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Carleman's Inequality Over The Values Of Euler Function And Sum Of Divisors Function*

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Abstract

In this paper we study Carleman's inequality over the values of Euler function, sum of divisors function, and their reciprocals. We show that the constant of Carleman's inequality over the values of these functions is not the best possible.

1 Introduction

For positive real numbers $a_1, \ldots, a_n$, Carleman's inequality [7] asserts that

$$\sum_{k=1}^{n} (a_1 \cdots a_k)^{\frac{1}{k}} \leqslant \mathrm{e} \sum_{k=1}^{n} a_k. \tag{1}$$

The constant e is the best possible. Based on the results in [5], recently in [2] we have studied Carleman's inequality over prime numbers and over reciprocal of the prime numbers, by proving

$$\frac{\sum_{k=1}^{n} (p_1 \cdots p_k)^{\frac{1}{k}}}{\sum_{k=1}^{n} p_k} = \frac{1}{\mathrm{e}} + O\left(\frac{1}{\log n}\right) \quad \text{and} \quad \frac{\sum_{k=1}^{n} (\frac{1}{p_1} \cdots \frac{1}{p_k})^{\frac{1}{k}}}{\sum_{k=1}^{n} \frac{1}{p_k}} = \mathrm{e} + O\left(\frac{1}{\log\log n}\right),$$

where p_k denote the kth prime.

In this paper, we are motivated by studying Carleman's inequality over the values of arithmetical functions, more precisely, over the values of Euler function φ, sum of divisors function σ, and their reciprocals. For each positive arithmetical function f let

$$C_f(n) = \frac{\sum_{k=1}^{n} (f(1) \cdots f(k))^{\frac{1}{k}}}{\sum_{k=1}^{n} f(k)}.$$

We prove the following results, providing non-trivial limit values $\lim_{n\to\infty} C_f(n)$ for the above mentioned arithmetical functions.

THEOREM 1. As $n \to \infty$ we have

$$C_\varphi(n) = \eta_\varphi + \eta_\varphi \frac{\log n}{n} + O\Big(\frac{(\log n)^{\frac{2}{3}} (\log\log n)^{\frac{4}{3}}}{n}\Big) \quad \text{with} \quad \eta_\varphi = \frac{\pi^2}{6\,\mathrm{e}} \prod_p \left(1 - \frac{1}{p}\right)^{\frac{1}{p}}, \tag{2}$$

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and

$$C_{\frac{1}{\varphi}}(n) = \eta_{\frac{1}{\varphi}} + O\Big(\frac{1}{\log n}\Big) \quad \text{with} \quad \eta_{\frac{1}{\varphi}} = \frac{2\mathrm{e}\,\pi^4}{315\zeta(3)} \prod_p \Big(1 - \frac{1}{p}\Big)^{-\frac{1}{p}}. \tag{3}$$

THEOREM 2. As $n \to \infty$ we have

$$C_\sigma(n) = \eta_\sigma + O\Big(\frac{1}{\log n}\Big) \quad \text{with} \quad \eta_\sigma = \frac{6}{\mathrm{e}\,\pi^2} \prod_p \Big(1 + \frac{1}{p}\Big)^{\frac{1}{p}} \prod_{\substack{p^\alpha \\ \alpha \geqslant 2}} \Big(\frac{p^{\alpha+1} - 1}{p^{\alpha+1} - p}\Big)^{\frac{1}{p^\alpha}}, \tag{4}$$

and

$$C_{\frac{1}{\sigma}}(n) = \eta_{\frac{1}{\sigma}} + O\Big(\frac{\log\log n}{\log n}\Big) \quad \text{with} \quad \eta_{\frac{1}{\sigma}} = \frac{\mathrm{e}\prod_p \Big(1 + \frac{1}{p}\Big)^{-\frac{1}{p}} \prod_{\substack{p^\alpha \\ \alpha \geqslant 2}} \Big(\frac{p^{\alpha+1}-1}{p^{\alpha+1}-p}\Big)^{-\frac{1}{p^\alpha}}}{\prod_p \Big(1 - \frac{(p-1)^2}{p} \sum_{\alpha=1}^{\infty} \frac{1}{(p^\alpha - 1)(p^{\alpha+1} - 1)}\Big)}. \tag{5}$$

