Applied Mathematics E-Notes, 19(2019), 535-551 © ISSN 1607-2510
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On The Fixed Points Of Large-Kannan Contraction Mappings And Applications*

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Received 10 August 2018

Abstract

In this paper, we establish some fixed point results for large-Kannan mappings in complete metric spaces. Our results are applied to solve some implicit integral equations.

1 Introduction and Preliminaries

It is well known that Banach's contraction principle (1922) is a powerful tool in analysis, that most mathematicians applying about to solve many of their problems. It is remarkable by its simplicity; this comes from the fact that the contractive condition on the mapping is easy to check and it requires only a complete metric space for its framework. Banach's contraction principle appeared in explicit form in Banach's thesis [2], where it was used to obtain the solution of an integral equation in the functional space $C([0,1])$. One of the other application of the Banach contraction principle is for example, the study of existence and uniqueness of solutions of the Itô stochastic equation with deviating argument. Indeed, it was proved (see [5]) that if the stochastic parameters satisfy Lipschitz condition in the second variable, then the associated integral operator turns out to be a contraction and the solution is obtained by using the method of successive approximations. In Banach spaces setting, recall that Krasnoselskii's fixed point result is a combination of Banach and Schauder's fixed point theorems, this result can be seen as a consequence of the fact that the measure of noncompactness is invariant by compact perturbations. The sum of two operators is a powerful tool used to solve delay integral equations, neutral functional equations, Cauchy problems for ordinary differential equations and partial differential equations modeled by Hammerstein integral operators in L_p-spaces. For more details, see [1, 3, 4].

In this work, we establish existence and uniqueness results for large-Kannan mappings extending those in [8, 9, 12]. In particular, we distinguish the continuous and

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*Mathematics Subject Classifications: 47H10.

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non-continuous cases. Finally, our results are explored to solve an implicit functional equation.

The classical statement of Banach fixed point theorem is the following:

THEOREM 1. Let (X, d) be a complete metric space and $T : X \to X$ be a contraction mapping, i.e.,

$$d(Tx, Ty) \leq \lambda d(x, y),$$

for all $x, y \in X$, where $0 < \lambda < 1$. Then T has a unique fixed point y_0 in X. Moreover for each $x_0 \in X$, the sequence of iterates $\{T^n x_0\}_n$ converges to y_0.

In 1968, R. Kannan [8, 9] obtained the following fixed point result.

THEOREM 2. Let (X, d) be a complete metric space and $T : X \to X$ be a selfmapping on X. Assume that there exists $\lambda \in \left[0, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq \lambda \left[d(x, Tx) + d(y, Ty)\right],$$

for all $x, y \in X$. Then T has a unique fixed point z_0 in X. Moreover for each $x_0 \in X$, the sequence of iterates $\{T^n x_0\}_n$ converges to z_0.

It has be seen that contraction mappings are continuous which is not in general the case of Kannan mappings as shown in the following examples.

EXAMPLE 1. Let $(X, d) = (\mathbb{R}, |\cdot|)$ and $T : \mathbb{R} \to \mathbb{R}$ be defined by

$$Tx = \begin{cases} 0 & \text{if } x \leq 2, \\ \frac{-1}{4} & \text{if } x > 2. \end{cases}$$

For all $x, y \in \mathbb{R}$, we prove easily that

$$|Tx - Ty| \leq \frac{1}{4}\left(|x - Tx| + |y - Ty|\right).$$

EXAMPLE 2. Let $(X, d) = ([0, 1], |\cdot|)$ and $T : [0, 1] \to [0, 1]$ be given by

$$Tx = \begin{cases} \frac{x}{16} & \text{if } x \in [0, 1[, \\ \frac{1}{18} & \text{if } x = 1. \end{cases}$$

Let $x, y \in [0, 1[$. Thus

$$|Tx - Ty| = \left|\frac{x}{16} - \frac{y}{16}\right| = \frac{1}{16}|x - y|,$$

and

$$|x - Tx| = \frac{15x}{16}, \; |y - Ty| = \frac{15y}{16},$$

which implies that

$$|Tx - Ty| = \frac{1}{16}|x - y| \leq \frac{1}{14}\left(|x - Tx| + |y - Ty|\right).$$

Now, if $x \in [0, 1[$ and $y = 1$, we get

$$|Tx - Ty| = \left|\frac{x}{16} - \frac{1}{18}\right|,$$

and

$$|x - Tx| = \frac{15x}{16},\ |T1 - 1| = \frac{17}{18}.$$

Consequently, for all $x, y \in [0, 1]$, we have

$$|Tx - Ty| \leq \frac{x}{16} + \frac{1}{18} \leq \frac{1}{14}\left(|x - Tx| + |y - Ty|\right).$$

EXAMPLE 3. Let $(X, d) = ([0, 1], |\cdot|)$ and $T : [0, 1] \to [0, 1]$ be defined by

$$Tx = \begin{cases} \frac{x}{6} & \text{if } 0 \leq x < \frac{1}{2}, \\ \frac{x}{4} & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

If $x, y \in \left[0, \frac{1}{2}\right[$, we get

$$|Tx - Ty| \leq \frac{1}{6}(x + y) = \frac{1}{5}\left(|x - Tx| + |y - Ty|\right).$$

On the other hand, if $x, y \in \left[\frac{1}{2}, 1\right]$, we get

$$|Tx - Ty| \leq \frac{1}{4}(x + y) = \frac{1}{3}\left(|x - Tx| + |y - Ty|\right).$$

Now, if $0 \leq x < \frac{1}{2} \leq y$, then

$$|Tx - Ty| = \left|\frac{1}{6}x - \frac{1}{4}y\right| \leq \frac{1}{4}(x + y) \leq \frac{1}{3}\left(|x - Tx| + |y - Ty|\right).$$

Consequently, for all $x, y \in [0, 1]$, we obtain that

$$|Tx - Ty| \leq \frac{1}{3}\left(|x - Tx| + |y - Ty|\right).$$

There is a large literature dealing with Kannan mappings and their generalizations, we can quote for examples [6, 7, 8, 9, 10] and [12]. In [3], Burton observed that Theorem 1 is more interesting in applications through some modification and formulated it as follows:

DEFINITION 1. Let (X,d) be a metric space and let $T : X \to X$ be a selfmapping on X. T is said to be a large contraction, if for $x, y \in X$, with $x \neq y$, we have $d(Tx, Ty) < d(x, y)$, and if for all $\epsilon > 0$, there exists $\delta < 1$ such that

$$[x, y \in X, \quad d(x, y) \geq \epsilon] \Longrightarrow d(Tx, Ty) \leq \delta d(x, y).$$

REMARK 1. We observe that every contraction mapping is a large contraction. The converse does not hold in general, as the following example shows [3].

