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Hyers-Ulam Stability And Exponential Dichotomy Of Discrete Semigroup*

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Abstract

In this manuscript we prove that the discrete semigroup $\mathcal{T} = \{T(p) : p \in \mathbb{Z}_+\}$ is Hyers–Ulam stable if and only if it has uniform exponential dichotomy. In fact, we prove that if the discrete semigroup $\mathcal{T}$ possesses uniform exponential dichotomy then for each $f_q \in P_0(\mathbb{N}, \mathcal{B})$, the discrete time equation $\psi_{p+1} = T(1)\psi_p + f_{p+1}$ have bounded solution, starting by a unique $x \in Ker\mathbb{P}_r$. Consequently the semigroup $\mathcal{T}$ will be Hyers–Ulam stable and vice versa.

1 Introduction

In 1940, Ulam presented some problems concerning the stability of functional equations, [17]. He asked: let $\mathcal{G}_1$ be a group and $(\mathcal{G}_2, d)$ be a metric group. For a given $\varepsilon > 0$, does there exists $\delta > 0$ such that if $f : \mathcal{G}_1 \to \mathcal{G}_2$ satisfies

$$d(f(xy), f(x)f(y)) < \delta, \quad \text{for all } x, y \in \mathcal{G}_1,$$

then there exists a homomorphism $T : \mathcal{G}_1 \to \mathcal{G}_2$ such that

$$d(f(x), T(x)) \leq \varepsilon \quad \text{for all } x \in \mathcal{G}_1?$$

In the next year, Hyers [6] answered Ulam's question, partially, by considering $\mathcal{G}_1$ and $\mathcal{G}_2$ Banach spaces. Afterwards, solution of such problem is known as Ulam–Hyers stability. In 1978, Rassias [15], provided a remarkable generalization of the Ulam–Hyers stability of mappings by considering variables. Obloza [14] was the first one, who extended the Hyers–Ulam ($\mathcal{HU}$) stability concept to differential equations. Alsina and Ger [1], studied the mentioned concept of stability for differential equation of the form $\dot{y} = y$. Since then, different researchers studied $\mathcal{HU}$ stability with different approaches, we refer the reader to [4, 7, 9, 10, 11, 13, 16, 18, 19, 20, 21, 22, 23, 24].

The notion of exponential stability and dichotomy plays a central role in the theory of dynamical systems. Development has been made to analyze the exponential stability and dichotomy of evolution equations with different approaches, see [2, 3, 5, 8, 12, 25, 26, 27].

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In this manuscript we relate the uniform exponential dichotomy and $\mathcal{HU}$ stability of exponentially bounded semigroup $\mathcal{T}$. Consider the following time-dependent discrete system

$$\Upsilon_{p+1} = T(1)\Upsilon_p, \quad \forall \;\; p \in \mathbb{Z}_+, \tag{1}$$

where $T(p)$ represent discrete semigroup on the Banach space $\mathcal{B}$.

The system (1) is said to be $\mathcal{HU}$ stable, if there exists a real constant $\mathcal{K} > 0$ such that for each $\epsilon > 0$ and each solution ψ_p of

$$\|\psi_{p+1} - \; T(1)\psi_p\| \leq \varepsilon, \quad \forall \;\; p \in \mathbb{Z}_+.$$

there exists a solution Υ_p of (1) such that

$$\sup_{p\in\mathbb{Z}_+} \|\psi_p - \Upsilon_p\| \leq \mathcal{K}\varepsilon.$$

We show that the family $\{T(p) : p \in \mathbb{Z}_+\}$ of discrete semigroup of operators is $\mathcal{HU}$ stable if and only if it is uniformly exponentially dichotomic.

The paper is arranged as follows: in section 2, we present some helpful notations and definitions regarding the family of one parameter discrete semigroup of operators. In section 3, we prove a result related to the exponential dichotomy of discrete semigroup which is helpful in the proof of our main result.

