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Stability Analysis Of Neutral Stochastic Differential Equations With Poisson Jumps And Variable Delays*

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Abstract

In this paper, we prove some results on the mean square asymptotic stability of the zero solution for a class of neutral stochastic differential with Poisson jumps and variable delays by using a contraction mapping principle. A mean square asymptotic stability theorem with a necessary and sufficient condition is proved, which improves and generalizes some previous results due to Dianli Zhao [35]. Finally, an example is exhibited to illustrate the effectiveness of the proposed results.

1 Introduction

In nature, physics, society, engineering, and so on we always meet two kinds of functions with respect to time: one is deterministic and another is random. Stochastic differential equations were first initiated and developed by K. Itô [9]. Today they have become a very powerful tool applied to mathematics, physics, biology, finance, and so forth. Real systems depend on not only present and past states but also involve derivatives with delays. As a result, these systems are often built in the form of neutral differential equations. Practical examples of neutral delay differential systems include the distributed networks containing lossless transmission lines [4], population ecology [15], and other engineering systems [13]. For neutral stochastic delay differential equations, we refer to [14, 22, 27].

It is well known that Lyapunov's method has been the classical technique to study stability of deterministic and stochastic differential equations and functional differential equations for more than 100 years, for example [26, 27]. However, there are a lot of difficulties to construct Lyapunov functions for examining stability. Burton in the monograph [3] and the works [1, 2, 5, 10, 11, 29, 35, 37, 38] have successfully applied fixed point theory to overcome these problems. In addition, there are some papers where the fixed point theory is used to investigate the stability of stochastic (delayed) differential equations (see for instance [17, 18, 20, 21, 33]). More precisely, Luo [17] studied the mean square asymptotic stability for a class of linear scalar neutral stochastic differential equations by means of fixed point theory. Furthermore, Luo [18–19]

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firstly considers the exponential stability for stochastic partial differential equations with delays by the fixed point method. In [30, 31] the fixed point theory is used to discuss the asymptotic stability in pth moment of mild solutions to nonlinear impulsive stochastic partial differential equations with bounded delays and infinite delays. Wu et al. [33] applied fixed point theory to study the stability of a class of nonlinear neutral stochastic differential equations with variable time delays.

It often happens in real problems that a stochastic system jumps from a "normal state" or "good state" to a "bad state," and the strength of system is random. For this class of systems, it is natural and necessary to include a jump term in them. The effect of Poisson jumps should be taken into account when studying the stability of stochastic differential equations (see for exemple [6, 7, 16, 28, 36]). Therefore, except stochastic and delay effects, Poisson jumps' effects are likely to exist widely in a variety of evolution processes in which states are changed abruptly at some moments of time, including such fields as finance, economy, medicine, electronics, and so forth. Then, it is natural to consider the effect of Poisson jumps when studying the stability of stochastic delayed differential equations.

So far, these topics have received a lot of attention and there are so many references about them. In detail, Guo and Zhu [7] studied the stochastic Voltera-Levin equation with Poisson jumps, and they obtained pth moment stability of this equations. In addition, Guo and Zhu have generalized and extended their works in [10]. Based on fixed point theory, Chen et al. in [6] proved that the mild solution to a class of impulsive stochastic differential equations with delays and Poisson jumps is not only existent and unique but also pth moment exponentially stable. Recently, Liu et al. [16], applied fixed point theory to study the mean square asymptotic stability for a class of nonlinear neutral stochastic differential equations with Poisson jumps and variable delays.

Motived by previous work mentioned above, in this paper we address the mean square asymptotic stability for a linear neutral stochastic differential equation with variable delays and Poisson jumps. An asymptotic mean square stability theorem with a necessary and sufficient condition is proved. Some well-known results are improved and generalized. More precisely, our model contains as a particular case the one analysed in Dianli Zhao [35], and therefore we ensure the validity of those results.

The content of this paper is organized as follows. In Section 2, we recall some results which are necessary for our analysis. In Section 3, we give the main result about mean square asymptotic stability and its proof. In Section 4, an illustrative example is analyzed to test our theory and our method. The last Section is the conclusion.

2 Model Description and Preliminaries

Throughout this paper, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with some filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, i.e., the filtration is continuous on the right and $\mathcal{F}_0$ contains all $\mathbb{P}$-zero sets. Let $\{w(t), t \geq 0\}$ denote a standard one-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Also, let $\widetilde{N}(t) := N(t) - \beta t$, where $\beta \in \mathbb{R}^+$ and $N(t)$ is a stationary $\mathcal{F}_t$-Poisson point process with intensity β. Obviously, $\widetilde{N}(t)$ is a compensated Poisson process. Here $C(S_1, S_2)$ denotes the set of all continuous functions $\phi : S_1 \to S_2$ with the supremum norm. Finally, $\mathbb{E}$ will denote expectation.

In this paper, we consider the linear neutral stochastic differential equation with variable delays and Poisson jumps:

$$
\begin{aligned}
& d\left(x(t)-\frac{c\left(t\right)}{1-\tau_1'\left(t\right)}x\left(t-\tau_1\left(t\right)\right)\right) \\
= & \left(-a(t)x(t-\tau_1\left(t\right))-\frac{d}{dt}\left(\frac{c\left(t\right)}{1-\tau_1'\left(t\right)}\right)x\left(t-\tau_1\left(t\right)\right)\right)dt \\
& +\Sigma\left(t\right)x(t-\tau_2\left(t\right))dw\left(t\right)+\Gamma\left(t\right)x(t-\tau_3\left(t\right))d\widetilde{N}\left(t\right),\ t\geq t_0, \qquad (1)
\end{aligned}
$$

denote $x\left(t\right)\in\mathbb{R}$ the solution to (1) with the intial condition

$$x(t)=\psi\left(t\right)\ \text{ for } t\in\left[m\left(t_0\right),t_0\right], \qquad (2)$$

and $\psi\in C\left(\left[m\left(t_0\right),t_0\right],\mathbb{R}\right),$ where $a,b,\Sigma,\Gamma\in C\left(\mathbb{R}^+,\mathbb{R}\right),$ $c\in C^1\left(\mathbb{R}^+,\mathbb{R}\right)$ and $\tau_i\in C\left(\mathbb{R}^+,\mathbb{R}^+\right)$ satisfy $t-\tau_i\left(t\right)\to\infty$ as $t\to\infty$, $i=1,2,3$ and for each $t_0\geq 0$,

$$m_i\left(t_0\right)=\inf\left\{t-\tau_i\left(t\right),t\geq t_0\right\},m\left(t_0\right)=\min\left\{m_i\left(t_0\right),i=1,2,3\right\}. \qquad (3)$$

Special cases of equation (1) have been investigated by many authors. For example, Ardjouni and Djoudi in [1], Dianli Zhao in [35] studied the equation

$$x'(t)=-a(t)x(t-\tau_1\left(t\right))+c\left(t\right)x'(t-\tau_1\left(t\right)), \qquad (4)$$

and have respectively proved the following theorems.