EXAMPLE 4. Let $(X,d) = (\mathbb{R}, |.|)$ and let $T : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $Tx = x - x^3$. Then for $x, y \in \mathbb{R}$, by applying the Mean Value Theorem, we get

$$|Tx - Ty| = \left|x - x^3 - y + y^3\right| \leq \left|1 - 3c^2\right| |x - y|,$$

where $c \in \left]\min\{x,y\}, \max\{x,y\}\right[$.

Afterwards, from the inequality given above, it is easy to observe that there exists δ sufficiently small such that for all $x, y \in [-\delta, \delta]$ $(x \neq y)$, we have $|Tx - Ty| < |x - y|$. Additionally, it was proved in [3] that for a given $\epsilon > 0$, if $|x - y| \geq \epsilon$, then

$$|Tx - Ty| \leq \left|1 - \frac{\epsilon^2}{4}\right| |x - y|,$$

moreover, since $T0 = 0$ and $\lim_{x \to 0} \left|\frac{x - x^3}{x}\right| = 1$, we deduce that T is not a contraction selfmapping on $[-\delta, \delta]$.

EXAMPLE 5. Let $f : [0,1] \to [0,1]$ be given by $f(x) = x - \frac{x^4}{4}$. Then

$$\begin{aligned} |f(x) - f(y)| &= \left|x - \frac{x^4}{4} - \left(y - \frac{y^4}{4}\right)\right| = \left|(x-y) - \frac{1}{4}\left(\left(x^2\right)^2 - \left(y^2\right)^2\right)\right| \\ &= \left|(x-y) - \frac{1}{4}\left(x^2 - y^2\right)\left(x^2 + y^2\right)\right| \\ &= \left|(x-y) - \frac{1}{4}(x-y)(x+y)\left(x^2 + y^2\right)\right| \\ &= \left|(x-y)\left[1 - \frac{1}{4}(x+y)\left(x^2 + y^2\right)\right]\right|. \end{aligned}$$

Since $|x - y| \leq |x + y|$ and $|x - y|^2 = x^2 + y^2 - 2xy \leq 2\left(x^2 + y^2\right)$, it follows that

$$\begin{aligned} |f(x) - f(y)| &= \left|(x-y)\left[1 - \frac{1}{4}(x+y)\left(x^2 + y^2\right)\right]\right| \\ &\leq |x - y|\left(1 - \frac{|x-y|^3}{8}\right). \end{aligned}$$

Next, if $|x - y| \geq \epsilon$, we infer that

$$|f(x) - f(y)| \leq |x - y|\left(1 - \frac{\epsilon^3}{8}\right).$$

Hence, to deduce that f is a large contraction, it suffices to take $\delta(\epsilon) = \left(1 - \frac{\epsilon^3}{8}\right)$.

Now, to show that f is not a contraction mapping, it suffices to see that

$$\lim_{x \to 0} \left| \frac{x - \frac{x^4}{4}}{x} \right| = 1.$$

Thus, there is no $k \in (0,1)$ such that

$$\left| x - \frac{x^4}{4} - \left(y - \frac{y^4}{4} \right) \right| \leq k \left| x - y \right|, \quad \forall x, y \in (0,1).$$

THEOREM 3 ([3]). Let (X, d) be a complete metric space and let $T : X \to X$ be a large contraction selfmapping. Assume that there exist $x_0 \in X$ and $L > 0$, such that $d(x_0, T^n x_0) \leq L$ for all $n \geq 1$. Then T has a unique fixed point in X.

REMARK 2. Notice that, if (X, d) is a compact metric space, then the assumption that there exist $x_0 \in X$ and $L > 0$, such that $d(x_0, T^n x_0) \leq L$ for all $n \geq 1$ can be dropped. Indeed, in this case, the existence and uniqueness of the fixed point is ensured by Edelstein's theorem.

REMARK 3. If (X, d) is a bounded complete metric space, then for all $x_0 \in X$ and for all integer $n \geq 1$, we have $d(x_0, T^n x_0) \leq \delta(X)$, where $\delta(X)$ is the diameter of X. So the boundedness assumption given above is trivially satisfied in this setting.

2 Main Results

We start this section by the following lemma which asserts that the set of contraction mappings has an infinite subset of Kannan mappings.

LEMMA 1. Let (X, d) be a metric space. Assume that $T : X \to X$ be a selfmapping on X satisfying that

$$d(Tx, Ty) \leq \alpha d(x, y) \text{ for all } x, y \in X,$$

where $\alpha \in \left[0, \frac{1}{3}\right[$. Then T is a Kannan mapping with a constant of contraction equals to $\frac{\alpha}{1-\alpha}$.

PROOF. Let $x, y \in X$. Then by assumption, we have

$$d(Tx, Ty) \leq \alpha d(x, y),$$

where $\alpha \in \left[0, \frac{1}{3}\right[$. On the other hand, by using the triangle inequality, we get

$$d(x, y) \leq d(x, Tx) + d(Tx, Ty) + d(Ty, y).$$

Multiplying the previous inequality by α, it follows that

$$d(Tx,Ty) \leq \alpha d(x,y) \leq \alpha(d(x,Tx) + d(Tx,Ty) + d(Ty,y)),$$

therefore,

$$d(Tx,Ty) \leq \frac{\alpha}{1-\alpha}(d(x,Tx) + d(y,Ty)).$$

Since $\alpha \in \left[0, \frac{1}{3}\right[$, then $\frac{\alpha}{1-\alpha} \in \left[0, \frac{1}{2}\right[$. Consequently, T is a Kannan mapping.

