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Common Fixed Point For Multivalued (ψ, θ, G)-Contraction Type Maps In Metric Spaces With A Graph Structure*

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Abstract

In this work, we introduce the notion of a common (ψ, θ, G)-contraction multivalued mapping in order to establish some new common fixed point theorems for these classes of mappings in complete metric spaces endowed with a graph. An example of application illustrates the main existence theorem. Our results generalize some recent known results.

1 Introduction and Preliminaries

Since the proof of Banach's fixed contraction principle [2] in 1922, many research works have considered different kinds of generalizations. Among them, the classical multivalued version was established by Covitz and Nadler [12] in 1969 using the Hausdorff-Pompeiu metric in a complete metric space.

In 2008, Jachymski [8] provided a new approach in metric fixed point theory by replacing the order structure with a graph structure on a metric space. He introduced the concept of G-contraction, where the contraction condition is only verified on the edge of the graph. Subsequently, many authors have extended the Banach G-contraction in different ways (we refer to [1], [3], [13], [15], [16], and references therein).

Recently, Jleli and Samet [10] introduced another definition called θ-contraction and proved a fixed point result as a generalization of the Banach contraction principle. Their result has then been extended by many authors (see, e.g., [6], [7], [9], [11], [17]). Given a metric space (X, d), a mapping $T : X \longrightarrow X$ is a θ-contraction if there exist $\theta \in \Theta$ and $k \in (0, 1)$ such that:

$$\theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k,$$

for all $x, y \in X$ with $d(Tx, Ty) > 0$. Here Θ refers to the set of all functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:

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(Θ_1) θ is non-decreasing,

(Θ_2) for each sequence $(t_n)_n \subset (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0^+$

(Θ_3) there exist $r \in (0,1)$ and $l \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = l$.

The aim of this paper is to prove some common fixed point results for a new class of multi-valued mappings called (ψ, θ, G)-contractions in a metric space endowed with a graph G.

Let us collect some basic notions and primary results we need to develop our existence results. Let (X, d) be a metric space. We denote by $CB(X)$ the family of nonempty closed bounded subsets of X and by $C(X)$ the family of nonempty closed subsets of X. For $A, B \in C(X)$, let

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\},$$

where $d(x, B) = \inf\{d(x, y) : y \in B\}$. H is called the Hausdorff-Pompeiu distance on $C(X)$. This is a metric on $CB(X)$.

A graph G is an ordered pair (V, E), where V is a set and $E \subset V \times V$ is a binary relation on V. Elements of E are called edges and are denoted by $E(G)$ while elements of V, denoted $V(G)$, are called vertices. If a direction is imposed in E, that is the edges are directed, then we get a digraph (directed graph). Hereafter, we assume that G has no parallel edges, i.e., two vertices cannot be connected by more than one edge. Thus, G can be identified with the pair $(V(G), E(G))$. If x and y are vertices of G, then a path in G from x to y of length $k \in \mathbb{N}$ is a finite sequence $(x_n)_n$, $n \in \{0, 1, 2, \ldots, k\}$ of vertices such that $x = x_0, \ldots, x_k = y$ and $(x_{n-1}, x_n) \in E(G)$ for $n \in \{1, 2, \ldots, k\}$. A graph G is connected if there is a path between any two vertices and it is weakly connected if $\widetilde{G}$ is connected, where $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Let G^{-1} be the graph obtained from G by reversing the direction of edges (the conversion of the graph G). We have

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

It is more convenient to treat $\widetilde{G}$ as a directed graph for which the set of edges is symmetric. Then

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}).$$

Let G_x be the component of G consisting of all edges and vertices which are contained in some path in G beginning at x. If G is such that $E(G)$ is symmetric, then for $x \in V(G)$, we may define the equivalence class $[x]_G$ on $V(G)$ by the relation xRy if there is a path in G from x to y. Then $V(G_x) = [x]_G$.

Throughout this paper, (X, d) denotes a metric space, $G = (V(G), E(G))$ is a directed graph without parallel edges with $V(G) = X$ and $(x, x) \notin E(G)$ (the graph does not contain loops). The following condition first appeared in [8]:

PROPERTY (A). For any sequence $(x_n)_n$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$, for all $n \in \mathbb{N}$, then $(x_n, x) \in E(G)$ for $n \in \mathbb{N}$.