THEOREM A (Ardjouni and Djoudi [1]). Suppose that τ_1 is twice differentiable and $\tau_1'\left(t\right)\neq 1$ for all $t\in\mathbb{R}^+$, and there exists a continuous function $h:\left[m\left(0\right),\infty\right[\to\mathbb{R}$ and a constant $\alpha\in(0,1)$ such that for $t\geq 0$,

$$\liminf_{t\to\infty}\int_0^t h(s)ds>-\infty,$$

and

$$
\begin{aligned}
& \left|\frac{c\left(t\right)}{1-\tau_1'\left(t\right)}\right|+\int_{t-\tau_1(t)}^t\left|h\left(s\right)\right|ds+\int_0^t e^{-\int_s^t h(u)du}\left|h\left(s\right)\right|\left(\int_{s-\tau_1(s)}^s\left|h\left(u\right)\right|du\right)ds \\
& +\int_0^t e^{-\int_s^t h(u)du}\left|a\left(s\right)+h\left(s-\tau_1\left(s\right)\right)\left(1-\tau_1'\left(s\right)\right)-r\left(s\right)\right|ds\leq\alpha,
\end{aligned}
$$

where

$$r\left(t\right)=\frac{\left(c\left(t\right)h\left(t\right)+c'(t)\right)\left(1-\tau_1'\left(t\right)\right)+c(t)\tau_1''\left(t\right)}{\left(1-\tau_1'\left(t\right)\right)^2}.$$

Then the zero solution of (4) is asymptotically stable if and only if

$$\int_0^t h(s)ds\to\infty\ as\ t\to\infty.$$

Under sufficient conditions, Dianli Zhao in [35] has addressed new criteria for asymptotic stability of equation (4) by using the Banach fixed point theorem method. The author obtain one result as follows.

THEOREM B (Dianli Zhao [35]). Let τ_1 be twice differentiable and suppose that $\tau_1'(t) \neq 1$ for all $t \in [m(t_0), \infty[$. Suppose that

(i) there exists a continuous function $h : [m(t_0), \infty[\to \mathbb{R}$ satisfying $\int_{t_0}^{t} h(s)ds \to \infty$ as $t \to \infty$;

(ii) there exists a bounded function $p : [m(t_0), \infty[\to (0, \infty)$ with $p(t) = 1$ for $t \in [m(t_0), t_0]$ and $p'(t)$ exists for all $t \in [m(t_0), \infty[$;

(iii) there exists a constant $\alpha \in (0, 1)$ such that for any $t \geq t_0$,

$$\begin{aligned}
&\left|\frac{c(t)\, p(t-\tau_1(t))}{p(t)\,(1-\tau_1'(t))}\right| + \int_{t-\tau_1(t)}^{t} \left| h(s) - \frac{p'(s)}{p(s)} \right| ds \\
&+ \int_0^t e^{-\int_s^t h(u)du} |h(s)| \left(\int_{s-\tau_1(s)}^{s} \left| h(u) - \frac{p'(u)}{p(u)} \right| du \right) ds \\
&+ \int_0^t e^{-\int_s^t h(u)du} \Bigg| -\overline{b}(s) + \left(h(s-\tau_1(s)) - \frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))} \right) (1-\tau_1'(s)) \\
&-\overline{k}(s) \Bigg| ds \leq \alpha, \qquad (5)
\end{aligned}$$

where

$$\overline{k}(t) = \frac{[C(t)\, h(t) + C'(t)]\,(1-\tau_1'(t)) + C(t)\, \tau_1''(t)}{(1-\tau_1'(t))^2}, \qquad (6)$$

$$\overline{b}(t) = \frac{a(t)\, p(t-\tau_1(t)) - c(t)p'(t-\tau_1(t))}{p(t)} \quad \text{and} \quad C(t) = \frac{c(t)\, p(t-\tau_1(t))}{p(t)}. \qquad (7)$$

Then the zero solution of (4) is asymptotically stable.

For each $t_0 \geq 0$ and $\psi \in C([m(t_0), t_0], \mathbb{R})$ fixed, we define X_ψ^l the complete metric space of all $\mathcal{F}_t$−adapted processes $x(t, \omega) : [m(t_0), \infty) \times \Omega \to \mathbb{R}$, which is almost surely continuous in t for fixed $\omega \in \Omega$ as follows

$$\begin{aligned}
X_\psi^l \quad = \quad &\Big\{ x(t,\omega) : [m(t_0), \infty) \times \Omega \to \mathbb{R}/\ x(t,.) = \psi(t) \text{ for } t \in [m(t_0), t_0] \\
&\|x\|_X \leq l \text{ for } t \geq t_0 \text{ and } \mathbb{E}\,|x(t)|^2 \to 0 \text{ as } t \to \infty \Big\}
\end{aligned}$$

with $\|x\|_X := \left(\mathbb{E} \left(\sup_{t \geq m(t_0)} |x(t)|^2 \right) \right)^{1/2}$, where $l > 0$ is positive number.

Let us now recall the definitions of stability that will be used in the next section.

DEFINITION 1.1. The zero solution of (1) is said to be:

i) stable if for any $\varepsilon > 0$ and $t_0 \geq 0$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that $\psi \in C([m(t_0), t_0], \mathbb{R})$ and $\|\psi\| < \delta$ imply $\mathbb{E}|x(t, t_0, \psi)|^2 < \varepsilon$ for $t \geq t_0$.

ii) asymptotically stable if the zero solution is stable and for any $\varepsilon > 0$ and $t_0 \geq 0$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that $\psi \in C([m(t_0), t_0], \mathbb{R})$ and $\|\psi\| < \delta$ imply $\mathbb{E}|x(t, t_0, \psi)|^2 \to 0$ as $t \to \infty$.

REMARK 2.1. When $p(t) = 1$ and sufficient conditions for stability are presented by the fixed point theory, Theorem B becomes theorem A.

Some of the results, such as Theorem A, depend on the constraint $\left|\frac{c(t)}{1-\tau_1'(t)}\right| < 1$. But in many environments, there are interesting examples where the constraint is not satisfied. However, in Theorem B, this condition is removed. Our objective here is to improve Theorem B and extend it to investigate a wide class of stochastic neutral differential equation with Poisson jumps and variable delays presented in (1). The stability problem for this class of equations has not yet been solved since Poisson jumps are considered. To this end, in this paper we make the first attempt to fill this gap and study mean square asymptotic stablity of (1) by fixed point theory. In particular, by employing two auxiliary continuous functions on the contraction condition, we get new criteria which can be applied in the case $\left|\frac{c(t)}{1-\tau_1'(t)}\right| \geq 1$ as well. The results of this article are new and they extend and improve previously known results in [35]. Finally, an illustrative example is given.

3 Main Results

For each $t_0 \in \mathbb{R}^+$, $C([m(t_0), t_0], \mathbb{R})$ is emdowed with the supremum norm

$$\|\psi\| = \max\{|\psi(s)| : t_0 \leq s \leq m(t_0)\}.$$

For each $(t_0, \psi) \in \mathbb{R}^+ \times C([m(t_0), t_0], \mathbb{R})$, a solution of (1) through (t_0, ψ) is a continuous function $x : [m(t_0), t_0 + \sigma) \to \mathbb{R}$ for some positive constant $\sigma > 0$ such that x satisfies (1) on $[t_0, t_0 + \sigma)$ and $x = \psi$ on $[m(t_0), t_0]$. We denote such a solution by $x(t) = x(t, t_0, \psi)$.