The above theorems imply that the constant of Carleman's inequality over the values of φ, $\frac{1}{\varphi}$, σ and $\frac{1}{\sigma}$ is not the best possible. More precisely, computations running on Maple, give

$$\eta_\varphi \approxeq 0.3388, \qquad \eta_{\frac{1}{\varphi}} \approxeq 2.2096, \qquad \eta_\sigma \approxeq 0.3493, \qquad \eta_{\frac{1}{\sigma}} \approxeq 2.2721.$$

2 Proofs

During the proofs, for given positive function f we let $G_f(n)$ denote the geometric mean of the numbers $f(1), f(2), \dots, f(n)$. Hence

$$G_f(n) := (f(1)f(2)\cdots f(n))^{\frac{1}{n}},$$

and

$$C_f(n) = \frac{\sum_{k=1}^n G_f(k)}{\sum_{k=1}^n f(k)}. \tag{6}$$

Also, we note that

$$G_{\frac{1}{f}}(n) := \Big(\frac{1}{f(1)} \frac{1}{f(2)} \cdots \frac{1}{f(n)}\Big)^{\frac{1}{n}} = \frac{1}{G_f(n)}.$$

Therefore

$$C_{\frac{1}{f}}(n) = \frac{\sum_{k=1}^n \frac{1}{G_f(k)}}{\sum_{k=1}^n \frac{1}{f(k)}}. \tag{7}$$

The asymptotic expansion

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^r x^r + O(x^{r+1}) \qquad (x \to 0), \tag{8}$$

which holds for any given integer $r \geqslant 0$, usually will be useful to obtain an asymptotic expansion for $G_{\frac{1}{f}}(n)$ by using asymptotic expansion of $G_f(n)$.

PROOF of THEOREM 1. Let

$$g_\varphi = \frac{1}{\mathrm{e}} \prod_p \left(1 - \frac{1}{p}\right)^{\frac{1}{p}},$$

where the product runs over all primes. Corollary 2.5 of [8] asserts that

$$G_\varphi(n) = g_\varphi n + \frac{g_\varphi}{2} \log n + O(\log\log n). \tag{9}$$

Thus

$$\begin{aligned}\sum_{k=1}^{n} G_\varphi(k) &= g_\varphi \sum_{k=1}^{n} k + \frac{g_\varphi}{2} \sum_{k=1}^{n} \log k + O\left(\sum_{k=2}^{n} \log\log k\right) \\ &= g_\varphi \left(\frac{n^2}{2} + \frac{n}{2}\right) + \frac{g_\varphi}{2} \log n! + O\left(\int_2^n \log\log t \,\mathrm{d}t\right).\end{aligned}$$

Stirling approximation asserts that

$$n! = \left(\frac{n}{\mathrm{e}}\right)^n \sqrt{2\pi n} \left(1 + O\Big(\frac{1}{n}\Big)\right).$$

Taking logarithm and simplifying, we get

$$\log n! = n \log n - n + O(\log n).$$

Also, since the function $t \mapsto \log\log t$ is strictly increasing on the interval $[2, n]$ of length $n - 2 < n$, we obtain

$$\int_2^n \log\log t \,\mathrm{d}t < n \log\log n.$$

Combining the above approximations, gives

$$\sum_{k=1}^{n} G_\varphi(k) = \frac{g_\varphi}{2} \left(n^2 + n \log n + O(n \log\log n)\right). \tag{10}$$

Chapter IV of [9] provides a proof of the following best known approximation

$$\sum_{k=1}^{n} \varphi(k) = \frac{3}{\pi^2} n^2 + O\Big(n (\log n)^{\frac{2}{3}} (\log\log n)^{\frac{4}{3}}\Big).$$

We use the last approximation, (10), (6) and asymptotic expansion (8) with $r = 0$, to write

$$\begin{aligned}C_\varphi(n) &= \frac{\frac{g_\varphi}{2}\left(n^2 + n \log n + O(n \log\log n)\right)}{\frac{3}{\pi^2} n^2 \Big(1 + O\Big(\frac{(\log n)^{\frac{2}{3}} (\log\log n)^{\frac{4}{3}}}{n}\Big)\Big)} \\ &= \Big(\eta_\varphi + \eta_\varphi \frac{\log n}{n} + O\Big(\frac{\log\log n}{n}\Big)\Big)\Big(1 + O\Big(\frac{(\log n)^{\frac{2}{3}} (\log\log n)^{\frac{4}{3}}}{n}\Big)\Big),\end{aligned}$$

from which we obtain (2).

