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A New Class of Pál Type Interpolation*

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Abstract

In this paper, we introduce a new class of Pál type interpolation problem which is obtained by adding a node to one set of interpolation points and simultaneously omitting a node from another set of interpolation points.

1 Introduction

Let π_n be the set of polynomials of degree less than or equal to n with complex coefficients. Let $A(z) \in \pi_m$ and $B(z) \in \pi_n$, then for a given positive integer r the problem of $(0, r)$ Pál type interpolation is to find a polynomial $P(z) \in \pi_{m+n-1}$, such that it assumes arbitrary prescribed values at the zeros of $A(z)$ and arbitrary prescribed values of the r^{th} derivative at the zeros of $B(z)$. The $(0, r)$ Pál type interpolation on the pair $\{A(z), B(z)\}$ is regular if and only if for any $P(z) \in \pi_{m+n-1}$ satisfying the set of conditions

$$\begin{aligned} P(y_i) &= 0;\ \text{where } A(y_i) = 0;\ i = 1, 2, \ldots, m, \\ P^{(r)}(z_j) &= 0;\ \text{where } B(z_j) = 0;\ j = 1, 2, \ldots, n, \end{aligned}$$

we have $P(z) \equiv 0$. Here, the zeros of $A(z)$, $B(z)$ are assumed to be simple[1]. R. Brueck [2] studied non-uniformly distributed nodes on the unit disk obtained by applying a certain Möbius transform to the set of zeros of $(z^n - 1)$ and $(z^n + 1)$ and defined the polynomials $v_n^\alpha(z)$ and $w_n^\alpha(z)$ by the equations

$$v_n^\alpha(z) = (z + \alpha)^n - (1 + \alpha z)^n, \tag{1}$$

$$w_n^\alpha(z) = (z + \alpha)^n + (1 + \alpha z)^n, \tag{2}$$

whose zeros are the nodes obtained after applying the transform. M. G. de Bruin [3] studied Pál type interpolation for nine different pairs of the zeros of polynomials given by (1) and (2), where one or two of the zeros of $w_n^\alpha(z)$ and/or $v_n^\alpha(z)$ are omitted from the set of interpolation points. Such types of Lacunary Polynomial Interpolation problems are different from the problems, which are studied by M. G. de Bruin and his associates, H. P. Dikshit, A. K. Pathak, A. Mandloi, Geeta Modi et. al. in [4, 5, 6, 7, 8, 9, 10, 11], where they added one or two real/complex zeros to the set of interpolation

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points. For studies in another significant direction in the relevant field, one may refer to [12, 13, 14]. However, in the course of our studies associated with regularity of Pál type interpolation, we revisited in [15] Pál type interpolation problems for the sets consisting of zeros of certain polynomials with complex coefficients with some additional value nodes. Also, we studied regularity of $(0, 1)$ 'incomplete' Pál type interpolation problem for several pairs of sets of nodes consisting of zeros of polynomials $w_n^\alpha(z)$ and $v_n^\alpha(z)$, where we omit the zeros $z = \pm 1$ or $z = \pm\zeta$ of $v_n^\alpha(z)$ and/or $z = -1$ or $z = \pm\zeta$ of $w_n^\alpha(z)$[16, 17].

In this paper, we study Pál type interpolation problems, where we add a node to one set of interpolation points and simultaneously omit a node from another set of interpolation points. We consider the polynomials $a_m(z) \in \pi_m$ and $b_n(z) \in \pi_n$ $(m \geq n)$ with simple zeros and take $A_m(z)$ and $B_n(z)$ as the sets of the zeros of these polynomials respectively with $B_n(z) \subseteq A_m(z)$.

2 $(0, 1)$ Pál Type Interpolation

In this section, we study regularity of $(0, 1)$ Pál type interpolation problems on the pair of zeros of certain polynomials.

THEOREM 2.1. $(0, 1)$ Pál type interpolation problem on $\left\{(z+1)a_m(z), \frac{b_n(z)}{(z-1)}\right\}$ for $-1 \notin A_m(z)$ and $1 \in B_n(z)$ is regular.