Our purpose here is to extend the work carried out in [35] by providing a necessary and sufficient condition for asymptotic stability of the zero solution of equation (1). B. Zhang [37, 38] was the first to establish necessary and sufficient condition for the stability of solutions of functional differential equation by the fixed point theory. The necessity of condition (10) below for the main stability result was first established in [37]. It is well known that studying the stability of an equation using a fixed point technique involves the construction of a suitable fixed point mapping. So, we construct a contraction mapping Q on a complete metric space X_ψ^l defined above, which may depend on the initial condition ψ. Using Banach's contraction mapping principle, we obtain a fixed point for this mapping and hence a solution for (1), which in addition is mean square asymptotically stable.

Now, we can state our main result.

THEOREM 3.1. Let τ_1 be twice differentiable and suppose that $\tau_1'(t) \neq 1$ for all $t \in [m(t_0), \infty[$. Suppose that

(i) there exists a bounded function $p : [m(t_0), \infty[\to (0, \infty)$ with $p(t) = 1$ for $t \in [m(t_0), t_0]$ and $p'(t)$ exists for all $t \in [m(t_0), \infty[$, and there exists an arbitrary continuous function $h : [m(t_0), \infty[\to \mathbb{R}$ and a constant $\gamma \in \left(0, \frac{1}{4}\right)$ such that for any $t \geq t_0$,

$$\left[\left|\frac{p(t-\tau_1(t))}{p(t)}\frac{c(t)}{(1-\tau_1'(t))}\right| + \int_{t-\tau_1(t)}^{t}\left|h(s) - \frac{p'(s)}{p(s)}\right| ds\right.$$
$$+\int_{t_0}^{t} e^{-\int_s^t h(u)du} |h(s)| \left(\int_{s-\tau_1(s)}^{s}\left|h(u) - \frac{p'(u)}{p(u)}\right| du\right) ds$$
$$+\int_{t_0}^{t} e^{-\int_s^t h(u)du}\left| -\overline{b}(s) + \left(h(s-\tau_1(s)) - \frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))}\right)(1-\tau_1'(s))\right.$$
$$\left.\left.-\overline{k}(s)\right| ds\right]^2 + 4\int_{t_0}^{t} e^{-2\int_s^t h(u)du}\left|\frac{\Sigma(s)\,p(s-\tau_2(s))}{p(s)}\right|^2 ds$$
$$+4\beta\int_{t_0}^{t} e^{-2\int_s^t h(u)du}\left|\frac{\Gamma(s)\,p(s-\tau_3(s))}{p(s)}\right|^2 ds \leq \gamma, \tag{8}$$

where $\overline{b}(s)$ and $\overline{k}(s)$ are defined as in (6) and (7);

(ii) and such that
$$\liminf_{t\to\infty} \int_{t_0}^{t} h(s)ds > -\infty. \tag{9}$$

Then the zero solution of (1) is mean-square asymptotic stable if and only if
$$\int_{t_0}^{t} h(s)ds \to \infty \text{ as } t \to \infty. \tag{10}$$

PROOF. The technique for constructing a contraction mapping comes from an idea in [35]. Let $z(t) = \psi(t)$ on $t \in [m(t_0), t_0]$, and for $t \geq t_0$,
$$x(t) = p(t)z(t). \tag{11}$$

Make substitution of (11) into (1) to show
$$z'(t) = -\frac{p'(t)}{p(t)}z(t) - \frac{a(t)p(t-\tau_1(t)) - c(t)\,p'(t-\tau_1(t))}{p(t)}z(t-\tau_1(t))$$

$$+\frac{c(t)\,p(t-\tau_1(t))}{p(t)}z'(t-\tau_1(t))$$
$$+\frac{\Sigma(t)\,p(t-\tau_2(t))}{p(t)}z(t-\tau_2(t))dw(t)$$
$$+\frac{\Gamma(t)\,p(t-\tau_3(t))}{p(t)}z(t-\tau_3(t))d\widetilde{N}(t),\ \ t\geq t_0, \tag{12}$$

then it can be verified that x satisfies (1). Since p is a bounded function and $\mathbb{E}\left|x(t)\right|^2 = p^2(t)\mathbb{E}\left|z(t)\right|^2$, to obtain mean square asymptotically stable of the zero solution of (1), it remains to prove that the zero solution of (12) is asymptotically stable. Multiply both sides of (12) by $\exp\left(-\int_{t_0}^{t}h(s)ds\right)$, and then inegrate from t_0 to t, we have

$$\begin{aligned}
z(t) \;=\;& \psi(t_0)\,e^{-\int_{t_0}^{t}h(s)ds}+\int_{t_0}^{t}\left(h(s)-\frac{p'(s)}{p(s)}\right)e^{-\int_{s}^{t}h(u)du}z(s)ds \\
&-\int_{t_0}^{t}e^{-\int_{s}^{t}h(u)du}\frac{a(s)p(s-\tau_1(s))-c(s)\,p'(s-\tau_1(s))}{p(s)}z(s-\tau_1(s))ds \\
&+\int_{t_0}^{t}e^{-\int_{s}^{t}h(u)du}\frac{c(s)\,p(s-\tau_1(s))}{p(s)}z'(s-\tau_1(s))ds \\
&+\int_{t_0}^{t}e^{-\int_{s}^{t}h(u)du}\frac{\Sigma(s)\,p(s-\tau_2(s))}{p(s)}z(s-\tau_2(s))dw(s) \\
&+\int_{t_0}^{t}e^{-\int_{s}^{t}h(u)du}\frac{\Gamma(s)\,p(s-\tau_3(s))}{p(s)}z(s-\tau_3(s))d\widetilde{N}(s).
\end{aligned} \tag{13}$$

Rewrite (13) in the following equivalent form

$$\begin{aligned}
z(t) \;=\;& \psi(t_0)\,e^{-\int_{t_0}^{t}h(s)ds}+\int_{t_0}^{t}e^{-\int_{s}^{t}h(u)du}d\left(\int_{s-\tau_1(s)}^{s}\left(h(u)-\frac{p'(u)}{p(u)}\right)z(u)\,du\right) \\
&+\int_{t_0}^{t}e^{-\int_{s}^{t}h(u)du}\left(h(s-\tau_1(s))-\frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))}\right)(1-\tau_1'(s))\,z(s-\tau_1(s))ds \\
&-\int_{t_0}^{t}e^{-\int_{s}^{t}h(u)du}\frac{a(s)p(s-\tau_1(s))-c(s)\,p'(s-\tau_1(s))}{p(s)}z(s-\tau_1(s))ds \\
&+\int_{t_0}^{t}e^{-\int_{s}^{t}h(u)du}\frac{c(s)\,p(s-\tau_1(s))}{p(s)}z'(s-\tau_1(s))ds \\
&+\int_{t_0}^{t}e^{-\int_{s}^{t}h(u)du}\frac{\Sigma(s)\,p(s-\tau_2(s))}{p(s)}z(s-\tau_2(s))dw(s)
\end{aligned}$$

$$+\int_{t_0}^{t} e^{-\int_s^t h(u)du} \frac{\Gamma(s)\, p(s-\tau_3(s))}{p(s)} z(s-\tau_3(s)) d\widetilde{N}(s).$$