2.1 Large-Kannan Contractions in the Continuous Sense

Now, we give the following definition of a large-Kannan contraction (in the continuous sense) as an extension of the classical ones.

DEFINITION 2. Let (X, d) be a metric space and let $T : X \to X$ be a selfmapping on X. T is said to be a large-Kannan contraction (in the continuous sense), if for $x, y \in X$, with $x \neq y$, we have $d(Tx,Ty) < d(x,y)$, and if for all $\epsilon > 0$, there exists $\delta < \frac{1}{2}$ such that

$$[x, y \in X, \quad d(x,y) \geq \epsilon] \Longrightarrow d(Tx,Ty) \leq \delta \left[d(x,Tx) + d(y,Ty)\right].$$

REMARK 4. Large-Kannan contractions (in the continuous sense) are continuous. This is an immediate consequence of the inequality $d(Tx,Ty) < d(x,y)$ for $x \neq y$.

REMARK 5. Let (X, d) be a metric space and let $T : X \to X$ be a large contraction on X. Assume that $\delta \in [0, \frac{1}{3}[$, then by Lemma 1, it is easy to conclude that T is a large-Kannan contraction mapping.

Now, we give the following fixed point result for large-Kannan contractions.

THEOREM 4. Let (X, d) be a complete metric space and $T : X \to X$ be a large-Kannan contraction mapping (in the continuous sense). Then T has a unique fixed point in X.

PROOF. Let $x_0 \in X$, if there exists an integer $m \geq 1$ such that $T^m(x_0) = T^{m+1}(x_0)$, then $T(T^m x_0) = T^m x_0$ and $T^m x_0$ is a fixed point of T.

Now, assume that $T^n x_0 \neq T^{n+1} x_0$ for every integer $n \geq 1$. Since T is large-Kannan contraction (in the continuous sense), then

$$d\left(T^{n+1}x_0, T^n x_0\right) < d\left(T^n x_0, T^{n-1} x_0\right) < ... < d(Tx_0, x_0).$$

This proves that the sequence $\zeta_n = d\left(T^{n+1}x_0, T^n x_0\right)$ is strictly decreasing, hence $\lim_{n \longrightarrow +\infty} \zeta_n = \gamma \geq 0$. If $\gamma > 0$, then for all $n \geq 1$, we get

$$d\left(T^{n+1}x_0, T^n x_0\right) \geq \gamma.$$

Consequently, there exists $\delta < \frac{1}{2}$ such that

$$\begin{aligned} d\left(T^{n+1}x_0, T^{n+2}x_0\right) &= d\left(T\left(T^n x_0\right), T\left(T^{n+1}x_0\right)\right) \\ &\leq \delta\left[d\left(T^n x_0, T^{n+1}x_0\right) + d\left(T^{n+1}x_0, T^{n+2}x_0\right)\right]. \end{aligned}$$

This implies that

$$(1-\delta)\, d\left(T^{n+1}x_0, T^{n+2}x_0\right) \leq \delta d\left(T^n x_0, T^{n+1}x_0\right).$$

Thus, we have

$$\begin{aligned} d\left(T^{n+1}x_0, T^{n+2}x_0\right) &\leq \frac{\delta}{1-\delta} d\left(T^n x_0, T^{n+1}x_0\right) \\ &\leq \left(\frac{\delta}{1-\delta}\right)^2 d\left(T^{n-1}x_0, T^n x_0\right) \\ &\vdots \\ &\leq \left(\frac{\delta}{1-\delta}\right)^n d\left(Tx_0, T^2 x_0\right) \\ &\leq \left(\frac{\delta}{1-\delta}\right)^{n+1} d\left(x_0, Tx_0\right). \end{aligned} \tag{1}$$

Since $\delta < \frac{1}{2}$, we see that $k = \frac{\delta}{1-\delta} < 1$. So, by using (1), it follows that

$$\lim_{n\to\infty} d\left(T^n x_0, T^{n+1}x_0\right) = 0, \tag{2}$$

which is a contradiction. Hence $\gamma = 0$ and achieves the proof of this step.

Now, we shall prove that $\{x_n\}_n$ given by $x_n = T^n x_0$ is a Cauchy sequence in X. Suppose, to the contrary that $\{x_n\}_n$ is not a Cauchy sequence. Thus, there exist $\epsilon > 0$ and subsequences of integers (N_k), (n_k), (m_k) such that

$$N_k \to \infty,\ m_k > n_k > N_k,$$

and

$$\epsilon \leq d\left(x_{m_k}, x_{n_k}\right). \tag{3}$$

Since T is large-Kannan mapping, by using (3), there exists $\delta < \frac{1}{2}$ such that

$$\epsilon \leq d\left(x_{m_k}, x_{n_k}\right) = d\left(Tx_{m_k-1}, Tx_{n_k-1}\right) \leq \delta\left[d\left(x_{m_k-1}, x_{m_k}\right) + d\left(x_{n_k-1}, x_{n_k}\right)\right].$$

Letting $k \to \infty$, from (2), follows that

$$\lim_{k\to\infty} d\left(x_{m_k-1}, x_{m_k}\right) = \lim_{k\to\infty} d\left(x_{n_k-1}, x_{n_k}\right) = 0.$$

Hence $\lim\limits_{k\to\infty} d\left(x_{m_k}, x_{n_k}\right) = 0$, which is a contradiction. Thus $\{x_n\}_n$ is a Cauchy sequence in X. Finally, since X is complete, then there exists $l \in X$ such that

$\lim_{n\to\infty} x_n = \lim_{n\to\infty} T^n x_0 = l$. The continuity of T implies that $T(l) = l$, which proves that l is a fixed point of T.