PROOF. Here, we have total $m + n$ interpolation points.
The problem is to find a polynomial $P(z) \in \pi_{m+n-1}$, which satisfies the following conditions:

$$\begin{aligned}
&P(y_i) = 0;\ y_i \text{ is a zero of } a_m(z);\ i = 1, 2, \ldots, m,\\
&P(-1) = 0,\\
&P'(z_j) = 0;\ z_j \text{ is a zero of } \frac{b_n(z)}{(z-1)};\ j = 1, 2, \ldots, (n-1).
\end{aligned}$$

Let $P(z) = (z+1)a_m(z)Q(z);\ Q(z) \in \pi_{n-2}$. Then $P(z) \in \pi_{m+n-1}$. The posed problem will be regular, if $P(z) \equiv 0$. Since $P'(z_j) = 0$, we have

$$(z_j + 1)a_m(z_j)Q'(z_j) + \{(z_j + 1)a'_m(z_j) + a_m(z_j)\} Q(z_j) = 0.$$

As $B_n(z) \subseteq A_m(z)$, we have $a_m(z_j) = 0$. Thus,

$$(z_j + 1)a'_m(z_j)Q(z_j) = 0.$$

Since $a_m(z)$ has simple zeros, $a'_m(z)$ cannot vanish at it's zeros $z_j \in B_n(z) \subseteq A_m(z)$. Hence $a'_m(z_j) \neq 0$. Further, $-1 \notin A_m(z)$ implies that $z_j \neq -1$. Hence, we get

$$Q(z_j) = 0;\ j = 1, 2, \ldots, (n-1).$$

As z_j has $(n-1)$ values and $Q(z) \in \pi_{n-2}$, it follows that $Q(z) \equiv 0$.

THEOREM 2.2. $(0,1)$ Pál type interpolation problem on $\left\{(z-1)a_m(z), \frac{b_n(z)}{(z+1)}\right\}$ for $1 \notin A_m(z)$ and $-1 \in B_n(z)$ is regular.

PROOF: The proof follows as that of Theorem 2.1.

THEOREM 2.3. $(0,1)$ Pál type interpolation problem on $\left\{\frac{a_m(z)}{(z-1)}, (z+1)b_n(z)\right\}$ for $\pm 1 \in A_m(z)$ and $\pm 1 \notin B_n(z)$ is regular.

PROOF. Here, we have total $m+n$ interpolation points. The problem is to find a polynomial $P(z) \in \pi_{m+n-1}$, which satisfies the following conditions:

$$P(y_i) = 0;\ y_i \text{ is a zero of } \frac{a_m(z)}{(z-1)};\ i = 1, 2, \ldots, (m-1),$$

$$P'(-1) = 0,$$

$$P'(z_j) = 0;\ z_j \text{ is a zero of } b_n(z);\ j = 1, 2, \ldots, n.$$

Let

$$P(z) = \frac{a_m(z)}{(z-1)} Q(z);\ \text{where } Q(z) \in \pi_n,$$

so that $P(z) \in \pi_{m+n-1}$. The posed problem will be regular, if $P(z) \equiv 0$.

Since $P'(z_j) = 0$ and $z_j \neq 1$, we have

$$\frac{a_m(z_j)}{(z_j-1)} Q'(z_j) + \left\{\frac{a'_m(z_j)}{(z_j-1)} - \frac{a_m(z_j)}{(z_j-1)^2}\right\} Q(z_j) = 0.$$

Also, since $z_j \in B_n(z) \subseteq A_m(z)$, we get

$$\frac{a'_m(z_j)}{(z_j-1)} Q(z_j) = 0.$$

As, $a_m(z)$ has simple zeros, it trivially gives

$$Q(z_j) = 0;\ j = 1, 2, \ldots, n. \tag{3}$$

Also, $P'(-1) = 0$, $-1 \in A_m(z)$ and $a'_m(z_j) \neq 0$ together imply that

$$Q(-1) = 0. \tag{4}$$

Our assumption that $Q(z) \in \pi_n$, on account of (3) and (4), gives that $Q(z) \equiv 0$.