Performing an integration by parts, we have for $t \geq t_0$,

$$\begin{aligned} z(t) &= \Bigg(\psi(t_0) - \frac{p(t_0-\tau_1(t_0))}{p(t_0)} \frac{c(t_0)}{(1-\tau_1'(t_0))} \psi(t_0-\tau_1(t_0)) \\ &\quad - \int_{t_0-\tau_1(t_0)}^{t_0} \left(h(u) - \frac{p'(u)}{p(u)} \right) z(u)du \Bigg) e^{-\int_{t_0}^t h(s)ds} \\ &\quad + \frac{p(t-\tau_1(t))}{p(t)} \frac{c(t)}{1-\tau_1'(t)} z(t-\tau_1(t)) \\ &\quad + \int_{t-\tau_1(t)}^{t} \left(h(s) - \frac{p'(s)}{p(s)} \right) z(s)ds + \int_{t_0}^t e^{-\int_s^t h(u)du} \Bigg\{ -\overline{b}(s) \\ &\quad + \left(h(s-\tau_1(s)) - \frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))} \right)(1-\tau_1'(s)) - \overline{k}(s) \Bigg\} \\ &\quad \times z(s-\tau_1(s))ds \\ &\quad - \int_{t_0}^{t} e^{-\int_s^t h(u)du} h(s) \left(\int_{s-\tau_1(s)}^{s} \left(h(u) - \frac{p'(u)}{p(u)} \right) z(u)du \right) ds \\ &\quad + \int_{t_0}^{t} e^{-\int_s^t h(u)du} \frac{\Sigma(s)\, p(s-\tau_2(s))}{p(s)} z(s-\tau_2(s)) dw(s) \\ &\quad + \int_{t_0}^{t} e^{-\int_s^t h(u)du} \frac{\Gamma(s)\, p(s-\tau_3(s))}{p(s)} z(s-\tau_3(s)) d\widetilde{N}(s), \end{aligned} \tag{14}$$

where $\overline{b}(s)$ and $\overline{k}(s)$ are defined as in (6) and (7) respectively.

Use (14) to define an operator $Q : X_\psi^l \to X_\psi^l$ as follows:

$$(Qz)(t) := \begin{cases} \psi(t), & t \in [m(t_0), t_0], \\ \sum_{i=1}^{7} I_i(t), & t \geq t_0, \end{cases} \tag{15}$$

where

$$\begin{aligned} I_1(t) &= \Bigg(\psi(t_0) - \frac{p(t_0-\tau_1(t_0))}{p(t_0)} \frac{c(t_0)}{(1-\tau_1'(t_0))} \psi(t_0-\tau_1(t_0)) \\ &\quad - \int_{t_0-\tau_1(t_0)}^{t_0} \left(h(u) - \frac{p'(u)}{p(u)} \right) z(u)du \Bigg) e^{-\int_{t_0}^t h(s)ds}, \end{aligned}$$

$$I_2(t) = \frac{p(t-\tau_1(t))}{p(t)} \frac{c(t)}{1-\tau_1'(t)} z(t-\tau_1(t)),$$

$$I_3(t) = \int_{t-\tau_1(t)}^{t} \left(h(s) - \frac{p'(s)}{p(s)} \right) z(s)ds,$$

$$\begin{aligned} I_4(t) \quad &= \quad \int_{t_0}^{t} e^{-\int_s^t h(u)du} \Bigg\{ -\bar{b}(s) + \left(h(s-\tau_1(s)) - \frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))} \right)(1-\tau_1'(s)) \\ &\quad -\bar{k}(s) \Bigg\} z(s-\tau_1(s))ds, \end{aligned}$$

$$I_5(t) = \int_{t_0}^{t} e^{-\int_s^t h(u)du} h(s) \left(\int_{s-\tau_1(s)}^{s} \left(h(u) - \frac{p'(u)}{p(u)} \right) z(u)du \right) ds,$$

$$I_6(t) = \int_{t_0}^{t} e^{-\int_s^t h(u)du} \frac{\Sigma(s)\, p(s-\tau_2(s))}{p(s)} z(s-\tau_2(s))dw(s),$$

$$I_7(t) = \int_{t_0}^{t} e^{-\int_s^t h(u)du} \frac{\Gamma(s)\, p(s-\tau_3(s))}{p(s)} z(s-\tau_3(s))d\widetilde{N}(s).$$

Now we split the rest of our proof into three steps.

First step: We need to prove that $Q\left(X_\psi^l\right) \subset X_\psi^l$. For $z \in X_\psi^l$, it is necessary to show that $Q(z) \in X_\psi^l$. At first, we suppose that (10) holds. From (9), it is true that $\exp\left(-\int_{t_0}^t h(s)ds\right)$ is bounded, which is denoted by

$$M = \sup_{t \geq t_0} \left\{ e^{-\int_{t_0}^t h(s)ds} \right\}.$$

We must prove the mean square continuity of Q on $[t_0, \infty)$. It is clear that Q is continuous on $[m(t_0), t_0]$. For fixed time $t_1 \geq t_0$, $z \in X_\psi^l$, and $|r|$ sufficiently small, we have

$$\mathbb{E}\left|(Qz)(t_1+r) - (Qz)(t_1)\right|^2 \leq 7\sum_{i=1}^{7} \mathbb{E}\left|I_i(t_1+r) - I_i(t_1)\right|^2.$$

It is easy to obtain that

$$\mathbb{E}\left|I_i(t_1+r) - I_i(t_1)\right|^2 \to 0, \text{ as } r \to 0, \ i = 1, ..., 5.$$

Furthermore,

$$\mathbb{E}\left|I_6(t_1+r) - I_6(t_1)\right|^2$$

$$\begin{aligned}
&\leq 2\mathbb{E}\left|\int_{t_0}^{t_1}\left(e^{-\int_{t_1}^{t_1+r}h(u)du}-1\right)e^{-\int_0^{t_1}h(u)du}\frac{\Sigma(s)\,p(s-\tau_2(s))z(s-\tau_2(s))}{p(s)}dw(s)\right|^2 \\
&\quad +2\mathbb{E}\left|\int_{t_1}^{t_1+r}e^{-\int_s^{t_1+r}h(u)du}\frac{\Sigma(s)\,p(s-\tau_2(s))z(s-\tau_2(s))}{p(s)}dw(s)\right|^2 \\
&\leq 2\mathbb{E}\left(\int_{t_0}^{t_1}\left(e^{-\int_{t_1}^{t_1+r}h(u)du}-1\right)^2 e^{-2\int_0^{t_1}h(u)du}\left|\frac{\Sigma(s)\,p(s-\tau_2(s))z(s-\tau_2(s))}{p(s)}\right|^2 ds\right) \\
&\quad +2\mathbb{E}\left(\int_{t_1}^{t_1+r}e^{-2\int_s^{t_1+r}h(u)du}\left|\frac{\Sigma(s)\,p(s-\tau_2(s))z(s-\tau_2(s))}{p(s)}\right|^2 ds\right)\to 0,
\end{aligned}$$