With this condition, Jachymski showed that in a complete metric space, a G-contraction mapping f has a fixed point if and only if

$$X_f = \{x \in X : (x, f(x)) \in E(G)\} \neq \emptyset, \tag{1}$$

that is the graph of f intersects the edge of the space graph.

Further to the set Φ, we consider the following classes of functions:

DEFINITION 1. We denote by Ψ the set of functions $\psi : (1, \infty) \to (1, \infty)$ satisfying the following conditions:

(i) ψ is non-decreasing;

(ii) For each sequence $(t_n)_n \subset (1, \infty)$, $\lim\limits_{n\to\infty} \psi(t_n) = 1$ if and only if $\lim\limits_{n\to\infty} t_n = 1$.

Now we give the following definition, extending the definitions of G-contraction [8], θ-contraction [10], and $(G - \psi)$-contraction [4].

DEFINITION 2. Let (X, d) be a metric space endowed with a graph G. Two mappings T_1, $T_2 : X \to C(X)$ are said to be a common (ψ, θ, G)-contraction if for all $x, y \in X$ such that $(x, y) \in E(G)$ and $a \in T_i(x)$, there exists $b \in T_j(y)$ for $i, j \in \{1, 2\}$ with $i \neq j$ such that $(a, b) \in E(G)$ and

$$\psi(\theta(d^p(a, b))) \leq \psi([\theta((M_p(T_i x, T_j y))]^{k(d(x,y))}) + LN_p(T_i x, T_j y),$$

where

$$N_p(T_i x, T_j y) = \min\{d^p(x, T_i(x)), d^p(y, T_j(y)), d^p(y, T_i(x)), d^p(x, T_j(y))\},$$

$$\begin{aligned} M_p(T_i x, T_j y) \quad = \quad \max\Big\{ & d^p(x, y), d^p(x, T_i(x)), d^p(y, T_j(y)), \\ & \frac{d^p(y, T_i(x)) + d^p(x, T_j(y))}{2^p}, \\ & \frac{d^p(x, T_i(x))d^p(y, T_j(y))}{1 + d^p(x, y)}, \frac{d^p(y, T_i(x))d^p(x, T_j(y))}{1 + d^p(x, y)} \Big\}, \end{aligned}$$

$k : (0, +\infty) \to [0, 1)$ satisfies $\limsup\limits_{s\to t^+} k(s) < 1$, for all $t \in [0, +\infty)$, $L \geq 0$, $\theta \in \Theta$, $\psi \in \Psi$, $\psi \circ \theta$ is lower semi continuity, and $1 \leq p < \frac{1}{r}$.

2 Main Result

Our existence results for common fixed points are collected in the following:

THEOREM 1. Let (X, d) be a complete metric space endowed with a directed graph G and suppose that the triple (X, d, G) has the property (A). Let T_1, $T_2 : X \to C(X)$ be a common (ψ, θ, G)-contraction. Then the following statements hold:

(i) For every $x \in X_{T_i}$, $i = 1$ or $i = 2$, the mappings T_1, $T_2\mid_{[x]_{\widetilde{G}}}$ have a common fixed point, where X_{T_i} is as defined in (1).

(ii) If $X_{T_i} \neq \emptyset$, $i = 1$ or $i = 2$, and G is weakly connected, then T_1 and T_2 have a common fixed point in X.

(iii) If $X' = \cup\{[x]_{\widetilde{G}} : x \in X_{T_i}\}$, $i = 1$ or $i = 2$, then T_1, $T_2\mid_{X'}$ have a common fixed point.

(iv) If $Graph(T_i) \subseteq E(G)$, $i = 1$ or $i = 2$, then T_1 and T_2 have a common fixed point.

PROOF.

Claim 1. (a) Construction of a Cauchy sequence $(x_n)_n$. Given $x_0 \in X_{T_i}$ ($i = 1$ or 2), there is an $x_1 \in T_i(x_0)$ such that $(x_0, x_1) \in E(G)$. Since T_1 and T_2 are a common (ψ,θ,G)-contraction, then there exists $x_2 \in T_j(x_1)$ ($j = 2$ or 1) such that $(x_1, x_2) \in E(G)$ and

$$\begin{aligned}\psi(\theta(d^p(x_1,x_2))) &\le \psi([\theta(M_p(T_ix_0,T_jx_1))]^{k(d(x_0,x_1))}) + LN_p(T_ix_0,T_jx_1))\\ &\le \psi([\theta(M_p(T_ix_0,T_jx_1))]^{k(d(x_0,x_1))}) + Ld^p(x_1,T_i(x_0)))\\ &= \psi([\theta(M_p(T_ix_0,T_jx_1))]^{k(d(x_0,x_1))}).\end{aligned}$$