To prove (3) we use (9) and the approximation (8) with $r = 1$, to write

$$G_{\frac{1}{\varphi}}(n) = \frac{1}{G_\varphi(n)} = \frac{1}{g_\varphi n\left(1 + \frac{\log n}{2n} + O\left(\frac{\log\log n}{n}\right)\right)} = \frac{1}{g_\varphi n} - \frac{1}{2g_\varphi}\frac{\log n}{n^2} + O\left(\frac{\log\log n}{n^2}\right).$$

Hence, by using the approximation $\sum_{k=1}^n \frac{1}{k} = \log n + O(1)$, we get

$$\sum_{k=1}^n G_{\frac{1}{\varphi}}(k) = \frac{1}{g_\varphi}\log n + O(1).$$

Also, it is known [3] that

$$\sum_{k=1}^n \frac{1}{\varphi(k)} = A_1 \log n + A_1(\gamma - B_1) + O\left(\frac{(\log n)^{\frac{2}{3}}}{n}\right),$$

where $A_1 = \frac{\zeta(2)\zeta(3)}{\zeta(6)} = \frac{315\zeta(3)}{2\pi^4}$, $B_1 = \sum_p \frac{\log p}{p^2-p+1}$, and γ is Euler's constant. The relation (7), the above approximations, and (8) with $r = 1$, give

$$\begin{aligned} C_{\frac{1}{\varphi}}(n) &= \frac{\frac{1}{g_\varphi}\log n + O(1)}{A_1 \log n + A_1(\gamma - B_1) + O\left(\frac{(\log n)^{\frac{2}{3}}}{n}\right)} \\ &= \frac{\frac{1}{g_\varphi}\log n\left(1 + O\left(\frac{1}{\log n}\right)\right)}{A_1\log n\left(1 + \frac{\gamma - B_1}{\log n} + O\left(\frac{1}{n(\log n)^{\frac{1}{3}}}\right)\right)} \\ &= \frac{1}{A_1 g_\varphi}\left(1 + O\left(\frac{1}{\log n}\right)\right)\left(1 + \frac{B_1 - \gamma}{\log n} + O\left(\frac{1}{(\log n)^2}\right)\right) = \eta_{\frac{1}{\varphi}} + O\left(\frac{1}{\log n}\right), \end{aligned}$$

concluding the proof.

PROOF of THEOREM 2. Theorem 1.1 of [4] asserts that

$$\sum_{k\leqslant n} \log\sigma(k) = n\log n + (E + c_1 + c_2 + \gamma - 1)n + O\left(\frac{n}{\log n}\right), \tag{11}$$

where E is the constant in Mertens' approximation, defined by

$$E = \lim_{x\to\infty}\sum_{p\leqslant x}\frac{\log p}{p} - \log x,$$

and c_1 and c_2 are absolute constants defined by

$$c_1 = \sum_p \frac{1}{p}\log\left(1 + \frac{1}{p}\right), \qquad \text{and} \qquad c_2 = \sum_{\substack{p^\alpha \\ \alpha\geqslant 2}} \frac{1}{p^\alpha}\log\frac{p^{\alpha+1}-1}{p^\alpha - 1}. \tag{12}$$

It is known (see (2.8) of [1]) that

$$E = -\gamma - \sum_{\substack{p^\alpha \\ \alpha \geqslant 2}} \frac{\log p}{p^\alpha}. \tag{13}$$

Note that $n \log G_\sigma(n) = \sum_{k \leqslant n} \log \sigma(k)$. Hence (11) implies that

$$G_\sigma(n) = g_\sigma n + O\Big(\frac{n}{\log n}\Big), \tag{14}$$

where $g_\sigma = \mathrm{e}^{E+c_1+c_2+\gamma-1}$. By using (12) and (13) we get

$$g_\sigma = \frac{1}{\mathrm{e}} \prod_p \left(1 + \frac{1}{p}\right)^{\frac{1}{p}} \prod_{\substack{p^\alpha \\ \alpha \geqslant 2}} \left(\frac{p^{\alpha+1}-1}{p^{\alpha+1}-p}\right)^{\frac{1}{p^\alpha}}.$$