as $r\to\infty$, and

$$\begin{aligned}
&\mathbb{E}\left|I_7(t_1+r)-I_7(t_1)\right|^2 \\
&\leq 2\mathbb{E}\left|\int_{t_0}^{t_1}\left(e^{-\int_{t_1}^{t_1+r}h(u)du}-1\right)e^{-\int_0^{t_1}h(u)du}\frac{\Gamma(s)\,p(s-\tau_3(s))}{p(s)}z(s-\tau_3(s))d\widetilde{N}(s)\right|^2 \\
&\quad +2\mathbb{E}\left|\int_{t_1}^{t_1+r}e^{-\int_s^{t_1+r}h(u)du}\frac{\Gamma(s)\,p(s-\tau_3(s))}{p(s)}z(s-\tau_3(s))d\widetilde{N}(s)\right|^2 \\
&\leq 2\beta\mathbb{E}\left(\int_{t_0}^{t_1}\left(e^{-\int_{t_1}^{t_1+r}h(u)du}-1\right)^2 e^{-2\int_0^{t_1}h(u)du}\left|\frac{\Gamma(s)\,p(s-\tau_3(s))}{p(s)}z(s-\tau_3(s))\right|^2 ds\right) \\
&\quad +2\beta\mathbb{E}\left(\int_{t_1}^{t_1+r}e^{-2\int_s^{t_1+r}h(u)du}\left|\frac{\Gamma(s)\,p(s-\tau_3(s))}{p(s)}z(s-\tau_3(s))\right|^2 ds\right)\to 0,
\end{aligned}$$

as $r\to 0$.

Therefore, Q is mean square continuous on $[t_0,\infty[$.

We verify that $\mathbb{E}\left|(Qz)(t)\right|^2\to 0$ as $t\to\infty$. Since $\mathbb{E}\left|z(t)\right|\to 0$, $t-\tau_i(t)\to\infty$ as $t\to\infty$, $i=1,2,3$, for each $\varepsilon>0$, there exists a $T_1>0$, such that $t\geq T_1$, implies $\mathbb{E}\left|z(t)\right|^2<\varepsilon$ and $\mathbb{E}\left|z(t-\tau_i(t))\right|^2<\varepsilon$, $i=1,2,3$. Hence

$$\begin{aligned}
\mathbb{E}\left|I_6(t)\right|^2 &\leq \mathbb{E}\left(\int_{t_0}^{T_1}e^{-2\int_s^t h(u)du}\left|\frac{\Sigma(s)\,p(s-\tau_2(s))}{p(s)}\right|^2\left|z(s-\tau_2(s))\right|^2 ds\right) \\
&\quad +\mathbb{E}\left(\int_{T_1}^{t}e^{-2\int_s^t h(u)du}\left|\frac{\Sigma(s)\,p(s-\tau_2(s))}{p(s)}\right|^2\left|z(s-\tau_2(s))\right|^2 ds\right) \\
&\leq \mathbb{E}\left(\sup_{\sigma\geq m(t_0)}\left|z(\sigma)\right|^2\right)\int_{t_0}^{T_1}e^{-2\int_s^t h(u)du}\left|\frac{\Sigma(s)\,p(s-\tau_2(s))}{p(s)}\right|^2 ds \\
&\quad +\varepsilon\int_{T_1}^{t}e^{-2\int_s^t h(u)du}\left|\frac{\Sigma(s)\,p(s-\tau_2(s))}{p(s)}\right|^2 ds.
\end{aligned}$$

Due to condition (10), there is $T' > T_1$, such that when $t \geq T'$, we obtain

$$\mathbb{E}\left(\sup_{\sigma\geq m(t_0)}|z(\sigma)|^2\right)\int_{t_0}^{T_1} e^{-2\int_s^t h(u)du}\left|\frac{\Sigma(s)\,p(s-\tau_2(s))}{p(s)}\right|^2 ds \leq (1-\gamma)\,\varepsilon.$$

By condition (8), we have $\mathbb{E}\left|I_6(t)\right|^2 < (1-\gamma)\,\varepsilon+\gamma\varepsilon = \varepsilon$. Thus $\mathbb{E}\left|I_6(t)\right|^2 \to 0$ as $t \to \infty$. Similarly,

$$\begin{aligned}
\mathbb{E}\left|I_7(t)\right|^2 &\leq \beta\mathbb{E}\left(\int_{t_0}^{T_1} e^{-2\int_s^t h(u)du}\left|\frac{\Gamma(s)\,p(s-\tau_3(s))}{p(s)}\right|^2 |z(s-\tau_3(s))|^2\,ds\right)\\
&\quad+\beta\mathbb{E}\left(\int_{T_1}^{t} e^{-2\int_s^t h(u)du}\left|\frac{\Gamma(s)\,p(s-\tau_3(s))}{p(s)}\right|^2 |z(s-\tau_3(s))|^2\,ds\right)\\
&\leq \beta\mathbb{E}\left(\sup_{\sigma\geq m(t_0)}|z(\sigma)|^2\right)\int_{t_0}^{T_1} e^{-2\int_s^t h(u)du}\left|\frac{\Gamma(s)\,p(s-\tau_3(s))}{p(s)}\right|^2 ds\\
&\quad+\varepsilon\beta\int_{T_1}^{t} e^{-2\int_s^t h(u)du}\left|\frac{\Gamma(s)\,p(s-\tau_3(s))}{p(s)}\right|^2 ds\\
&< (1-\gamma)\,\varepsilon+\gamma\varepsilon=\varepsilon.
\end{aligned}$$

Thus $\mathbb{E}\left|I_7(t)\right|^2 \to 0$ as $t \to \infty$, and it is very easy to get $\mathbb{E}\left|I_i(t)\right|^2 \to 0$ as $t \to \infty$, $i = 1, 2, ..., 5$. This implies $\mathbb{E}\left|(Qz)(t)\right|^2 \to 0$ as $t \to \infty$.

Next, we verify that $\|Q(z)\|_X \leq l$. Let ψ be a small bounded initial function with $\|\psi\| < \delta$, we choose $\delta > 0$, ($\delta < l$) such that

$$\begin{aligned}
&4\delta M^2\left(1+\left|\frac{p(t_0-\tau_1(t_0))}{p(t_0)}\frac{c(t_0)}{(1-\tau_1'(t_0))}\right|+\textstyle\int_{t_0-\tau_1(t_0)}^{t_0}\left|h(u)-\frac{p'(u)}{p(u)}\right|du\right)^2\\
&< (1-4\gamma)\,l^2, \qquad (16)
\end{aligned}$$

where γ is the left hand side of (8). Let $z \in X_\psi^l$, then, $\|z\|_X < l$. It follows from (15), condition (8) in Theorem 3.1 and L^p-Doob inequality that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\geq m(t_0)}\left|(Qz)(t)\right|^2\right]\\
&\leq 4\left|\psi(t_0)\right|^2\left(1+\left|\frac{p(t_0-\tau_1(t_0))}{p(t_0)}\frac{c(t_0)}{(1-\tau_1'(t_0))}\right|\right.\\
&\quad\left.+\int_{t_0-\tau_1(t_0)}^{t_0}\left|h(u)-\frac{p'(u)}{p(u)}\right|du\right)^2 e^{-2\int_{t_0}^t h(s)ds}
\end{aligned}$$