Since $(x_1, x_2) \in E(G)$ and T_1, T_2 are common (ψ,θ,G)-contraction, there exists $x_3 \in T_i(x_2)$ such that $(x_2, x_3) \in E(G)$ and

$$\begin{aligned}\psi(\theta(d^p(x_2,x_3))) &\le \psi([\theta(M_p(T_jx_1,T_ix_2))]^{k(d(x_1,x_2))}) + LN_p(T_jx_1,T_ix_2)\\ &\le \psi([\theta(M_p(T_jx_1,T_ix_2))]^{k(d(x_1,x_2))}) + Ld^p(x_2,T_j(x_1)))\\ &= \psi([\theta(M_p(T_jx_1,T_ix_2))]^{k(d(x_1,x_2))}).\end{aligned}$$

By induction, we construct a sequence $(x_n)_n$ such that $x_{2n+1} \in T_i(x_{2n})$, $x_{2n+2} \in T_j(x_{2n+1})$, $(x_n, x_{n+1}) \in E(G)$, and

$$\psi(\theta(d^p(x_n,x_{n+1}))) \le \begin{cases}\psi([\theta(M_p(T_ix_{n-1},T_jx_n))]^{k(d(x_{n-1},x_n))}) & \text{for odd } n,\\ \psi([\theta(M_p(T_jx_{n-1},T_ix_n))]^{k(d(x_{n-1},x_n))}) & \text{for even } n.\end{cases}$$

Let us distinguish between two cases:

- *Case 1: n is odd.*

$$\begin{aligned}&M_p(T_ix_{2k},T_jx_{2k+1})\\ =\;& \max\Big\{d^p(x_{2k},x_{2k+1}), d^p(x_{2k},T_i(x_{2k})), d^p(x_{2k+1},T_j(x_{2k+1})),\\ &\frac{d^p(x_{2k+1},T_i(x_{2k})) + d^p(x_{2k},T_j(x_{2k+1}))}{2^p}, \frac{d^p(x_{2k},T_i(x_{2k}))d^p(x_{2k+1},T_j(x_{2k+1}))}{1+d^p(x_{2k},x_{2k+1})},\\ &\frac{d^p(x_{2k+1},T_i(x_{2k}))d^p(x_{2k},T_j(x_{2k+1}))}{1+d^p(x_{2k},x_{2k+1})}\Big\}\\ \le\;& \max\Big\{d^p(x_{2k},x_{2k+1}), d^p(x_{2k},x_{2k+1}), d^p(x_{2k+1},x_{2k+2}),\end{aligned}$$

$$\frac{d^p(x_{2k+1},x_{2k+1})+d^p(x_{2k},x_{2k+2})}{2^p},\frac{d^p(x_{2k},x_{2k+1})d^p(x_{2k+1},x_{2k+2})}{1+d^p(x_{2k},x_{2k+1})},$$
$$\left.\frac{d^p(x_{2k+1},x_{2k+1})d^p(x_{2k},x_{2k+2})}{1+d^p(x_{2k},x_{2k+1})}\right\}$$
$$= \max\left\{d^p(x_{2k},x_{2k+1}),d^p(x_{2k+1},x_{2k+2}),\frac{d^p(x_{2k},x_{2k+2})}{2^p}\right\}.$$

Since for all $a,b\geq 0$ and $p\geq 1$, we have

$$(a+b)^p\leq 2^{p-1}(a^p+b^p).$$

Then

$$\begin{aligned}\frac{d^p(x_{2k},x_{2k+2})}{2^p} &\leq \frac{(d(x_{2k},x_{2k+1})+d(x_{2k+1},x_{2k+2}))^p}{2^p}\\ &\leq \frac{d^p(x_{2k},x_{2k+1})+d^p(x_{2k+1},x_{2k+2})}{2}.\end{aligned}$$

We deduce that

$$M_p(T_ix_{2k},T_jx_{2k+1})=\max\left\{d^p(x_{2k},x_{2k+1}),d^p(x_{2k+1},x_{2k+2})\right\}.$$

If $M_p(T_ix_{2k},T_jx_{2k+1})=d^p(x_{2k+1},x_{2k+2})$, then

$$\begin{aligned}\psi(\theta(d^p(x_{2k+1},x_{2k+2}))) &\leq \psi([\theta(d^p(x_{2k+1},x_{2k+2}))]^{k(d(x_{2k},x_{2k+1}))})\\ &< \psi(\theta(d^p(x_{2k+1},x_{2k+2}))),\end{aligned}$$

which is a contradiction. Therefore $M_p(T_ix_{2k},T_jx_{2k+1})=d^p(x_{2k},x_{2k+1})$.