Now, suppose that l' is another fixed point for T such that $l \neq l'$. Thus $d(l, l') \geq \epsilon_0$ for some $\epsilon_0 > 0$. Since T is Kannan-large mapping, there exists $\delta_0 < \frac{1}{2}$ such that

$$d(l, l') = d(T(l), T(l')) \leq \delta_0 [d(l, T(l)) + d(l', T(l'))].$$

Hence, we get $d(l, l') = 0$, which is a contradiction. Thus, we must have $l = l'$.

REMARK 6. For the case of large-Kannan mappings (in the continuous sense), the existence of fixed points is proved without any assumption on the boundedness of the set $\{d(x_0, T^n x_0)\}$ for some $x_0 \in X$.

COROLLARY 1. Let (X, d) be a complete metric space and $T : X \to X$ be a selfmapping on X such that T^{m_0} is a large-Kannan mapping (in the continuous sense) for some integer $m_0 \geq 1$. Then T has a unique fixed point in X.

PROOF. From Theorem 4, there exists $z_0 \in X$ such that $T^{m_0} z_0 = z_0$, then

$$T(T^{m_0} z_0) = T^{m_0+1} z_0 = T z_0.$$

This gives $T^{m_0}(T z_0) = T z_0$ and implies that $T z_0$ is a fixed point for $T^{m_0} z_0$. The uniqueness of the fixed point for the mapping T^{m_0} (given by Theorem 4) shows that $T z_0 = z_0$. Now, if z_1 is another fixed point for T, then z_1 is a fixed point for T^{m_0}. Hence $z_0 = z_1$, which achieves the proof.

EXAMPLE 6. Let $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = -x^3$. If $x \geq 0$, we have $|x| = x \leq x + x^3 = x - \left(-x^3\right) = |x - f(x)|$. If $x \leq 0$, we have $|x| = -x \leq -x - x^3 = -x + \left(-x^3\right) = |x - f(x)|$. Let us prove that f is not a Kannan mapping. For all $x, y \in \mathbb{R}$, we have

$$|f(x) - f(y)| = \left|x^3 - y^3\right|.$$

Thus

$$|x - f(x)| + |y - f(y)| = \left|x + x^3\right| + \left|y + y^3\right|.$$

By taking $y = 0$ and letting $x \to +\infty$, we get

$$\lim_{x\to+\infty} \frac{|f(x) - f0|}{|x - f(x)| + |0 - f0|} = \lim_{x\to+\infty} \frac{|f(x)|}{|x + x^3|} = \lim_{x\to+\infty} \frac{|x^3|}{|x + x^3|} = 1 > \frac{1}{2}.$$

Consequently, f need not be a Kannan mapping.

Now, set

$$\Omega = \left\{(x, y) \in [-1, 1]^2 : \left|x^2 + y^2 + xy\right| + \frac{1}{2}|x - y|^2 \leq \frac{1}{2}\right\}.$$

If $x, y \in \Omega$, then

$$\begin{aligned} |f(x) - f(y)| &= \left|x^3 - y^3\right| \\ &= |x-y|\left|x^2+y^2+xy\right| \\ &\leq (|x|+|y|)\left|x^2+y^2+xy\right| \\ &\leq (|x-f(x)|+|y-f(y)|)\left|x^2+y^2+xy\right| \\ &\leq (|x-f(x)|+|y-f(y)|)\left(\frac{1-|x-y|^2}{2}\right). \end{aligned}$$

So, for a given (sufficiently small) $\epsilon > 0$, if $x, y \in \Omega$ satisfying that $|x-y| \geq \epsilon$, we have

$$|f(x) - f(y)| \leq (|x-f(x)|+|y-f(y)|)\left(\frac{1-\epsilon^2}{2}\right).$$

Finally, to conclude that f is a large-Kannan mapping, it suffices to take $\delta(\epsilon) = \frac{1-\epsilon^2}{2}$ which achieves the proof.

By Rakotch [11], let Σ denote the class of real-valued control functions (not necessarily continuous) which satisfy the condition

$$\Sigma = \left\{ f : (0,\infty) \to \left[0, \frac{1}{2}\right[, \; f(t_n) \mapsto \frac{1}{2} \Rightarrow t_n \to 0\,(n \to \infty) \right\}.$$

Now, we are in position to prove a general version of Theorem 4 given as follows:

THEOREM 5. Let (X, d) be a complete metric space and $T : X \to X$ be a selfmapping such that, for $x, y \in X$, with $x \neq y$, we have $d(Tx, Ty) < d(x, y)$ and for all $\epsilon > 0$, there exists $f_\epsilon \in \Sigma$ such that

$$[x, y \in X, \quad d(x,y) \geq \epsilon] \Longrightarrow d(Tx, Ty) \leq f_\epsilon(d(x,y))\,[d(x,Tx) + d(y,Ty)].$$

Then T has a unique fixed point z_0 in X.

PROOF. Let $x_0 \in X$, if there exists an integer $m_0 \geq 1$ such that $T^{m_0}(x_0) = T^{m_0+1}(x_0)$, then $T(T^{m_0}x_0) = T^{m_0}x_0$ and $T^{m_0}x_0$ is a fixed point of T.

Now, assume that $T^n x_0 \neq T^{n+1}x_0$ for every integer $n \geq 1$. Define the sequence $\{x_n\}_n$ by $x_n = T^n x_0$. Hence

$$d(x_n, x_{n+1}) = d\left(T^n x_0, T^{n+1}x_0\right) < d\left(T^n x_0, T^{n-1}x_0\right) = d(x_{n-1}, x_n),$$

this implies that the sequence $\zeta_n = d(x_n, x_{n+1})$ is strictly decreasing, consequently $\lim\limits_{n \longrightarrow +\infty} \zeta_n = \gamma \geq 0$. If $\gamma > 0$, by assumption, there exists $f_\gamma \in \Sigma$ such that

$$d(x_n, x_{n+1}) = d\left(T^n x_0, T^{n+1}x_0\right) \leq f_\gamma(d(x_{n-1}, x_n))\,[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)].$$

Therefore

$$\frac{d(x_n, x_{n+1})}{d(x_n, x_{n+1}) + d(x_{n-1}, x_n)} \leq f_\gamma(d(x_{n-1}, x_n)) < \frac{1}{2}.$$

Letting $n \to \infty$, it follows that

$$\lim_{n\to\infty} \frac{d(x_n, x_{n+1})}{d(x_n, x_{n+1}) + d(x_{n-1}, x_n)} = \frac{\gamma}{2\gamma} = \frac{1}{2} \leq \lim_{n\to\infty} f_\gamma (d(x_{n-1}, x_n)) < \frac{1}{2}, \qquad (4)$$

which is a contradiction, then we must have $\gamma = 0$.