$$
\begin{aligned}
& +4\mathbb{E}\left(\sup_{t\geq m(t_0)}|z(t)|^2\right)\left\{\sup_{t\geq t_0}\left[\left|\frac{p(t-\tau_1(t))}{p(t)}\frac{c(t)}{(1-\tau_1'(t))}\right|\right.\right. \\
& +\int_{t-\tau_1(t)}^{t}\left|h(s)-\frac{p'(s)}{p(s)}\right|ds \\
& +\int_{t_0}^{t}e^{-\int_s^t h(u)du}\left|-\bar{b}(s)+\left(h(s-\tau_1(s))-\frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))}\right)(1-\tau_1'(s))-\bar{k}(s)\right|ds \\
& \left.+\int_{t_0}^{t^*}e^{-\int_s^t h(u)du}|h(s)|\left(\int_{s-\tau_1(s)}^{s}\left|h(u)-\frac{p'(u)}{p(u)}\right|du\right)\ ds\right]^2 \\
& +4\sup_{t\geq t_0}\left(\int_{t_0}^{t}e^{-2\int_s^t h(u)du}\left|\frac{\Sigma(s)\,p(s-\tau_2(s))}{p(s)}\right|^2 ds\right) \\
& \left.+4\beta\sup_{t\geq t_0}\left(\int_{t_0}^{t}e^{-2\int_s^t h(u)du}\left|\frac{\Gamma(s)\,p(s-\tau_3(s))}{p(s)}\right|^2 ds\right)\right\} \\
\leq\ & 4\delta\left(1+\left|\frac{p(t_0-\tau_1(t_0))}{p(t_0)}\frac{c(t_0)}{(1-\tau_1'(t_0))}\right|\right. \\
& \left.+\int_{t_0-\tau_1(t_0)}^{t_0}\left|h(u)-\frac{p'(u)}{p(u)}\right|du\right)^2 e^{-2\int_{t_0}^t h(s)ds}+4\gamma l^2 \\
\leq\ & (1-4\gamma)\,l^2+4\gamma l^2=l^2.
\end{aligned}
$$

By (16), we see that

$$
\mathbb{E}\left[\sup_{t\geq m(t_0)}|(Qz)(t)|^2\right]\leq(1-4\gamma)\,l^2+4\gamma l^2=l^2.
$$

Hence, $\|Qz\|_X\leq l$ for $t\in[m(t_0),\infty)$ because $\|Qz\|_X=\|\psi\|\leq l$ for $t\in[m(t_0),t_0]$. Then $Q\left(X_\psi^l\right)\subset X_\psi^l$.

Second step: Now, we will show that Q has a unique fixed point z in X_ψ^l. For any $x,y\in X_\psi^l$,

$$
\begin{aligned}
& \mathbb{E}\left(\sup_{t\geq m(t_0)}|(Qx)(t)-(Qy)(t)|^2\right) \\
\leq\ & \mathbb{E}\left(\sup_{t\geq t_0}\left|\frac{p(t-\tau_1(t))}{p(t)}\frac{c(t)}{1-\tau_1'(t)}\left[x(t-\tau_1(t))-y(t-\tau_1(t))\right]\right.\right. \\
& +\int_{t-\tau_1(t)}^{t}\left(h(s)-\frac{p'(s)}{p(s)}\right)[x(s)-y(s)]\,ds
\end{aligned}
$$

$$+\int_{t_0}^{t} e^{-\int_s^t h(u)du} \left\{-\bar{b}(s) + \left(h(s-\tau_1(s)) - \frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))}\right)(1-\tau_1'(s)) - \overline{k}(s)\right\}$$
$$\times [x(s-\tau_1(s)) - y(s-\tau_1(s))]\, ds$$
$$-\int_{t_0}^{t} e^{-\int_s^t h(u)du} h(s) \left(\int_{s-\tau_1(s)}^{s} \left(h(u) - \frac{p'(u)}{p(u)}\right)[x(u)-y(u)]\, du\right) ds$$
$$+\int_{t_0}^{t} e^{-\int_s^t h(u)du} \frac{\Sigma(s)\, p(s-\tau_2(s))}{p(s)} [x(s-\tau_2(s)) - y(s-\tau_2(s))]\, dw(s)$$
$$+\int_{t_0}^{t} e^{-\int_s^t h(u)du} \frac{\Gamma(s)\, p(s-\tau_3(s))}{p(s)} [x(s-\tau_3(s)) - y(s-\tau_3(s))]\, d\widetilde{N}(s) \Big|^2\Big).$$

By using the Doob L^p-inequality (see [12]),

$$\begin{aligned}
&\mathbb{E}\left|\sup_{t\geq t_0}\left(\int_{t_0}^{t} e^{-\int_s^t h(u)du} \frac{\Sigma(s)\, p(s-\tau_2(s))}{p(s)} [x(s-\tau_2(s)) - y(s-\tau_2(s))]\, dw(s)\right)\right|^2 \\
\leq\ & 4\mathbb{E}\sup_{t\geq t_0}\left(\int_{t_0}^{t} e^{-2\int_s^t h(u)du} \left|\frac{\Sigma(s)\, p(s-\tau_2(s))}{p(s)}\right|^2 |x(s-\tau_2(s)) - y(s-\tau_2(s))|^2\, ds\right) \\
\leq\ & 4\sup_{t\geq t_0}\left(\int_{t_0}^{t} e^{-2\int_s^t h(u)du} \left|\frac{\Sigma(s)\, p(s-\tau_2(s))}{p(s)}\right|^2 ds\right) \\
& \times\mathbb{E}\left(\sup_{s\geq t_0} |x(s-\tau_2(s)) - y(s-\tau_2(s))|^2\right),
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left|\sup_{t\geq t_0}\left(\int_{t_0}^{t} e^{-\int_s^t h(u)du} \frac{\Gamma(s)\, p(s-\tau_3(s))}{p(s)} [x(s-\tau_3(s)) - y(s-\tau_3(s))]\, d\widetilde{N}(s)\right)\right|^2 \\
\leq\ & 4\beta\mathbb{E}\sup_{t\geq t_0}\left(\int_{t_0}^{t} e^{-2\int_s^t h(u)du} \left|\frac{\Gamma(s)\, p(s-\tau_3(s))}{p(s)}\right|^2 |x(s-\tau_3(s)) - y(s-\tau_3(s))|^2\, ds\right) \\
\leq\ & 4\beta\sup_{t\geq t_0}\left(\int_{t_0}^{t} e^{-2\int_s^t h(u)du} \left|\frac{\Gamma(s)\, p(s-\tau_3(s))}{p(s)}\right|^2 ds\right) \\
& \times\mathbb{E}\left(\sup_{s\geq t_0} |x(s-\tau_3(s)) - y(s-\tau_3(s))|^2\right).
\end{aligned}$$