- *Case 2: n is even.* In an analogous manner, we can show that

$$M_p(T_jx_{2k+2},T_ix_{2k+1})=d^p(x_{2k+1},x_{2k+2}).$$

Hence for all $n\in\mathbb{N}$, we have

$$\psi(\theta(d^p(x_n,x_{n+1})))\leq\psi([\theta(d^p(x_{n-1},x_n))]^{k(d(x_{n-1},x_n))}). \tag{2}$$

Since $0<k(d(x_{n-1},x_n))<1$ for all $n\in\mathbb{N}$, then

$$\psi(\theta(d^p(x_n,x_{n+1})))<\psi(\theta(d^p(x_{n-1},x_n))),$$

that is $(d(x_n,x_{n+1}))_n$ is a decreasing sequence of positive numbers. Hence the sequence $(d(x_n,x_{n+1}))_n$ is convergent.

(b) $(x_n)_n$ **is a Cauchy sequence in** (X,d)**.** Since $\limsup_{s\to t^+}k(s)<1$ and the sequence $(d(x_n,x_{n+1}))$ is convergent, then there exists $a\in(0,1)$ and $n_0\in\mathbb{N}$ such that $k(d(x_n,x_{n+1}))<a$, for all $n\geq n_0$. From the inequality in (2), we obtain that for $n\geq n_0$

$$1<\psi(\theta(d^p(x_n,x_{n+1})))\leq\psi([\theta(d^p(x_{n-1},x_n))]^a)\leq\ldots\leq\psi([\theta(d^p(x_{n_0},x_{n_0+1}))]^{a^{n-n_0}}). \tag{3}$$

Taking the limit as $n \to \infty$, we get $\psi(\theta(d^p(x_n, x_{n+1}))) \to 1$. By definition of θ and ψ, $d^p(x_n, x_{n+1}) \to 0$, as $n \to \infty$. By (Θ_3), there exist $r \in (0,1)$ and $l \in (0, +\infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(d^p(x_n, x_{n+1})) - 1}{[d^p(x_n, x_{n+1})]^r} = l.$$

• If $l < \infty$, let $B = \frac{l}{2}$. By the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\Big|\frac{\theta(d^p(x_n, x_{n+1})) - 1}{[d^p(x_n, x_{n+1})]^r} - l\Big| \leq B.$$

This implies that, for all $n \geq n_0$

$$\frac{\theta(d^p(x_n, x_{n+1})) - 1}{[d^p(x_n, x_{n+1})]^r} \geq B.$$

Then, for all $n \geq n_0$

$$n[d^p(x_n, x_{n+1})]^r \leq An[\theta(d^p(x_n, x_{n+1})) - 1],$$

where $A = \frac{1}{B}$.

• If $l = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$

$$\frac{\theta(d^p(x_n, x_{n+1})) - 1}{[d^p(x_n, x_{n+1})]^r} \geq B.$$

Then, for all $n \geq n_0$

$$n[d^p(x_n, x_{n+1})]^r \leq An[\theta(d^p(x_n, x_{n+1})) - 1],$$

where $A = \frac{1}{B}$.
Therefore, in all cases, there exist $A > 0$ and $n_0 \in \mathbb{N}$ such that

$$n[d^p(x_n, x_{n+1})]^r \leq An[\theta(d^p(x_n, x_{n+1})) - 1].$$

By (3) and since ψ is non-decreasing, we obtain

$$n[d^p(x_n, x_{n+1})]^r \leq An\Big[[\theta(d^p(x_{n_0}, x_{n_0+1}))]^{a^{n-n_0}} - 1\Big],$$

for all $n \geq n_0$. Taking the limit as $n \to \infty$, we get

$$\lim_{n\to+\infty} n[d^p(x_n, x_{n+1})]^r = 0.$$

From the definition of the limit, there exists $n_1 \in \mathbb{N}$ such that, for all $n \geq n_1$

$$n[d^p(x_n, x_{n+1})]^r \leq 1.$$

Therefore, for all $n \geq n_1$

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{pr}}}.$$

Hence for each $m, n \in \mathbb{N}$ with $m > n \geq n_1$, we have

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{\frac{1}{pr}}}.$$

As $n, m \to \infty$, we get $d(x_n, x_m) \to 0$ (since $\frac{1}{pr} > 1$), showing that $(x_n)_n$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there exists $x \in X$ such that $\lim\limits_{n \to +\infty} x_n = x$.

