$

which implies that $\psi(d(z,Az))=0$ or $\phi(d(z,Az))=0$, i.e., $d(z,Az)=0$ and so $z=Az$. Finally, we obtain $z=Az=Bz$. So z is a common fixed point of A and B.

COROLLARY 1. Taking $B=I$ in Theorem 2, one gets a unique fixed point of A.

REMARK 2. Theorem 7 in [5] and Theorem 3.1 in [44] are special cases of Theorem 2.

3 An Application in Dynamical Programming

In this section, we will use Theorem 2 in order to show the existence and uniqueness of solutions to the following functional equations:

$$\begin{cases} w(a) = \sup_{b\in E}\{h(a,b) + H(a,b,z(G(a,b)))\}, \\ z(a) = \sup_{b\in E}\{h(a,b) + H(a,b,w(G(a,b)))\}, \end{cases} \tag{14}$$

where E is a state space, S is a decision space, $a \in S$, $b \in E$, $w, z : S \to \mathbb{R}$, $h : S \times E \to \mathbb{R}, G : S \times E \to S$ and $H : S \times E \times \mathbb{R} \to \mathbb{R}$ are considered operators (see also [20, 21, 36, 44]).

We denote by $B(S)$ the set of all bounded functionals on S. Define also $\|.\|_\infty$ as

$$\|v\|_\infty = \sup_{x\in S} |v(x)|, \forall v \in B(S).$$

REMARK 3 ([44]). $(B(S), \|.\|_\infty)$ is a Banach space, where the distance function on $B(S)$ is defined as

$$d_\infty(T_1, T_2) = \sup_{x\in S} |T_1(x) - T_2(x)|, \quad \forall T_1, T_2 \in B(S).$$

LEMMA 2 ([5]). For all $T_1, T_2 \in B(S)$, we have

$$|\sup_{x\in S} T_1(x) - \sup_{x\in S} T_2(x)| \le \sup_{x\in S} |T_1(x) - T_2(x)|. \tag{15}$$

PROPOSITION 1 ([44]). Suppose that h, $H(.,.,0)$, $H(.,.,1) : S \times E \to \mathbb{R}$ are three bounded functionals. Suppose also there exists $C \ge 0$ such that

$$|H(a,b,t_1) - H(a,b,t_2)| \le C|t_1 - t_2|,\ \forall a \in S,\ b \in E \text{ and } \ t_1, t_2 \in \mathbb{R}. \tag{16}$$

Consider the operator $O : B(S) \to B(S)$ defined as

$$(Ow)(a) = \sup_{b\in E}\{h(a,b) + H(a,b,z(G(a,b))\}, \forall a \in S, \tag{17}$$

where

$$z(a) = \sup_{b\in E}\{h(a,b) + H(a,b,w(G(a,b)))\},\ \forall a \in S,$$

for $w \in B(S)$ and $b \in E$. Then O is well defined.

THEOREM 3. Consider the assumptions of Proposition 1. Assume in addition that

$$\begin{aligned} &\psi\left(d_\infty(H(a,b,z(w_1(G(a,b))), H(a,b,z(w_2(G(a,b))))\right) \\ \le\ &\mathcal{F}\left(\psi(\mathrm{M}(\mathrm{w}_1,\mathrm{w}_2)), \phi(\mathrm{M}(\mathrm{w}_1,\mathrm{w}_2))\right) \end{aligned} \tag{18}$$

where $\psi, \phi \in \Psi$, $\mathcal{F} \in C$ and

$$\begin{aligned} M(w_1, w_2) &= \max\left\{ d_\infty(zw_1, zw_2), \frac{d_\infty(zw_1, Ow_1)(d_\infty(zw_2, Ow_2)+1)}{1+d_\infty(zw_1, zw_2)}, \right. \\ &\left. \frac{d_\infty(zw_2, Ow_2)(d_\infty(zw_1, Ow_1)+1)}{1+d_\infty(zw_1, zw_2)} \right\}, \end{aligned}$$

for all $w_1, w_2 \in B(S)$, $a \in S$ and $b \in E$. Then the functional equations (14) have a unique common solution $w_0 \in B(S)$.

PROOF. First, we show that the mappings in (17) satisfy the condition (4). Indeed, by using Lemma **??**, for all $w_1, w_2 \in B(S)$, we have

$$\begin{aligned} \psi\left(d_\infty(Ow_1, Ow_2)\right) &\leq \psi(\sup_{b\in E} |H(a, b, z(w_1) - H(a, b, z(w_2)|) \\ &\leq \mathcal{F}\left(\psi(\mathrm{M}(\mathrm{w}_1, \mathrm{w}_2)), \phi(\mathrm{M}(\mathrm{w}_1, \mathrm{w}_2))\right). \end{aligned}$$

So all conditions of Theorem 2 are satisfied, hence the system (14) has a unique solution.

COROLLARY 2 ([44]). Consider the assumptions of Proposition 1. Assume in addition that

$$\begin{aligned} &\psi\left(d(F(a, b, z(w_1(G(a, b))), F(a, b, z(w_2(G(a, b))))\right) \\ \leq\ &\varphi(M(w_1, w_2)) \end{aligned} \tag{19}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\varphi(0) = 0$ and $\varphi(t) < t$ for $t > 0$, and

$$\begin{aligned} M(w_1, w_2) &= \max\left\{ d_\infty(zw_1, zw_2), \frac{d_\infty(zw_1, Ow_1)(d_\infty(zw_2, Ow_2)+1)}{1+d_\infty(zw_1, zw_2)}, \right. \\ &\left. \frac{d_\infty(zw_2, Ow_2)(d_\infty(zw_1, Ow_1)+1)}{1+d_\infty(zw_1, zw_2)} \right\}, \end{aligned}$$

for all $w_1, w_2 \in B(S)$, $a \in S$ and $b \in E$. Then the functional equations (14) have a unique common solution $w_0 \in B(S)$.

PROOF. It suffices to choose $F(s, t) = \varphi(s)$ in Theorem 3.

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