Then we have

$$\left\{\mathbb{E}\left(\sup_{t\geq m(t_0)} |(Qx)(t) - (Qy)(t)|^2\right)\right\}^{\frac{1}{2}}$$

$$\leq \sqrt{3}\left\{\mathbb{E}\left(\sup_{t\geq m(t_0)}|x(t)-y(t)|^2\right)\right\}^{\frac{1}{2}} \times \left\{\sup_{t\geq t_0}\left[\left|\frac{p(t-\tau_1(t))}{p(t)}\frac{c(t)}{1-\tau_1'(t)}\right|\right.\right.$$
$$+\int_{t-\tau_1(t)}^{t}\left|h(s)-\frac{p'(s)}{p(s)}\right|ds$$
$$+\int_{t_0}^{t}e^{-\int_s^t h(u)du}\left|-\bar{b}(s)+\left(h(s-\tau_1(s))-\frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))}\right)(1-\tau_1'(s))-\bar{k}(s)\right|ds$$
$$\left.+\int_{t_0}^{t}e^{-\int_s^t h(u)du}|h(s)|\left(\int_{s-\tau_1(s)}^{s}\left|h(u)-\frac{p'(u)}{p(u)}\right|du\right)ds\right]^2$$
$$+4\int_{t_0}^{t}e^{-2\int_s^t h(u)du}\left|\frac{\Sigma(s)p(s-\tau_2(s))}{p(s)}\right|^2 ds$$
$$\left.+4\beta\int_{t_0}^{t}e^{-2\int_s^t h(u)du}\left|\frac{\Gamma(s)p(s-\tau_3(s))}{p(s)}\right|^2 ds\right\}^{\frac{1}{2}}.$$

So

$$\left\{\mathbb{E}\left(\sup_{t\geq m(t_0)}|(Qx)(t)-(Qy)(t)|^2\right)\right\}^{\frac{1}{2}} \leq \sqrt{3\gamma}\left\{\mathbb{E}\left(\sup_{t\geq m(t_0)}|x(t)-y(t)|^2\right)\right\}^{\frac{1}{2}}.$$

By condition (8), Q is a contraction mapping with constant $\sqrt{3\gamma}$. Thanks to the contraction mapping principle (Smart [32], p. 2), we deduce that $Q : X_\psi^l \to X_\psi^l$ possesses a unique fixed point z in X_ψ^l, which is a solution of (14) with $z(t) = \psi(t)$ on $t \in [m(t_0), t_0]$ and $\mathbb{E}|z(t,t_0,\psi)|^2 \to 0$ as $t \to \infty$.

Third step: To prove stability at t_0, let $\varepsilon > 0$ be given, then we choose $m > 0$ such that $m < \min\{l, \varepsilon\}$. Replacing l with m in X_ψ^l beginning with (16), we see that there is a $\delta > 0$ such that $\|\psi\| < \delta$ implies that the unique solution of (12) with $z(t) = \psi(t)$ on $t \in [m(t_0), t_0]$ satisfies $\|Qz\|_X \leq m < \varepsilon$ for all $t \geq m(t_0)$. This shows that the zero solution of (12) is asymptotic stable if (10) holds.

Conversely, we suppose that (10) fails. From (9), there exists a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} \int_0^{t_n} h(u)du = l$ for some $l \in \mathbb{R}$. We may also choose a positive constant J satisfying

$$-J \leq \int_0^{t_n} h(u)du \leq +J, \tag{17}$$

for all $n \geq 1$. To simplify the expression, we define

$$F(s) := \left|-\bar{b}(s)+\left(h(s-\tau_1(s))-\frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))}\right)(1-\tau_1'(s))-\bar{k}(s)\right|$$
$$+|h(s)|\int_{s-\tau_1(s)}^{s}\left|h(u)-\frac{p'(u)}{p(u)}\right|du, \tag{18}$$

for all $s \geq 0$. From (8), we have

$$\int_0^{t_n} e^{-\int_s^{t_n} h(u)du} F(s)\, ds \leq \sqrt{\gamma}, \tag{19}$$

wich implies that

$$\int_0^{t_n} e^{\int_0^s h(u)du} F(s)\, ds \leq \sqrt{\gamma} e^{\int_0^{t_n} h(u)du} \leq \sqrt{\gamma} e^M. \tag{20}$$

Therefore, the sequence $\left\{\int_0^{t_n} e^{\int_0^s h(u)du} F(s)\, ds\right\}$ has a convergent subsequence. For brevity of notation, we may assume that

$$\lim_{n\to\infty} \int_0^{t_n} e^{\int_0^s h(u)du} F(s)\, ds = \xi, \tag{21}$$

for some $\xi \in \mathbb{R}^+$ and choose a positive integer m so large that

$$\int_{t_m}^{t_n} e^{\int_0^s h(u)du} F(s)\, ds \leq \frac{\delta_0}{8M}, \tag{22}$$

for all $n \geq m$, where $\delta_0 > 0$ satisfies

$$4\delta_0^2 M^2 e^{2J} \left(1 + \left|\frac{p(t_m - \tau_1(t_m))}{p(t_m)} \frac{c(t_m)}{(1 - \tau_1'(t_m))}\right| + \int_{t_m - \tau_1(t_m)}^{t_m} \left|h(u) - \frac{p'(u)}{p(u)}\right| du\right)^2 < (1 - 4\gamma).$$

Now, we consider the solution $z(t) = z(t, t_m, \psi)$ of (12) with $\|\psi(t_m)\| = \delta_0$ and $\|\psi(s)\| < \delta_0$ for $s < t_m$. If we replace l by 1 in the proof of $\|Qz\|_X \leq l$, we have $\mathbb{E}|z(t)|^2 < 1$ for $t \geq t_m$. We may choose ψ so that

$$\begin{aligned} G(t_m) \quad : \quad &= \psi(t_m) - \frac{p(t_m - \tau_1(t_m))}{p(t_m)} \frac{c(t_m)}{(1 - \tau_1'(t_m))} \psi(t_m - \tau_1(t_m)) \\ &- \int_{t_m - \tau_1(t_m)}^{t_m} \left(h(u) - \frac{p'(u)}{p(u)}\right) \psi(u) du \geq \frac{\delta_0}{2}. \end{aligned} \tag{23}$$

So, it follows from (23) with $z(t) = (Qz)(t)$ that for $n \geq m$,

$$\begin{aligned} &\mathbb{E}\Bigg| z(t_n) - \frac{p(t_n - \tau_1(t_n))}{p(t_n)} \frac{c(t_n)}{(1 - \tau_1'(t_n))} z(t_n - \tau_1(t_n)) \\ &- \int_{t_n - \tau_1(t_n)}^{t_n} \left(h(u) - \frac{p'(u)}{p(u)}\right) z(u) du \Bigg|^2 \\ \geq \quad & G^2(t_m) e^{-2\int_{t_m}^{t_n} h(u)du} - 2G(t_m) e^{-\int_{t_m}^{t_n} h(u)du} \int_{t_m}^{t_n} e^{-\int_s^{t_n} h(u)du} F(s)\, ds \end{aligned}$$

$$\geq \quad \frac{\delta_0}{2} e^{-2\int_{t_m}^{t_n} h(u)du} \left(\frac{\delta_0}{2} - 2M \int_{t_m}^{t_n} e^{\int_0^s h(u)du} F(s)\, ds \right) \geq \frac{\delta_0^2}{8} e^{-2M} > 0. \tag{24}$$

On the other hand, suppose that the solution of (12) $\mathbb{E}\left|z(t)\right|^2 = \mathbb{E}\left|z(t, t_m, \psi)\right|^2 \to 0$ as $t \to \infty$. Since $t_n - \tau_i(t_n) \to \infty$ as $n \to \infty$, for $i = 1, 2, 3$ and the condition (8) holds, we have

$$\begin{aligned} \mathbb{E}\Bigg| & z(t_n) - \frac{p(t_n - \tau_1(t_n))}{p(t_n)} \frac{c(t_n)}{(1 - \tau_1'(t_n))} z(t_n - \tau_1(t_n)) \\ & - \int_{t_m - \tau_1(t_m)}^{t_m} \left(h(u) - \frac{p'(u)}{p(u)} \right) z(u) du \Bigg|^2 \to 0, \end{aligned}$$

as $n \to \infty$, which contradicts (24). Hence condition (10) is necessary in order that (12) has a solution $\mathbb{E}\left|z(t, t_0, \psi\right|^2 \to 0$ as $t \to \infty$. Thus, the zero solution of (12) is mean square asymptotically stable, and hence the zero solution of (1) is asymptotically stable. The proof is complete.