(c) x is a common fixed point of T_1 and T_2. By the property (A), $(x_n, x) \in E(G)$, for each $n \in \mathbb{N}$. Again two cases are discussed separately.

• *Case 1: $n = 2k$ is even.* Suppose that $d(x, T_j(x)) > 0$. Since T_i and T_j are common (ψ, θ, G)-contraction, then there exists $y_k \in T_j(x)$ such that for all $k \in \mathbb{N}$

$$\begin{aligned}
\psi(\theta(d^p(x_{2k+1}, y_k))) &\leq \psi([\theta(M_p(T_i x_{2k}, T_j x))]^{k(d(x_{2k},x))}) + LN_p(T_i x_{2k}, T_j x) \\
&\leq \psi([\theta(M_p(T_i x_{2k}, T_j x))]^{k(d(x_{2k},x))}) + Ld^p(x, T_i(x_{2k}) \\
&\leq \psi([\theta(M_p(T_i x_{2k}, T_j x))]^{k(d(x_{2k},x))}) + Ld^p(x, x_{2k+1}),
\end{aligned}$$

where

$$\begin{aligned}
M_p(T_i x_{2k}, T_j x) &= \max\Big\{d^p(x_{2k}, x), d^p(x_{2k}, T_i(x_{2k})), d^p(x, T_j(x)), \\
&\quad \frac{d^p(x, T_i(x_{2k})) + d^p(x_{2k}, T_j(x))}{2^p}, \frac{d^p(x_{2k}, T_i(x_{2k}))d^p(x, T_j(x))}{1 + d^p(x_{2k}, x)}, \\
&\quad \frac{d^p(x, T_i(x_{2k}))d^p(x_{2k}, T_j(x))}{1 + d^p(x_{2k}, x)}\Big\} \\
&\leq \max\Big\{d^p(x_{2k}, x), d^p(x_{2k}, x_{2k+1}), d^p(x, T_j(x)), \\
&\quad \frac{d^p(x, x_{2k+1}) + d^p(x_{2k}, T_j(x))}{2^p}, \frac{d^p(x_{2k}, x_{2k+1})d^p(x, T_j(x))}{1 + d^p(x_{2k}, x)}, \\
&\quad \frac{d^p(x, x_{2k+1})d^p(x_{2k}, T_j(x))}{1 + d^p(x_{2k}, x)}\Big\}.
\end{aligned}$$

Then we can choose $k_0 \in \mathbb{N}$ such that $M_p(T_i x_{2k}, T_j x) = d^p(x, T_j(x))$ for each $k \geq k_0$. Since $y_k \in T_j(x)$, we have for each $k \geq k_0$

$$\psi(\theta(d^p(x_{2k+1}, y_k))) \leq \psi([\theta(d^p(x, T_j(x)))]^{k(d(x_{2k},x))}) + Ld^p(x, x_{2k+1}).$$

Taking into account the property of the function k, there exist $a \in (0, 1)$ and $k_1 \in \mathbb{N}$ such that for all $k \geq \max\{k_0, k_0\}$

$$\psi(\theta(d^p(x_{2k+1}, T_j(x)))) \quad \leq \quad \psi(\theta(d^p(x_{2k+1}, y_k)))$$

$$\leq \quad \psi([\theta(d^p(x,T_j(x)))]^a)+Ld^p(x,x_{2k+1}).$$

Taking the lower limit as $k\to\infty$, we deduce that

$$\limsup\inf_{k\to\infty}\psi(\theta(d^p(x_{2k+1},T_j(x))))\leq\psi([\theta(d^p(x,T_j(x)))]^a).$$

Since $\psi\circ\theta$ is lower semi-continuity and $a\in(0,1)$, we see that

$$\begin{aligned}\psi(\theta(d^p(x,T_j(x)))) &\leq \limsup\inf_{k\to\infty}\psi(\theta(d^p(x_{2k+1},T_j(x))))\\ &\leq \psi([\theta(d^p(x,T_j(x)))]^a)\\ &< \psi(\theta(d^p(x,T_j(x)))),\end{aligned}$$

which is a contradiction. Thus we have $d^p(x,T_j(x))=0$ which implies that $x\in T_j(x)$.