Now, we shall prove that $\{x_n\}_n$ given by $x_n = T^n x_0$ is a Cauchy sequence in X.

Suppose, to the contrary, that $\{x_n\}_n$ is not a Cauchy sequence in X. Thus, there exist $\epsilon_0 > 0$, subsequences of positive integers (N_k), (n_k) and (m_k) such that

$$N_k \to \infty,\ m_k > n_k > N_k,$$

and

$$\epsilon_0 \leq d(x_{m_k}, x_{n_k}) = d(Tx_{n_k-1}, Tx_{m_k-1}). \qquad (5)$$

This last inequality shows that $x_{m_k-1} \neq x_{n_k-1}$. Now, by assumptions and using the relation 5, there exists $f_\epsilon \in \Sigma$ such that

$$\begin{aligned} \epsilon_0 &\leq d(x_{n_k}, x_{m_k}) = d(Tx_{n_k-1}, Tx_{m_k-1}) \\ &\leq f_\epsilon (d(x_{n_k-1}, x_{m_k-1})) [d(x_{n_k-1}, x_{n_k}) + d(x_{m_k-1}, x_{m_k})] \\ &< \frac{1}{2} [d(x_{n_k-1}, x_{n_k}) + d(x_{m_k-1}, x_{m_k})]. \end{aligned}$$

Letting $k \to \infty$, it follows that

$$\epsilon_0 \leq \lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = 0,$$

which is a contradiction. Thus $\{x_n\}_n$ is a Cauchy sequence in X. Since X is a complete metric space, there exists $z_0 \in X$ such that $\lim_{n \longrightarrow +\infty} x_n = z_0$. To prove that z_0 is a fixed point for T, we argue as follows:

Select an arbitrary integer $n \geq 1$. By using the triangle inequality, we get

$$\begin{aligned} d(z_0, Tz_0) &\leq d(z_0, x_{n+1}) + d(Tz_0, x_{n+1}) \\ &= d(z_0, x_{n+1}) + d(Tz_0, Tx_n). \end{aligned}$$

Without loss of generality, we assume that $x_n \neq z_0$. Hence

$$\begin{aligned} d(z_0, Tz_0) &\leq d(z_0, x_{n+1}) + d(Tz_0, x_{n+1}) \\ &\leq d(z_0, x_{n+1}) + \tfrac{1}{2}(d(x_n, x_{n+1}) + d(z_0, Tz_0)). \end{aligned}$$

Similarly

$$0 \leq \tfrac{1}{2} d(z_0, Tz_0) \leq d(z_0, x_{n+1}) + \tfrac{1}{2} d(x_n, x_{n+1}).$$

Letting $n \longrightarrow +\infty$, it follows that

$$0 \leq \tfrac{1}{2} d(z_0, Tz_0) \leq \lim_{n \longrightarrow +\infty} d(z_0, x_{n+1}) + \frac{1}{2} \lim_{n \longrightarrow +\infty} d(x_n, x_{n+1}) = 0.$$

Consequently, z_0 is a fixed point for T.

If T has two fixed points $z_0, z_1 \in X, z_0 \neq z_1$, thus, by our assumption and putting $\epsilon = d(z_0, z_1)$, there exists $f_{\frac{\epsilon}{2}}$ (taking its values in the interval $[0, \frac{1}{2}[$) such that

$$\begin{aligned} 0 &< \frac{\epsilon}{2} < d(z_0, z_1) = d(Tz_0, Tz_1) \leq f_{\frac{\epsilon}{2}}(d(z_0, z_1))\,(d(z_0, Tz_0) + d(z_1, Tz_1)) \\ &< \frac{1}{2}(d(z_0, Tz_0) + d(z_1, Tz_1)) = 0. \end{aligned}$$

This is a contradiction. Consequently, we must have $z_0 = z_1$ which achieves the proof.

REMARK 7. It is worth noting that the condition $d(Tx, Ty) < d(x, y)$, $(x \neq y)$ does not imply the existence of the fixed point. To see this, it suffices to take $(X, d) = (\mathbb{R}, |\cdot|)$ and $Tx = \sqrt{x^2 + 1}$.

DEFINITION 3. Let (X, d) be a metric space and T be a selfmapping on X. T is said to be asymptotically regular if for each $x \in X$ we have $\lim\limits_{n \to \infty} d\left(T^n x, T^{n+1} x\right) = 0$.

Define now the class of functions Σ' by

$$\Sigma' = \{f : (0, \infty) \to [0, 1[\,,\; f(t_n) \mapsto 1 \Rightarrow t_n \to 0\,(n \to \infty)\}$$

In the following fixed point theorem, we drop the condition $d(Tx, Ty) < d(x, y)$, $(x \neq y)$ and we replace it by T continuous and asymptotically regular.

THEOREM 6. Let (X, d) be a metric space and let T be a continuous selfmapping on X which is asymptotically regular. Assume that for every $\epsilon > 0$, there exist $f_\epsilon \in \Sigma'$ such that

$$[x, y \in X, \quad d(x, y) \geq \epsilon] \Longrightarrow d(Tx, Ty) \leq f_\epsilon(d(x, y))\,[d(x, Tx) + d(y, Ty)]\,.$$

Then T has a unique fixed point z_0 in X. Moreover for each $x_0 \in X$, the sequence of iterates $\{T^n x_0\}_n$ converges to z_0.

