

# Nontrivial Solutions Of Second-Order Nonlinear Boundary Value Problems\*

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## Abstract

In this paper, some sufficient conditions for the existence of nontrivial solutions to Dirichlet boundary value problems of a class of nonlinear second order differential equations are given. Our approach is based on fixed point theorem for Leray-Schauder nonlinear alternative.

## 1 Introduction

In this paper, we study the existence and uniqueness of nontrivial solution for the following second-order boundary value problems (BVP):

$$\begin{cases} u'' = \lambda f(t, u), & 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases} \quad (1)$$

where  $\lambda > 0$  is a parameter,  $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous and  $\mathbb{R} = (-\infty, +\infty)$ .

The second-order BVPs arise in the study of natural problems. This problem was initiated by Choi [1]. By shooting method, he obtained the following results:

Let  $f(t, u) = g(t)e^u$ . Assume that  $g \in C^1(0, 1)$ ,  $g(t) > 0$  in  $(0, 1)$  and  $g(t)$  can be singular at  $t = 0$ , but is at most  $O(\frac{1}{t^{2-\delta}})$  as  $t \rightarrow 0^+$  for some  $\delta > 0$ . Then there exists a  $\lambda^* > 0$  such that (1) has a positive solution for  $0 < \lambda < \lambda^*$ , while for  $\lambda > \lambda^*$ , there is no solutions.

Wong [2] later gave the similar results when  $f(t, u) = g(t)h(u)$  where  $p(t) > 0$  is singular at 0 and at most  $O(\frac{1}{t^\alpha})$  as  $t \rightarrow 0^+$  for some  $\alpha \in [0, 2)$ ;  $h$  is locally Lipschitz continuous and is a increasing function. Ha and Lee [3], Agarwal et al. [4] improved the above results: when  $0 < f(t, u) \leq M_\eta p(t)$  and  $p(t) \in C([0, 1], [0, \infty))$ ,  $M_\eta$  is a positive constant for each given  $\eta > 0$  and satisfies  $\int_0^1 tp(t)dt < \infty$ .

Very recently, Young [6] investigated the existence of solutions to the nonlinear, singular second order Bohr boundary value problems by implementing an innovative differential inequality, Lyapunov functions and topological techniques; the authors [7–11] studied linearized domain decomposition approaches, the Quintic B-spline method

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and the iterative method for nonlinear boundary value problems; the authors [12–15] considered the existence results for second-order boundary-value problems with variable exponents, with Neumann character, with periodicity and with differential systems, respectively.

However, most of the above mentioned work is concerned only with the nonlinearity having nonnegative values and being nonsingular. And their methods doesn't work otherwise. Motivated by the work and the reasoning mentioned above, in this paper, without any monotone-type and nonnegative assumption, we obtain several sufficient conditions of the existence and uniqueness of nontrivial solution of BVPs when  $\lambda$  is in some interval. Our results are new and different from those of [1–5,13,15]. We do not use the method of lower and upper solutions which was essential for the technique used in [1–5] and we do not need any monotonicity and nonnegative assumptions on  $f$ .

## 2 Preliminaries and Lemmas

We put  $X = C[0, 1]$  endowed with the ordering  $x \leq y$  if  $x(t) \leq y(t)$  for all  $t \in [0, 1]$ , and  $\|u\| = \max_{t \in [0, 1]} |u(t)|$  is defined as usual by maximum norm. It follows that  $(X, \|\cdot\|)$  is a Banach space.

LEMMA 2.1 ([1]). Let  $y(t) \in X$ , then the BVP

$$\begin{cases} u'' - y(t) = 0, & 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases}$$

has a unique solution

$$u(t) = \int_0^1 G(t, s)y(s)ds,$$

where

$$G(t, s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1, \end{cases}$$

is the Green's function of the BVP

$$\begin{cases} u'' = 0 & 0 < t < 1, \\ u(0) = u(1) = 0. \end{cases}$$

REMARK 2.1. It is obvious that the Green's function  $G(t, s)$  is continuous and  $G(t, s) \geq 0$  for any  $0 \leq t, s \leq 1$ .

In addition, we also have

$$\max_{0 \leq t, s \leq 1} G(t, s) \leq \frac{1}{2}.$$

In fact, since

$$\begin{aligned} G(t, s) &= \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1, \end{cases} \\ &\leq \begin{cases} t - \frac{1}{2}t^2, & 0 \leq t \leq s \leq 1, \\ \frac{1}{2}s^2, & 0 \leq s \leq t \leq 1, \end{cases} \\ &\leq \frac{1}{2}, \end{aligned}$$

we see that  $\max_{0 \leq t, s \leq 1} G(t, s) \leq \frac{1}{2}$ .