• *Case 2: $n=2k+1$ is odd.* Arguing as in Case 1, we obtain $x\in T_i(x)$.
Since $(x_n,x_{n+1})\in E(G)$ and $(x_n,x)\in E(G)$, for $n\in\mathbb{N}$, we conclude that

$$(x_0,x_1,x_2,\ldots,x_n,x)$$

is a path in $\widetilde{G}$ and so $x\in[x_0]_{\widetilde{G}}$.

Claim 2. Since $X_{T_i}\neq\emptyset$, then there exists some $x_0\in X_{T_i}$. In addition, since G is weakly connected, then $[x_0]_{\widetilde{G}}=X$ and by Claim 1, T_1 and T_2 have a common fixed point in X.

Claim 3. The result follows from Claim 1 and Claim 2.

Claim 4. $Graph(T_i)\subseteq E(G)$ implies that all $x\in X$ are such that there exists some $u\in T_i(x)$ with $(x,u)\in E(G)$, so $X_{T_i}=X$ which imply that T_1 and T_2 have a common fixed point.

3 Example

Let $X=\{\frac{1}{2^n},\ n\in\mathbb{N}\}\cup\{0,1\}$ and $d(x,y)=|x-y|$ for all $x,y\in X$. Let $E(G)=\{(\frac{1}{2^n},0),(\frac{1}{2^n},\frac{1}{2^{n+1}}),\ n\in\mathbb{N}\}\cup\{(1,0)\}$, $\theta(t)=e^{(te^t)^{\frac{1}{4}}}$, $L=0$, $\psi(t)=\ln(t)+1$, $1\leq p<4$, and

$$k(t)=\begin{cases}(e^{\frac{1}{2^{np+p}}-\frac{1}{2^{np}}})^{\frac{1}{4}}, & \text{if} \quad t=\frac{1}{2^n},\ n\in\{0,1,2,\ldots\},\\ 0, & \text{if} \quad \text{otherwise}.\end{cases}$$

Let T_1 and $T_2: X\to C(X)$ be two mappings defined by

$$T_1(x)=\begin{cases}\{0\}, & \text{if} \quad x=0,\\ \{\frac{1}{2}\}, & \text{if} \quad x=1,\\ \{\frac{1}{2^{n+3}},\frac{1}{2^{n+4}},\ldots\}, & \text{if} \quad x=\frac{1}{2^n},\ n\in\mathbb{N},\end{cases}$$

$$T_2(x)=\begin{cases}\{0\}, & \text{if} \quad x=0,\\ \{\frac{1}{2^3}\}, & \text{if} \quad x=1,\\ \{\frac{1}{2^{n+3}},\frac{1}{2^{n+4}},\ldots\}, & \text{if} \quad x=\frac{1}{2^n},\ n\in\mathbb{N}.\end{cases}$$

Then T_1 and T_2 are a common (ψ,θ-G) contraction and $0 \in T_1(0) \cap T_2(0)$. To check the contraction type condition, we have to show that

$$\frac{d^p(x,y)}{M_p(T_ix,T_jy)} e^{d^p(x,y)-M_p(T_ix,T_jy)} \leq k^4(d(x,y)).$$

For this, let $x, y \in X$ be such that $(x,y) \in E(G)$ and consider three cases:

• *Case 1.* If $(x,y) = \left(\frac{1}{2^n}, 0\right)$, then
(i) $T_1\left(\frac{1}{2^n}\right) = \{\frac{1}{2^{n+3}}, \frac{1}{2^{n+4}}, \ldots\}$ and $T_2(0) = \{0\}$. For $a = \frac{1}{2^{n+s}}$ where $s \in \{3,4,\ldots\}$, let $b = 0$ and

$$\begin{aligned} \frac{d^p(x,y)}{M_p(T_1x,T_2y)} e^{d^p(x,y)-M_p(T_1x,T_2y)} &= \frac{2^{np}}{2^{np+sp}} e^{\frac{1}{2^{np+sp}} - \frac{1}{2^{np}}} \\ &< e^{\frac{1}{2^{np+sp}} - \frac{1}{2^{np}}} \\ &\leq e^{\frac{1}{2^{np+p}} - \frac{1}{2^{np}}} \\ &= k^4(d(x,y)). \end{aligned}$$