PROOF. Let $x_0 \in X$ and define the sequence $\{x_n\}_n$ by $x_n = T^n x_0$ for all integer $n \geq 1$. If there exists $m_0 \geq 1$ such that $T^{m_0}(x_0) = T^{m_0+1}(x_0)$, then $T(T^{m_0} x_0) = T^{m_0} x_0$ and $T^{m_0} x_0$ is a fixed point of T.
Now, suppose that $T^n x_0 \neq T^{n+1} x_0$ for all $n \geq 1$. We shall prove that $\{x_n\}_n$ is a Cauchy sequence in X.
If it's not the case, there exist $\epsilon_0 > 0$ and subsequences of positive integers (N_k), (n_k) and (m_k) such that $\lim\limits_{k \longrightarrow +\infty} N_k = +\infty$, $m_k > n_k > N_k$ and $d(x_{m_k}, x_{n_k}) \geq \epsilon_0$. Thus, by using our assumptions and the triangle inequality, we get

$$\begin{aligned} d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{m_k+1}) + f_{\epsilon_0}(d(x_{n_k}, x_{m_k}))\,[d(x_{n_k}, x_{n_k+1}) + d(x_{m_k}, x_{m_k+1})] \\ &\quad + d(x_{n_k+1}, x_{n_k})\,, \end{aligned}$$

for some $f_{\epsilon_0} \in \Sigma'$.
Then,

$$\begin{aligned}\left[1 - f_{\epsilon_0}\left(d\left(x_{n_k}, x_{m_k}\right)\right)\right] d\left(x_{n_k}, x_{m_k}\right) &\leq d\left(x_{n_k}, x_{m_k}\right) \\ &\leq \left[1 + f_{\epsilon_0}\left(d\left(x_{n_k}, x_{m_k}\right)\right)\right] \\ &\times \left(d\left(x_{n_k}, x_{n_k+1}\right) + d\left(x_{m_k}, x_{m_k+1}\right)\right). \end{aligned} \tag{6}$$

Dividing each right side in (6) by

$$\left[1 - f_{\epsilon_0}\left(d\left(x_{n_k}, x_{m_k}\right)\right)\right] \times \left(d\left(x_{n_k}, x_{n_k+1}\right) + d\left(x_{m_k}, x_{m_k+1}\right)\right),$$

and using the fact that $d\left(x_{m_k}, x_{n_k}\right) \geq \epsilon_0$, we conclude

$$\begin{aligned}\frac{\epsilon_0}{d\left(x_{n_k}, x_{n_k+1}\right) + d\left(x_{m_k}, x_{m_k+1}\right)} &\leq \frac{d\left(x_{n_k}, x_{m_k}\right)}{d\left(x_{n_k}, x_{n_k+1}\right) + d\left(x_{m_k}, x_{m_k+1}\right)} \\ &\leq \frac{1 + f_{\epsilon_0}\left(d\left(x_{n_k}, x_{m_k}\right)\right)}{1 - f_{\epsilon_0}\left(d\left(x_{n_k}, x_{m_k}\right)\right)}.\end{aligned}$$

Letting $k \to \infty$, thus

$$\lim_{k \to +\infty} \frac{1 + f_{\epsilon_0}\left(d\left(x_{n_k}, x_{m_k}\right)\right)}{1 - f_{\epsilon_0}\left(d\left(x_{n_k}, x_{m_k}\right)\right)} = +\infty.$$

Consequently,

$$\limsup_{n,m \to \infty} f_{\epsilon_0}\left(d\left(x_n, x_m\right)\right) = 1. \tag{7}$$

Since $f_{\epsilon_0} \in \Sigma'$, we obtain

$$\lim_{n,m \to \infty} d\left(x_n, x_m\right) = 0,$$

which is a contradiction. Hence $\{x_n\}_n$ must be a Cauchy sequence.

Therefore, since X is a complete metric space, there exists $z_0' \in X$ such that $\lim x_n = z_0'$. The continuity of T implies that $Tz_0' = z_0'$.

If z_1' is another fixed point for T such that $z_0' \neq z_1'$. Set $d\left(z_0', z_1'\right) = \epsilon_1 > \frac{\epsilon_1}{2}$. Then by assumption there exists $f_{\frac{\epsilon_1}{2}} \in \Sigma'$ such that

$$\left[x, y \in X, \quad d\left(x, y\right) \geq \frac{\epsilon_1}{2}\right] \Longrightarrow d\left(Tx, Ty\right) \leq f_{\frac{\epsilon_1}{2}}\left(d\left(x, y\right)\right)\left[d\left(x, Tx\right) + d\left(y, Ty\right)\right].$$

It follows that

$$d\left(z_0', z_1'\right) \leq f_{\epsilon_1}\left(d\left(z_0', z_1'\right)\right) \times \left[d\left(z_0', Tz_0'\right) + d\left(z_1', Tz_1'\right)\right] = 0,$$

which is a contradiction. Hence $z_0' = z_1'$ which achieves the proof.

2.2 Large-Kannan Contractions in the Non (Necessarily) Continuous Sense

In this section, we focus our study on fixed point results for Kannan-large mappings which are not necessarily continuous. First of all, we start by following definition

DEFINITION 4. Let (X, d) be a metric space and let $T : X \to X$ be a selfmapping. T is said to be a large-Kannan contraction (in the non-necessarily-continuous sense), if for $x, y \in X$, with $x \neq y$, we have $d(Tx, Ty) < \frac{1}{2}(d(x, Tx) + d(y, Ty))$, and if for all $\epsilon > 0$, there exists $\delta < \frac{1}{2}$ such that

$$[x, y \in X, \quad d(x, y) \geq \epsilon] \Longrightarrow d(Tx, Ty) \leq \delta\,[d(x, Tx) + d(y, Ty)]\,.$$

REMARK 8. The following example given in [7] shows that mappings satisfying that $d(Tx, Ty) < \frac{1}{2}(d(x, Tx) + d(y, Ty))$ may fail to have fixed points.