LEMMA 2.2 ([16]). Let  $X$  be a real Banach space,  $\Omega$  be a bounded open subset of  $X$ ,  $0 \in \Omega$  and  $T : \overline{\Omega} \rightarrow X$  be a completely continuous operator. Then, either there exist  $x \in \partial\Omega$ ,  $\mu > 1$  such that  $T(x) = \mu x$ , or there exists a fixed point  $x^* \in \overline{\Omega}$ .

### 3 Main Results

In this section, we present and prove our main results.

THEOREM 3.1. Suppose that  $f(t, 0) \not\equiv 0$ , there exist nonnegative functions  $p, r \in L[0, 1]$  such that

$$|f(t, u)| \leq p(t)|u| + r(t), \quad a.e. (t, u) \in [0, 1] \times \mathbb{R}, \quad (2)$$

and there exists  $t_0 \in [0, 1]$  such that  $p(t_0) \neq 0$ . Then there exists a constant  $\lambda^* > 0$  such that for any  $0 < \lambda \leq \lambda^*$ , the BVP (1) has at least one nontrivial solution  $u^* \in C[0, 1]$ .

PROOF. By Lemma 2.1, problem (1) has a solution  $u = u(t)$  if and only if  $u$  solves the operator equation

$$u(t) = Tu(t) := \lambda \int_0^1 G(t, s)f(s, u(s))ds$$

in  $X$ . So we need to seek a fixed point of  $T$  in  $X$ . By Ascoli-Arzelà Theorem, it is well known that this operator  $T : X \rightarrow X$  is a completely continuous operator.

Since  $|f(t, 0)| \leq r(t)$ , a.e.  $t \in [0, 1]$ , we know  $\int_0^1 r(t)dt > 0$ . From  $p(t_0) \neq 0$ , we easily obtain  $\int_0^1 p(s)ds > 0$ . Let

$$m = \frac{\int_0^1 r(s)ds}{\int_0^1 p(s)ds} \text{ and } \Omega = \{u \in C[0, 1] : \|u\| \leq m\}.$$

Suppose  $u \in \partial\Omega$  and  $\mu > 1$  such that  $Tu = \mu u$ . Then

$$\mu m = \mu \|u\| = \|Tu\|.$$

Thus

$$\begin{aligned} \|Tu\| &= \max_{0 \leq t \leq 1} |Tu(t)| \leq \max_{0 \leq t \leq 1} \lambda \int_0^1 G(t, s) |f(s, u(s))| ds \\ &\leq \frac{\lambda}{2} \int_0^1 |f(s, u(s))| ds \\ &\leq \frac{\lambda}{2} \int_0^1 [p(s)|u(s)| + r(s)] ds \\ &\leq \frac{\lambda}{2} \int_0^1 p(s) ds \|u\| + \frac{\lambda}{2} \int_0^1 r(s) ds. \end{aligned}$$

Choose  $\lambda^* = \left(\int_0^1 p(s) ds\right)^{-1}$ . Then when  $0 < \lambda \leq \lambda^*$ , we have

$$\mu \|u\| \leq \frac{1}{2} \|u\| + \frac{\int_0^1 r(s) ds}{2 \int_0^1 p(s) ds}.$$

Consequently,

$$\mu \leq \frac{1}{2} + \frac{\int_0^1 r(s) ds}{2m \int_0^1 p(s) ds} = 1.$$

This contradicts  $\mu > 1$ , by Lemma 2.2,  $T$  has a fixed point  $u^* \in \bar{\Omega}$ , since  $f(t, 0) \not\equiv 0$ , we see that when  $0 < \lambda \leq \lambda^*$ , the BVP (1) has a nontrivial solution  $u^* \in C[0, 1]$ . This completes the proof.

If we use the following stronger condition than (2), we can obtain the following Corollary.

**COROLLARY 3.1.** Suppose that  $f(t, 0) \not\equiv 0$ , and there exist a nonnegative function  $p \in L[0, 1]$  such that

$$|f(t, u_1) - f(t, u_2)| \leq p(t)|u_1 - u_2|, \quad a.e. (t, u_i) \in [0, 1] \times \mathbb{R} (i = 1, 2),$$

and there exists  $t_0 \in [0, 1]$  such that  $p(t_0) \neq 0$ . Then there exists a constant  $\lambda^* > 0$  such that for any  $0 < \lambda \leq \lambda^*$ , the BVP(1) has unique nontrivial solution  $u^* \in C[0, 1]$ .