(ii) $T_2\left(\frac{1}{2^n}\right) = \{\frac{1}{2^{n+3}}, \frac{1}{2^{n+4}}, \ldots\}$ and $T_1(0) = \{0\}$. For $a = \frac{1}{2^{n+s}}$ where $s \in \{3,4,\ldots\}$, let $b = 0$ and

$$\begin{aligned} \frac{d^p(x,y)}{M_p(T_2x,T_1y)} e^{d^p(x,y)-M_p(T_2x,T_1y)} &= \frac{2^{np}}{2^{np+sp}} e^{\frac{1}{2^{np+sp}} - \frac{1}{2^{np}}} \\ &< e^{\frac{1}{2^{np+sp}} - \frac{1}{2^{np}}} \\ &\leq e^{\frac{1}{2^{np+p}} - \frac{1}{2^{np}}} \\ &= k^4(d(x,y)). \end{aligned}$$

• *Case 2.* If $(x,y) = \left(\frac{1}{2^n}, \frac{1}{2^{n+1}}\right)$, then
(i) $T_1\left(\frac{1}{2^n}\right) = \{\frac{1}{2^{n+3}}, \frac{1}{2^{n+4}}, \ldots\}$ and $T_2\left(\frac{1}{2^{n+1}}\right) = \{\frac{1}{2^{n+4}}, \frac{1}{2^{n+5}}, \ldots\}$. For $a = \frac{1}{2^{n+s}}$ where $s \in \{3,4,\ldots\}$, let $b = \frac{1}{2^{n+s+1}}$ where $s \in \{3,4,\ldots\}$ and

$$\begin{aligned} \frac{d^p(x,y)}{M_p(T_1x,T_2y)} e^{d^p(x,y)-M_p(T_1x,T_2y)} &= \frac{2^{np+p}}{2^{np+sp+p}} e^{\frac{1}{2^{np+sp+p}} - \frac{1}{2^{np+p}}} \\ &< e^{\frac{1}{2^{np+sp+p}} - \frac{1}{2^{np+p}}} \\ &\leq e^{\frac{1}{2^{p(n+2)}} - \frac{1}{2^{p(n+1)}}} \\ &= k^4(d(x,y)). \end{aligned}$$

(ii) $T_2\left(\frac{1}{2^n}\right) = \{\frac{1}{2^{n+3}}, \frac{1}{2^{n+4}}, \ldots\}$ and $T_1\left(\frac{1}{2^{n+1}}\right) = \{\frac{1}{2^{n+4}}, \frac{1}{2^{n+5}}, \ldots\}$. For $a = \frac{1}{2^{n+s}}$ where $s \in \{3,4,\ldots\}$, let $b = \frac{1}{2^{n+s+1}}$ where $s \in \{3,4,\ldots\}$ and

$$\begin{aligned} \frac{d^p(x,y)}{M_p(T_2x,T_1y)} e^{d^p(x,y)-M_p(T_2x,T_1y)} &= \frac{2^{np+p}}{2^{np+sp+p}} e^{\frac{1}{2^{np+sp+p}} - \frac{1}{2^{np+p}}} \\ &\leq e^{\frac{1}{2^{p(n+2)}} - \frac{1}{2^{p(n+1)}}} \\ &= k^4(d(x,y)). \end{aligned}$$

• *Case 3.* If $(x, y) = (1, 0)$, then
(i) $T_1(1) = \{\frac{1}{2}\}$ and $T_2(0) = \{0\}$. For $a = \frac{1}{2}$, let $b = 0$ and

$$\begin{aligned} \frac{d^p(x,y)}{M_p(T_1x,T_2y)} e^{d^p(x,y)-M_p(T_1x,T_2y)} &= \frac{1}{2^p} e^{\frac{1}{2^p}-1} \\ &< e^{\frac{1}{2^p}-1} \\ &= k^4(d(x,y)). \end{aligned}$$

(ii) $T_2(1) = \{\frac{1}{2^3}\}$ and $T_1(0) = \{0\}$. For $a = \frac{1}{2^3}$, let $b = 0$ and

$$\begin{aligned} \frac{d^p(x,y)}{M_p(T_1x,T_2y)} e^{d^p(x,y)-M_p(T_1x,T_2y)} &= \frac{1}{2^{3p}} e^{\frac{1}{2^{3p}}-1} \\ &< e^{\frac{1}{2^p}-1} \\ &= k^4(d(x,y)). \end{aligned}$$