Thus

$$\begin{aligned} \sum_{k=1}^n G_\sigma(k) &= g_\sigma \sum_{k=1}^n k + O\left(\sum_{k=2}^n \frac{k}{\log k}\right) \\ &= g_\sigma \left(\frac{n^2}{2} + \frac{n}{2}\right) + O\left(\int_2^n \frac{t}{\log t}\,\mathrm{d}t\right). \end{aligned}$$

Note that

$$\int_2^n \frac{t}{\log t}\,\mathrm{d}t \leqslant 2\int_2^n \Big(\frac{t}{\log t} - \frac{t}{2\log^2 t}\Big)\,\mathrm{d}t = \frac{t^2}{\log t}\Big|_2^n < \frac{n^2}{\log n}.$$

Hence

$$\sum_{k=1}^n G_\sigma(k) = \frac{g_\sigma}{2} n^2 + O\Big(\frac{n^2}{\log n}\Big). \tag{15}$$

Chapter III, Section 2 of [9] provides a proof of the following best known approximation

$$\sum_{k=1}^n \sigma(k) = \frac{\pi^2}{12} n^2 + O\Big(n(\log n)^{\frac{2}{3}}\Big).$$

Considering this approximation and (15), using (6) and asymptotic expansion (8) with $r = 0$, we obtain (4).

Now, we prove (5). The approximation (14) and the asymptotic expansion (8) with $r = 0$ give

$$G_{\frac{1}{\sigma}}(n) = \frac{1}{G_\sigma(n)} = \frac{1}{g_\sigma n + O\big(\frac{n}{\log n}\big)} = \frac{1}{g_\sigma n} + O\Big(\frac{1}{n \log n}\Big).$$

Hence

$$\sum_{k=1}^{n} G_{\frac{1}{\sigma}}(k) = \frac{1}{g_\sigma}\sum_{k=1}^{n}\frac{1}{k} + O\left(\sum_{k=2}^{n}\frac{1}{k\log k}\right)$$
$$= \frac{1}{g_\sigma}\log n + O(1) + O\left(\int_2^n \frac{\mathrm{d}t}{t\log t}\right) = \frac{1}{g_\sigma}\log n + O(\log\log n).$$

Corollary 4.1 of [6] asserts that

$$\sum_{k=1}^{n}\frac{1}{\sigma(k)} = A_2\log n + A_2(\gamma + B_2) + O\Big(\frac{(\log n)^{\frac{2}{3}}(\log\log n)^{\frac{4}{3}}}{n}\Big),$$

where

$$A_2 = \prod_p\left(1 - \frac{(p-1)^2}{p}S_1(p)\right) \quad \text{and} \quad B_2 = \sum_p \frac{(p-1)^2(\log p)S_2(p)}{p-(p-1)^2S_1(p)},$$

with

$$S_1(p) = \sum_{\alpha=1}^{\infty}\frac{1}{(p^\alpha - 1)(p^{\alpha+1}-1)} \quad \text{and} \quad S_2(p) = \sum_{\alpha=1}^{\infty}\frac{\alpha}{(p^\alpha - 1)(p^{\alpha+1}-1)}.$$

Therefore, the above approximations and asymptotic expansion (8) with $r = 0$ imply

$$C_{\frac{1}{\sigma}}(n) = \frac{\frac{1}{g_\sigma}\log n + O(\log\log n)}{A_2\log n + A_2(\gamma + B_2) + O\big(\frac{(\log n)^{\frac{2}{3}}(\log\log n)^{\frac{4}{3}}}{n}\big)}$$
$$= \frac{1}{g_\sigma A_2}\left(1 + O\Big(\frac{\log\log n}{\log n}\Big)\right)\left(1 - \frac{\gamma + B_2}{\log n} + O\Big(\frac{1}{(\log n)^2}\Big)\right) = \eta_{\frac{1}{\sigma}} + O\Big(\frac{\log\log n}{\log n}\Big).$$

This completes the proof.

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