2 Notations and Preliminaries

By $\mathbb{N}$, $\mathbb{Z}_+$, $\mathcal{B}$ we denote the set of all natural numbers, all positive integers and Banach space of all bounded linear operators, respectively, the norm on $\mathcal{B}$ will be denoted by $\|\cdot\|$ and $\mathbb{B}(\mathcal{B})$ denote the Banach algebra of all bounded linear operators on $\mathcal{B}$. We define the following spaces: By $L^\infty(\mathbb{N}, \mathcal{B})$ we denote the space such that if $f \in L^\infty(\mathbb{N}, \mathcal{B})$, then $\sup_{p\in N} \|f(p)\| < \infty$ and $P(\mathbb{N}, \mathcal{B})$ denotes the space such that if $f \in P(\mathbb{N}, \mathcal{B})$ then $\lim\limits_{p\to\infty} f(p) = 0$. $P_0(\mathbb{N}, \mathcal{B})$ denotes the space such that if $f \in P(\mathbb{N}, \mathcal{B})$ then $f(0) = 0$. It is obvious that the defined spaces are Banach spaces. The spectrum of a given operator is denoted by $\sigma(.)$.

DEFINITION 1. The one parameter family $\mathcal{T} = \{T(p)\}_{p\geq 0} \subset \mathbb{B}(\mathcal{B})$ is said to be semigroup of operators if $T(0) = I$ and $T(t + s) = T(t)T(s)$, for all $t, s \in \mathbb{Z}_+$.

DEFINITION 2. The one parameter family $\mathcal{T} = \{T(p)\}_{p\geq 0} \subset \mathbb{B}(\mathcal{B})$ will be exponentially bounded if there exist $M \geq 1$ and $\xi > 0$ such that $\|T(p)\| \leq Me^{\xi p}$ for all $p \geq 0$.

DEFINITION 3. If there exists a projection $\mathbb{P}_r \in \mathbb{B}(\mathcal{B})$ and $M \geq 1$, $v > 0$, then the one parameter family $\mathcal{T} = \{T(p)\}_{p\geq 0}$ is uniformly exponentially dichotomic if the following holds:

1. $T(p)\mathbb{P}_r = \mathbb{P}_r T(p)$, for all $p \geq 0$;

2. $T(p)_{|} : Ker\mathbb{P}_r \to Ker\mathbb{P}_r$ is an isomorphism, for all $p \geq 0$;
3. $\|T(p)x\| \leq Me^{-vp}\|x\|$, for all $x \in Im\mathbb{P}_r$ and all $p \geq 0$;
4. $\|T(p)x\| \geq \frac{1}{M}e^{vp}\|x\|$, for all $x \in Ker\mathbb{P}_r$ and all $p \geq 0$.

DEFINITION 4. Let $\mathcal{T} = \{T(p)\}_{p\geq 0}$ be the one parameter family of operators on the Banach space $\mathcal{B}$ and let $\mathbb{Y} \subset \mathcal{B}$. $\mathbb{Y}$ is said to be $\mathcal{T}$-invariant if $T(p)\mathbb{Y} \subset \mathbb{Y}$, for all $p \geq 0$.

Consider the discrete time equation:

$$\psi_{p+1} = T(1)\psi_p + f_{p+1}, \tag{2}$$

where $p \in \mathbb{N}$, $\psi \in L^\infty(\mathbb{N}, \mathcal{B})$ and $f \in P_0(\mathbb{N}, \mathcal{B})$. The solution of (2) with initial condition $\psi_0 = x_0$ is given by:

$$\psi_p = T(p)x_0 + \sum_{q=0}^{p} T(p-q)f_q. \tag{3}$$

Let $\mathcal{B}_1 = \{x \in \mathcal{B} : \sup_{p\geq 0} \|T(p)x\| < \infty\}$. $\mathcal{T}$ will denote an exponentially bounded discrete semigroup, $\mathcal{B}_2$ will denote $\mathcal{T}$-invariant(closed) complement of the closed linear subspace $\mathcal{B}_1$ such that $\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2$, and the corresponding decomposition, due to $\mathbb{P}_r$, will be denoted by $Im\mathbb{P}_r = \mathcal{B}_1$ and $Ker\mathbb{P}_r = \mathcal{B}_2$.
The solution (3) can be written as:

$$\psi_p = \sum_{q=0}^{p} T(p-q)Im\mathbb{P}_r f_q - \sum_{q=p+1}^{\infty} T(p-q)^{-1}Ker\mathbb{P}_r f_q.$$

REMARK 1. $T(p)\mathbb{P}_r = \mathbb{P}_r T(p)$, for all $p \geq 0$.

3 Exponential Dichotomy of Discrete Semigroup

In this section we prove that if a discrete semigroup posses uniform exponential dichotomy then the solution of non–homogenous discrete time equation will be bounded, starting by a unique $x \in Ker\mathbb{P}_r$.