EXAMPLE 7. Let $X = \{1 + \frac{1}{n}, n = 1, 2, ...\}$ and $d_0 : X \times X \longrightarrow [0, +\infty[$ defined by

$$d_0(x, y) = \begin{cases} 0 & \text{if } x = y, \\ x + y & \text{if } x \neq y. \end{cases}$$

Thus (X, d_0) is a complete metric space. On the other hand, let $T : (X, d_0) \longrightarrow (X, d_0)$ defined by $T(1 + \frac{1}{n}) = 1 + \frac{1}{n+1}$. In [7], it was proved that T satisfies the inequality $d_0(Tx, Ty) < \frac{1}{2}(d_0(x, Tx) + d_0(y, Ty))$ for $x \neq y$ but T has no fixed points.

THEOREM 7. Let (X, d) be a complete metric space and let $T : (X, d) \longrightarrow (X, d)$ be a large-Kannan mapping (in the non-necessarily continuous sense). Then T has a unique fixed point.

PROOF. **The uniqueness:** If T has two fixed points $x_0, x_1 \in X, x_0 \neq x_1$. Thus

$$0 \leq d(x_0, x_1) = d(Tx_0, Tx_1) < \frac{1}{2}\left(d(x_0, Tx_0) + d(x_1, Tx_1)\right) = 0,$$

which is a contradiction.

The existence: *Step 1:* Let $x_0 \in X$ and define the Picard sequence $\{x_n\}_n$ by $x_n = T^n x_0$ for all integer $n \geq 1$. If there exists an integer $m_0 \geq 1$ such that $T^{m_0}x_0 = T^{m_0+1}x_0$, thus $T^{m_0}x_0$ is a fixed point point for T and the proof is achieved. Now, assume that $x_n = T^n x_0 \neq T^{n+1}x_0 = x_{n+1}$ for all $n \geq 1$. We shall prove that the sequence $\epsilon_n = d(x_n, x_{n+1})$ is strictly decreasing.

We have

$$\begin{aligned} d(x_n, x_{n+1}) = d(TT^{n-1}x_0, TT^n x_0)) &< \tfrac{1}{2}(d(T^{n-1}x_0, T^n x_0) + d(T^n x_0, T^{n+1}x_0)) \\ &= \tfrac{1}{2}(d(x_{n-1}, x_n) + d(x_n, x_{n+1})). \end{aligned}$$

So, we conclude that $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$, which proves that $\epsilon_n = d((x_{n-1}, x_n)$ is strictly decreasing. Hence, there exists $\epsilon_0 \geq 0$ such that $\lim_{n \longrightarrow +\infty} d(x_n, x_{n+1}) = \epsilon_0$.

Step 2: $\epsilon_0 = 0$: Suppose that $\epsilon_0 > 0$, since the sequence $\epsilon_n = d(x_n, x_{n+1})$ is decreasing, we get $\epsilon_0 < d(x_n, x_{n+1})$ for all integer $n \geq 1$. Thus, by assumption there exists $0 < \delta_0 < \frac{1}{2}$ such that

$$d(x_n,x_{n+1}) = d(TT^{n-1}x_0, TT^nx_0)) \leq \delta_0(d(T^{n-1}x_0, T^nx_0) + d(T^nx_0, T^{n+1}x_0))$$
$$= \delta_0(d(x_{n-1},x_n) + d(x_n,x_{n+1})),$$

which gives

$$d(x_n,x_{n+1}) \leq \tfrac{\delta_0}{1-\delta_0} d(x_{n-1},x_n).$$

By induction, it follows that

$$d(x_n,x_{n+1}) \leq (\tfrac{\delta_0}{1-\delta_0})^n d(x_0,x_1).$$

Afterwards, since $0 < \delta_0 < \frac{1}{2}$, then $\frac{\delta_0}{1-\delta_0} < 1$. This proves that $\lim\limits_{n\longrightarrow+\infty}(\dfrac{\delta_0}{1-\delta_0})^n = 0$ and implies $\lim\limits_{n\longrightarrow+\infty} d(x_n,x_{n+1}) = 0$ which is a contradiction. Consequently $\epsilon_0 = 0$.

Step 3: $\{x_n\}_n$ is a Cauchy sequence in X:

If it is not the case, then there exists α_0 and subsequences of integers $\{N_k\}, \{n_k\}$ and $\{m_k\}$ with $m_k > n_k > N_k$ such that $\alpha_0 \leq d(x_{n_k}, x_{m_k}) = d(Tx_{n_k-1}, Tx_{m_k-1})$, which leads to deduce that $x_{m_k-1} \neq x_{n_k-1}$.

Thus, by assumption and using the fact that the sequence $\epsilon_n = d(x_n,x_{n+1})$ is decreasing, we get

$$\alpha_0 \leq d(x_{m_k},x_{n_k}) = d(Tx_{n_k-1}, Tx_{m_k-1})$$
$$\leq \tfrac{1}{2}(d(x_{n_k-1},x_{n_k}) + d(x_{m_k-1},x_{m_k}))$$
$$\leq d(x_{n_k-1},x_{n_k}).$$

Letting $k \longrightarrow +\infty$, it follows that

$$\alpha_0 \leq \lim_{k\to+\infty} d(x_{m_k},x_{n_k}) \leq \lim_{k\to+\infty} d(x_{n_k},x_{n_k-1}) = 0,$$

which is a contradiction. Hence $\{x_n\}_n$ is a Cauchy sequence.

Since X is a complete metric space, there exists $z_0 \in X$ such that $\lim\limits_{n\longrightarrow+\infty} x_n = z_0$.