4 Consequences

If we let $p = 1$ in Theorem 1, we obtain

COROLLARY 1. Let (X, d) be a complete metric space endowed with a directed graph G and suppose that the triple (X, d, G) has the property (A). Suppose that the mappings $T_1, T_2 : X \to C(X)$ satisfy the following conditions:

(i) For all $x, y \in X$ such that $(x, y) \in E(G)$ and $a \in T_i(x)$, there exists $b \in T_j(y)$ for $i, j \in \{1, 2\}$ with $i \neq j$ such that $(a, b) \in E(G)$ and

$$\psi(\theta(d(a,b))) \leq \psi([\theta((M(T_ix,T_jy))]^{k(d(x,y))}) + LN(T_ix,T_jy),$$

where

$$N(T_ix,T_jy) = \min\{d(x,T_i(x)), d(y,T_j(y)), d(y,T_i(x)), d(x,T_j(y))\},$$

$$\begin{aligned} M(T_ix,T_jy) = \max\Bigg\{ & d(x,y), d(x,T_i(x)), d(y,T_j(y)), \frac{d(y,T_i(x)) + d(x,T_j(y))}{2}, \\ & \frac{d(x,T_i(x))d(y,T_j(y))}{1+d(x,y)}, \frac{d(y,T_i(x))d(x,T_j(y))}{1+d(x,y)} \Bigg\}. \end{aligned}$$

$k : (0, +\infty) \to [0, 1)$ satisfies $\limsup_{s \to t^+} k(s) < 1$, for all $t \in [0, +\infty)$, $L \geq 0$, $\theta \in \Theta$, $\psi \in \Psi$ and $\psi \circ \theta$ is lower semi-continuity.

(ii) There is $x_0 \in X$ such that $(x_0, y) \in E(G)$ for some $y \in T_i(x_0)$, $i = 1$ or $i = 2$.

If G is weakly connected, then T_1 and T_2 have a common fixed point.

If $\theta(t) = e^{\sqrt{t}}$, $\psi(t) = (\ln(t))^2 + 1$ and $k(t) = \sqrt{\alpha(t)}$ in Corollary 1, then we obtain

COROLLARY 2. Let (X, d) be a complete metric space endowed with a directed graph G and suppose that the triple (X, d, G) has the property (A). Suppose that the mappings $T_1, T_2 : X \to C(X)$ satisfy the following conditions:

(i) For all $x, y \in X$ such that $(x, y) \in E(G)$ and $a \in T_i(x)$, there exists $b \in T_j(y)$ for $i, j \in \{1, 2\}$ with $i \neq j$ such that $(a, b) \in E(G)$ and

$$d(a, b) \leq \alpha(d(x, y))M(T_i x, T_j y) + LN(T_i x, T_j y),$$

where $\alpha : (0, +\infty) \to [0, 1)$ satisfies $\limsup_{s \to t^+} \alpha(s) < 1$, for all $t \in [0, +\infty)$.

(ii) There is $x_0 \in X$ such that $(x_0, y) \in E(G)$ for some $y \in T_i(x_0)$, $i = 1$ or $i = 2$.

If G is weakly connected, then T_1 and T_2 have a common fixed point.

The following result is also an immediate consequence of Corollary 2.

COROLLARY 3. Let (X, d) be a complete metric space. Assume that the mappings $T_1, T_2 : X \to CB(X)$ satisfy

$$H(T_1(x), T_2(y)) \leq \alpha(d(x, y))M(T_1 x, T_2 y) + LN(T_1 x, T_2 y),$$

for all $x, y \in X$ such that $x \neq y$, where $\alpha : (0, +\infty) \to [0, 1)$ satisfies $\limsup_{s \to t^+} \alpha(s) < 1$, for all $t \in [0, +\infty)$. Then T_1 and T_2 have a common fixed point.

5 Remark

(1) Taking $T_1 = T_2$ in Theorem 1, we obtain fixed point results for (ψ, θ, G)-contraction maps.

(2) If in Corollary 1, we let $T_1 = T_2$, $\psi(t) = t$, and $E(G) = X \times X - \Delta$, then G is connected and Corollary 1 improves Theorem 4 by Durmaz [5] and Theorem 2.1 by Jleli *et al.* [9].

(3) Corollary 3 extends Theorem 3.1 by Rouhani *et al.* [14].

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