THEOREM 1. If the semigroup $\mathcal{T} = \{T(p)\}_{p\geq 0}$ is uniformly exponentially dichotomic then for each $f_q \in P_0(\mathbb{N}, \mathcal{B})$ there exists $\psi_p \in L^\infty(\mathbb{N}, \mathcal{B})$ bounded solution of discrete-time equation (2), starting by a unique $x \in Ker\mathbb{P}_r$.

PROOF. Let the semigroup $\mathcal{T} = \{T(p)\}_{p\geq 0}$ is uniformly exponentially dichotomic. The solution of the discrete-time equation (2) may be written as

$$\psi_p = \sum_{q=0}^{p} T(p-q)Im\mathbb{P}_r f_q - \sum_{q=p+1}^{\infty} T(p-q)^{-1}Ker\mathbb{P}_r f_q. \tag{4}$$

As

$$\begin{aligned}\sup_{p\geq 0}\|\sum_{q=0}^{p}T(p-q)Im\mathbb{P}_r f_q\| &\leq \sum_{q=0}^{p}Me^{-v(p-q)}\sup_{p\geq 0}\|f_q\| \\ &= \frac{M(1-e^{-vp})}{e^v-1}\sup_{p\geq 0}\|f_q\| \\ &\leq \frac{M}{e^v-1}\sup_{p\geq 0}\|f_q\|,\end{aligned}$$

and

$$\begin{aligned}\sup_{p\geq 0}\|\sum_{q=p}^{\infty}T(q-p)Ker\mathbb{P}_r f_q\| &\geq \sum_{q=p}^{\infty}\frac{1}{M}e^{v(q-p)}\sup_{p\geq 0}\|f_q\| \\ &= \frac{1}{M(1-e^{-v})}\sup_{p\geq 0}\|f_q\|.\end{aligned}$$

Equivalently,

$$\sup_{p\geq 0}\|\sum_{q=p}^{\infty}T(q-p)^{-1}Ker\mathbb{P}_r f_q\| \leq M(1-e^{-v})\sup_{p\geq 0}\|f_q\|.$$

Thus (4) implies

$$\sup_{p\geq 0}\|\psi_p\| \leq \Big(\frac{M}{e^v-1}+Me^{v(q-p)}\Big)\sup_{p\geq 0}\|f_q\|.$$

Let there exist two bounded solutions ψ_{1p} and ψ_{2p}, of the discrete-time equation (2) having their start in $Ker\mathbb{P}_r$. Then

$$\psi_{1p} = T(p)x_1 + \sum_{q=0}^{p}T(p-q)f_q, \quad x_1 \in Ker\mathbb{P}_r$$

and

$$\psi_{2p} = T(p)x_2 + \sum_{q=0}^{p}T(p-q)f_q, \quad x_2 \in Ker\mathbb{P}_r.$$

The difference $\psi_{1p}-\psi_{2p} = T(p)(x_1-x_2)$ is clearly bounded. Since $x_1, x_2 \in Ker\mathbb{P}_r$ so $x_1 - x_2 \in Ker\mathbb{P}_r$. Therefore $x_1 = x_2$.

4 $\mathcal{HU}$ Stability and Exponential Dichotomy for Discrete-Time Equation

Consider a discrete semigroup $\mathcal{T} = \{T(p)\}_{p\geq 0}$, which appears in the solutions of the discrete system $\Upsilon_{p+1} = T(1)\Upsilon_p$ and let ψ_p is the approximate solution of the considered system, then $\psi_{p+1} \approx T(1)\psi_p$. Let f_q is the force term then ψ_p is an exact solution of (2) corresponding to the forced term f_q, bounded by ε. Therefore, we have the following definition.

DEFINITION 5. The semigroup $\mathcal{T} = \{T(p)\}_{p\geq 0}$ is said to be $\mathcal{HU}$ stable if for any ε, the inequality $\|f_q\| \leq \varepsilon$ holds and there exists an exact solution Υ_p of $\Upsilon_{p+1} = T(1)\Upsilon_p$ and $\mathcal{K} \geq 0$ such that

$$\|\psi_p - \Upsilon_p\| \leq \mathcal{K}\varepsilon.$$

THEOREM 2. The semigroup $\mathcal{T} = \{T(p)\}_{p\geq 0}$ is $\mathcal{HU}$ stable if and only if it is uniformly exponentially dichotomic.