**PROOF.** In fact, if  $u_2 = 0$ , then we have  $|f(t, u)| \leq p(t)|u| + f(t, 0)$ , *a.e.*  $(t, u) \in [0, 1] \times \mathbb{R}$ . From Theorem 3.1, we know that the BVP (1) has a nontrivial solution  $u^* \in C[0, 1]$ . But in this case, we prefer to concentrate uniqueness of nontrivial solution for the BVP (1). Let  $T$  be given in Theorem 3.1. We shall show that  $T$  is a contraction.

In fact,

$$\begin{aligned} \|Tu_1 - Tu_2\| &= \max_{0 \leq t \leq 1} \lambda \left| \int_0^1 G(t,s)(f(t,u_1(s)) - f(t,u_2(s)))ds \right| \\ &\leq \frac{1}{2} \lambda \int_0^1 |f(t,u_1(s)) - f(t,u_2(s))| ds \\ &\leq \frac{1}{2} \lambda \int_0^1 [p(s)|u_1(s) - u_2(s)|] ds \\ &\leq \frac{1}{2} \lambda \int_0^1 p(s) ds \|u_1 - u_2\|. \end{aligned}$$

If we choose  $\lambda^* = (\int_0^1 p(s) ds)^{-1}$ , then

$$\|Tu_1 - Tu_2\| \leq \frac{1}{2} \|u_1 - u_2\|.$$

So  $T$  is indeed a contraction. Finally, we use the Banach fixed point theorem to deduce the existence of a unique solution to the *BVP* (1).

**THEOREM 3.2.** Suppose that  $f(t,0) \neq 0$  and

$$0 \leq M = \limsup_{|u| \rightarrow +\infty} \max_{0 \leq t \leq 1} \frac{|f(t,u)|}{|u|} < +\infty. \quad (3)$$

Then there exists a constant  $\lambda^* > 0$  such that for any  $0 < \lambda \leq \lambda^*$ , the *BVP*(1) has at least one nontrivial solution  $u^* \in C[0,1]$ .

**PROOF.** Let  $\varepsilon > 0$  such that  $M + 1 - \varepsilon > 0$ . By (3), there exists  $H > 0$  such that

$$|f(t,u)| \leq (M + 1 - \varepsilon)|u|, \text{ for } |u| \geq H \text{ and } 0 \leq t \leq 1.$$

Let  $N = \max_{t \in [0,1], |u| \leq H} |f(t,u)|$ . Then for any  $(t,u) \in [0,1] \times \mathbb{R}$ , we have

$$|f(t,u)| \leq (M + 1 - \varepsilon)|u| + N.$$

From Theorem 3.1, we know that the *BVP*(1) has at least one nontrivial solution  $u^* \in C[0,1]$ .

## 4 Examples

In this section, we give two examples.

**EXAMPLE 4.1.** Consider the following second-order boundary value problem(*BVP*):

$$\begin{cases} y'' = \lambda \frac{y}{2}, & 0 < t < 1, \\ y(0) = y(1) = 0. \end{cases} \quad (4)$$

Choose  $\lambda^* = 2$ , then by Corollary 3.1, the *BVP* (4) has a unique nontrivial solution  $y^* \in C[0, 1]$  for any  $\lambda \in (0, 2]$ .

EXAMPLE 4.2. Consider the following second-order boundary value problem(*BVP*):

$$\begin{cases} y'' = \lambda \left( \frac{yt \sin t}{t^2+1} + t(1+t) \right), & 0 < t < 1, \\ y(0) = y(1) = 0. \end{cases} \quad (5)$$

Choose  $\lambda^* = \frac{2}{\ln 2}$ , then by Theorem 3.1, the *BVP* (5) has a nontrivial solution  $y^* \in C[0, 1]$  for any  $\lambda \in (0, \frac{2}{\ln 2}]$ .

## 5 Conclusion

In this paper, without any monotone-type and nonnegative assumptions, several sufficient conditions of the existence and uniqueness of nontrivial solutions of BVPs are obtained when  $\lambda$  is in some interval. However, the nonnegative assumptions on  $f$  is essential in the references [1, 2]. Therefore, our results extend some existing ones.

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