Step 4: z_0 is a fixed point for T:

Select an arbitrary integer $n \geq 1$ and using the triangle inequality, we obtain

$$d(z_0,Tz_0) \leq d(z_0,x_{n+1}) + d(Tz_0,x_{n+1}).$$
$$= d(z_0,x_{n+1}) + d(Tz_0,Tx_n).$$

Without loss of generality, we assume that $x_n \neq z_0$. Thus

$$d(z_0,Tz_0) \leq d(z_0,x_{n+1}) + d(Tz_0,x_{n+1}).$$
$$\leq d(z_0,x_{n+1}) + \tfrac{1}{2}(d(x_n,x_{n+1}) + d(z_0,Tz_0)).$$

Similarly

$$0 \leq \tfrac{1}{2}d(z_0, Tz_0) \leq d(z_0, x_{n+1}) + \tfrac{1}{2}d(x_n, x_{n+1}).$$

Letting $n \longrightarrow +\infty$, we deduce

$$0 \leq \tfrac{1}{2}d(z_0, Tz_0) \leq \lim_{n \longrightarrow +\infty} d(z_0, x_{n+1}) + \frac{1}{2} \lim_{n \longrightarrow +\infty} d(x_n, x_{n+1}) = 0.$$

Hence $d(z_0, Tz_0) = 0$. Consequently, z_0 is a fixed point for T which achieves the proof.

REMARK 9. Following the same way given in Theorem 7, we can establish a variants of Corollary 1 and Theorem 6 for the case of large-Kannan mappings (in the non-necessarily continuous sense).

3 Applications

In this last section, we prove that our results established in the previous section enable us to solve some implicit functional integral equations.

Let (8) be the integral equation formulated as a fixed point problem of the following nonlinear mapping

$$Tx(t) = \gamma x(t) + \int_{-1}^{t} \kappa(t, s, x(s), Tx(s))\, ds \text{ where } 0 < \gamma < \frac{1}{3}, \tag{8}$$

in Banach space $E = C([-1,1], \mathbb{R})$ of scalar continuous functions where the investigation is essentially based on the properties of the kernel $\kappa(\cdot,\cdot,\cdot,\cdot)$.

Under the following assumptions:

1. $\kappa(t, s, x(s), Tx(s)) \geq 0$ for $t, s \in [-1,1]$ such that $\kappa(\cdot,\cdot,0,\cdot) \neq 0$ and $T(M) \subseteq M$ where $M = \{z \in E : -1 \leq z(t) \leq 1\}$.
2. The mapping G defined by $Gx(t) = \displaystyle\int_{-1}^{t} \kappa(t, s, x(s), Tx(s))\, ds$ satisfies $Gx \in M$ for all $x \in M$ and
$$\|Gx - Gy\| < (1-\gamma)\|x - y\|, \quad \forall x, y \in M, \quad (x \neq y).$$
3. For a given $\epsilon > 0$, there exists $\delta < \frac{1-3\gamma}{2}$ such that if $x, y \in M$ and $\|x - y\| \geq \epsilon$, we have for all $t \in [-1,1]$
$$|Gx(t) - Gy(t)| \leq \delta(|x(t) - Tx(t)| + |y(t) - Ty(t)|).$$

Then T has unique fixed point in M.

PROOF. We have

$$x(t) - Tx(t) = (1-\gamma)x(t) - \int_{-1}^{t} \kappa(t, s, x(s), Tx(s))\, ds.$$

Let $x, y \in M$ with $\|x-y\| \geq \epsilon$. Then, by using our assumptions, we get

$$\begin{aligned}
& |Tx(t) - Ty(t)| \\
= & \left|\gamma(x(t) - y(t)) + \int_{-1}^{t} \kappa(t, s, x(s), Tx(s))\, ds - \int_{-1}^{t} \kappa(t, s, y(s), Ty(s))\, ds\right| \\
\leq & \gamma |x(t) + Tx(t) - Tx(t) + Ty(t) - Ty(t) - y(t)| \\
& + \left|\int_{-1}^{t} [\kappa(t, s, x(s), Tx(s)) - \kappa(t, s, y(s), Ty(s))]\, ds\right| \\
\leq & \gamma(\|x - Tx\| + \|y - Ty\| + \|Tx - Ty\|) \\
& + \delta(\|x - Tx\| + \|y - Ty\|),
\end{aligned}$$

which gives that

$$\|Tx - Ty\| \leq \gamma \|Tx - Ty\| + (\gamma + \delta)(\|x - Tx\| + \|y - Ty\|).$$

Hence

$$\|Tx - Ty\| \leq \left(\frac{\gamma + \delta}{1 - \gamma}\right)(\|x - Tx\| + \|y - Ty\|).$$

Now, since $0 < \delta < \frac{1-3\gamma}{2}$, then $\frac{\gamma+\delta}{1-\gamma} < \frac{1}{2}$ and the result is an immediate consequence of Theorem 4.

REMARK 10. In equation (8), if T is a large-Kannan mapping (in the non-necessarily continuous sense) which is satisfied under assumptions (1) and (3) and if (2) is replaced by the following;

(2′) The mapping G defined by $Gx(t) = \int_{-1}^{t} \kappa(t, s, x(s), Tx(s))\, ds$ satisfies that $Gx \in M$ for all $x \in M$ and for all $x, y \in M$ with $x \neq y$, we have

$$\|Gx - Gy\| < \tfrac{1-3\gamma}{2}(\|x - Gx\| + \|y - Gy\|).$$

Thus, by the same reasoning given above and using Theorem 7, we prove that the equation (8) has a unique solution.

REMARK 11. If G is continuous and $G(M)$ is a compact set in M, then the fixed point of T can be deduced as an immediate consequence of Krasnoselskii's theorem since the mapping $A : M \to M$ defined by $A\psi = \gamma\psi$ is a contraction and satisfies that $A(M) \subset M$.

4 Conclusion

In applications, the implicit functional equation (8) is related to a large class of interesting problems. Several authors have studied important properties of their solutions (stability, controlability, ...). A major problem appears when the inversion of a perturbed differential operator does not yield a contraction and a compact mapping or when it is hard to check this fact, since in this situation the classical Krasnoselskii's fixed point result or analog does not apply. To overcome this constrainst, we solve this equation by treating the continuous and non-continuous cases provided that the kernel or its associated mapping satisfies large-Kannan assumptions which is the main motivation of this work.

Acknowledgment. The authors would like to thank the anonymous referees for their valuable remarks and suggestions which help us to improve the paper.

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