PROOF. ***Sufficiency***: Suppose on contrary that $\mathcal{T} = \{T(p)\}_{p\geq 0}$ is not dichotomic. Then there exists $\lambda \in \sigma(T)$ with $|\lambda| = 1$ and $y \neq 0$ such that $T(0)y = \lambda y$.
In general

$$T(p)y = \lambda^p y, \qquad \forall \;\; p \in \mathbb{N}.$$

Suppose that $f_q = \lambda^q y$ for all $q \neq 0$ and $f_q = 0$ for $q = 0$.
Let $\varepsilon \geq 0$ and ψ_p is the approximate solution of (1) such that $\sup\limits_{p\geq 0}\|\psi_{p+1} - T(1)\psi_p\| = \sup\limits_{p\geq 0}\|f(p+1)\|$, $\psi(0) = x_0$ with $\sup\limits_{q\geq 0}\|f_q\| \leq \varepsilon$ and let Υ_p is the exact solution of $\Upsilon_{p+1} = T(1)\Upsilon_p$.
As we assumed that the one parameter family $\mathcal{T} = \{T(p)\}_{p\geq 0}$ is $\mathcal{HU}$ stable. Thus,

$$\sup_{p\geq 0}\|\psi_p - \Upsilon_p\|$$

is bounded by $\mathcal{K}\varepsilon$.
So

$$\begin{aligned}
\sup_{p\geq 0}\|\psi_p - \Upsilon_p\| &= \sup_{p\geq 0}\|T(p)x_0 + \sum_{q=0}^{p} T(p-q)f_q - T(p)x_0\| \\
&= \sup_{p\geq 0}\|\sum_{q=0}^{p} T(p-q)f_q\| \\
&= \sup_{p\geq 0}\|\sum_{q=1}^{p} T(p-q)\lambda^q y\| \\
&= \sup_{p\geq 0}\|\sum_{q=1}^{p} T(p-q)\Upsilon_q \; y\| \\
&= \sup_{p\geq 0}\|\sum_{q=1}^{p} T(p)y\| \\
&= \sup_{p\geq 0}\|n\lambda^p y\|,
\end{aligned}$$

which is clearly unbounded. The contradiction arises due to our wrong supposition. So $\mathcal{T} = \{T(p)\}_{p\geq 0}$ is dichotomic.

Necessity: Let $\mathbb{P}_r$ is given by Definition 3. and $\mathcal{T}$ be exponentially dichotomic, then equation(2) have unique bounded solution.
Let $f \in P_0(\mathbb{N}, \mathcal{B})$, with $\sup_{p\geq 0}\|f_q\| \leq \varepsilon$, and let Υ_p is the exact solution of $\Upsilon_{p+1} = T(1)\Upsilon_p$ and ψ_p is the approximate solution which is an exact solution of equation(2) with $\psi(0) = x_0$, i.e.

$$\psi_p = T(p)x_0 + \sum_{q=0}^{p} T(p-q)f_q.$$

Then

$$\begin{aligned}
\sup_{p\geq 0}\|\psi_p - \Upsilon_p\| &= \sup_{p\geq 0}\|T(p)x_0 + \sum_{q=0}^{p} T(p-q)f_q - T(p)x_0\| \\
&= \sup_{p\geq 0}\|\sum_{q=0}^{p} T(p-q)f_q\| \\
&= \sup_{p\geq 0}\|\sum_{q=0}^{p} T(p-q)Ker\mathbb{P}_r f_q - \sum_{q=p+1}^{\infty} T(q-p))^{-1} Im\mathbb{P}_r f_q\| \\
&\leq \Big(\frac{M}{e^v - 1} + M(e^v - 1)\Big)\varepsilon \\
&= \mathcal{K}\varepsilon, \qquad \text{by choosing } \mathcal{K} = \Big(\frac{M}{e^v - 1} + M(e^v - 1)\Big).
\end{aligned}$$

Thus $\sup_{p\geq 0}\|\psi_p - \Upsilon_p\| \leq \mathcal{K}\varepsilon$. Which implies that the discrete semigroup $\mathcal{T} = \{T(p)\}_{p\geq 0}$ is $\mathcal{HU}$ stable.

Conclusion

We proved that the system $\Upsilon_{p+1} = T(1)\Upsilon_p, \ \forall \ p \in \mathbb{Z}_+$ is Hyers–Ulam stable if and only if it is dichotomic.

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