REMARK 3.1. It follows from the first part of the proof of Theorem 3.1 that the zero solution of (1) is stable under (8). Moreover, Theorem 3.1 still holds if (10) is satisfied for $t \geq t_\rho$ with some $t_\rho \in \mathbb{R}^+$.

REMARK 3.2**.** In the current paper we extend the results in [35] to the linear stochastic differential equation with Poisson jumps and variable delays (1). Notice that when $\Gamma(t) = 0$ and $\Sigma(t) = 0$, then (1) reduces to (4). Thus, our results are more general than those obtained in [35]. But we would like to emphasize that the computations in D. Zhao [35] are not completely correct since on page 5, we obtain that (8) actually should be

$$\begin{aligned} z(t) \quad = \quad & \psi(t_0) e^{-\int_{t_0}^{t} h(s)ds} + \int_{t_0}^{t} e^{-\int_{s}^{t} h(u)du} \left(h(s) - \frac{p'(s)}{p(s)} \right) z(s) ds \\ & - \int_{t_0}^{t} e^{-\int_{s}^{t} h(u)du} \frac{a(s)p(s - \tau_1(s)) - c(s)\, p'(s - \tau_1(s))}{p(s)} z(s - \tau_1(s)) ds \\ & + \int_{t_0}^{t} e^{-\int_{s}^{t} h(u)du} \frac{c(s)\, p(s - \tau_1(s))}{p(s)} z'(s - \tau_1(s)) ds, \end{aligned}$$

which is special form of (14), also the correct condition (iii) in Theorem 3.1 on page 4 of D. Zhao [35] should be the condition (5) in our paper.

4 An Example

In this section, we analyse an example to illustrate two facts. On the one hand, we will show how to apply our main result in this paper, Theorem 3.1. On the other hand

and most importantly, we will highlight the real interest and importance of our result because the previous theory developed by Ardjouni and Djoudi [1] cannot be applied to this example.

EXAMPLE 4.1. Consider the following linear stochastic delay differential equation

$$\begin{aligned} dx(t) &= \left(-a(t)x(t-\tau_1(t)) + c(t)\,x'(t-\tau_1(t))\right)dt \\ &\quad + \Sigma(t)\,x(t-\tau_2(t))dw(t) + \Gamma(t)\,x(t-\tau_3(t))d\widetilde{N}(t), \end{aligned} \tag{25}$$

for $t \geq 0$. Corresponding to equation (1), let

$$c(t) = \ln\left(\frac{0.95t+0.1}{5(t+0.1)}\right), \quad \Sigma(t) = \sqrt{\frac{0.15(t+0.5)^2}{(t+0.1)^3}},$$

$$\tau_1(t) = 0.05t, \quad \tau_2(t) = 0.5t, \quad \Gamma(t) = 0,$$

and $a(t)$ satisfies

$$\left|-\overline{b}(t) + \left(h(t-\tau_1(t)) - \frac{p'(t-\tau_1(t))}{p(t-\tau_1(t))}\right)(1-\tau_1'(t)) - \overline{k}(t)\right| \leq \frac{0.1}{t+0.1},$$

where $\overline{b}(t)$ and $\overline{k}(t)$ are defined as in (6) and (7) respectively. Then the zero solution of (25) is mean square asymptotically stable.

PROOF. Choosing $h(t) = \dfrac{2}{t+0.1}$ and $p(t) = \dfrac{0.1}{t+0.1}$ in Theorem 3.1. By straightforward computations, we can check that condition (8) in Theorem 3.1 holds. As $t \to \infty$, we have

$$\left|\frac{p(t-\tau_1(t))}{p(t)}\frac{c(t)}{1-\tau_1'(t)}\right| \leq \left|\frac{t+0.1}{0.95t+0.1}\frac{0.95t+0.1}{5(t+0.1)}\right| \leq 0.2,$$

$$\int_{t-\tau_1(t)}^{t}\left|h(s) - \frac{p'(s)}{p(s)}\right|ds \leq 0.026,$$

$$\int_{t_0}^{t} e^{-\int_s^t h(u)du}\,|h(s)|\left(\int_{s-\tau_1(s)}^{s}\left|h(u) - \frac{p'(u)}{p(u)}\right|du\right)ds \leq 0.026,$$

$$\begin{aligned} &\int_0^t e^{-\int_s^t h(u)du}\left|-\overline{b}(s) + \left(h(s-\tau_1(s)) - \frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))}\right)(1-\tau_1'(s)) - \overline{k}(s)\right|ds \\ = &\int_0^t e^{-\int_s^t \frac{2}{u+0.1}du}\frac{0.1}{s+0.1}ds \leq 0.05, \end{aligned}$$

and

$$4\int_0^t e^{-2\int_s^t h(u)du}\left|\Sigma(s)\frac{p(s-\tau_2(s))}{p(s)}\right|^2 ds$$
$$\leq \quad 4\int_0^t e^{-2\int_s^t \frac{2}{u+0.1}du}\left|\frac{0.15(s+0.5)^2}{(s+0.1)^3}\right|\left|\frac{s+0.1}{s+0.5}\right|^2 ds < 0.15.$$

Since $\int_0^t h(s)ds \to \infty$ as $t \to \infty$, $p(t) \leq 1$, it is easy to see that all the conditions of Theorem 3.1 are satisfied for $\gamma = (0.302)^2 + 0.15 = 0.241 < \frac{1}{4}$. Thus, Theorem 3.1 implies that the zero solution of (25) is mean square asymptotic stable.

Note that

$$\left|\frac{c(t)}{1-\tau_1'(t)}\right| = \left|\frac{1}{0.95}\ln\left(\frac{0.95t+0.1}{5(t+0.1)}\right)\right| = 1.74 \quad \text{as } t \to \infty.$$

Therfore the result in [1] is not applicable.

Conclusion: *This work studies the problem of mean square asymptotic stability of a linear stochastic neutral differential equation with Poisson jumps and variable delays. As the main tool, it used the contraction mapping principle to obtain asymptotic stability results. As the main result, the paper establishes an asymptotic stability theorem with a necessary and sufficient condition. This improves and extends some previous results due to Dianli Zhao [35]. Actually, the methods used in proofs are an appropriate modification of those in [1, 17, 35] and other cited references. Moreover, an example is given to illustrate our results.*

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