COROLLARY 2.3.1. $(0,1)$ Pál type interpolation problem on $\left\{\frac{v^{\alpha}_{2n}(z)}{(z-1)}, (z+1)v^{\alpha}_n(z)\right\}$ is regular, when n is an odd number.

COROLLARY 2.3.2. $(0,1)$ Pál type interpolation problem on $\left\{\frac{v^{\alpha}_{2n}(z)}{(z-1)}, (z+1)w^{\alpha}_n(z)\right\}$ is regular, when n is an even number.

THEOREM 2.4. $(0,1)$ Pál type interpolation problem on $\left\{\frac{a_m(z)}{(z+1)}, (z-1)b_n(z)\right\}$ for $\pm 1 \in A_m(z)$ and $\pm 1 \notin B_n(z)$ is regular.

PROOF: The proof follows as that of Theorem 2.3.

COROLLARY 2.4.1. $(0,1)$ Pál type interpolation problem on $\left\{\frac{v_{2n}^{\alpha}(z)}{(z+1)}, (z-1)w_n^{\alpha}(z)\right\}$ is regular.

COROLLARY 2.4.2. $(0,1)$ Pál type interpolation problem on $\left\{\frac{v_{2n}^{\alpha}(z)}{(z+1)}, (z+1)w_n^{\alpha}(z)\right\}$ is regular, when n is an even number.

3 $(0,2)$ Pál Type Interpolation

In this section, we study regularity of $(0,2)$ Pál type interpolation problems on the pair of zeros of certain polynomials.

THEOREM 3.1. $(0,2)$ Pál type interpolation problem on $\left\{(z+1)a_m(z), \frac{b_n(z)}{(z-1)}\right\}$ for $-1 \notin A_m(z)$ and $1 \in B_n(z)$ is regular.

PROOF. Here, we have total $m+n$ interpolation points. The problem is to find a polynomial $P(z) \in \pi_{m+n-1}$, which satisfies the following conditions:

$$P(y_i) = 0;\ y_i \text{ is a zero of } a_m(z);\ i = 1, 2, \ldots, m,$$

$$P(-1) = 0,$$

$$P''(z_j) = 0;\ z_j \text{ is a zero of } \frac{b_n(z)}{(z-1)};\ j = 1, 2, \ldots, (n-1).$$

Let $P(z) = (z+1)a_m(z)Q(z);$ where $Q(z) \in \pi_{n-2}$, so that $P(z) \in \pi_{m+n-1}$. The posed problem will be regular, if $P(z) \equiv 0$.

Since $P''(z_j) = 0$, we have

$$\begin{aligned}&(z_j+1)a_m(z_j)Q''(z_j) + 2\left\{(z_j+1)a_m'(z_j) + a_m(z_j)\right\}Q'(z_j)\\&+\left\{(z_j+1)a_m''(z_j) + 2a_m'(z_j)\right\}Q(z_j) = 0.\end{aligned}$$

As $z_j \in B_n(z) \subseteq A_m(z)$ and $a_m(z)$ has simple zeros, we have

$$2(z_j+1)a_m'(z_j)Q'(z_j) + \left\{(z_j+1)a_m''(z_j) + 2a_m'(z_j)\right\}Q(z_j) = 0.$$

As z_j has $(n-1)$ values, the differential equation is given by

$$2(z+1)a_m'(z)Q'(z) + \left\{(z+1)a_m''(z) + 2a_m'(z)\right\}Q(z) = C_1\frac{b_n(z)}{(z-1)}, \text{ or}$$

$$Q'(z) + \left\{\frac{1}{2}\frac{a_m''(z)}{a_m'(z)} + \frac{1}{(z+1)}\right\}Q(z) = C\frac{b_n(z)}{a_m'(z)(z-1)}. \tag{5}$$

Integrating factor of the differential equation (5) is given by

$$\Phi(z) = exp \int \left\{ \frac{1}{2} \frac{a''_m(z)}{a'_m(z)} + \frac{1}{(z+1)} \right\} dz,$$

$$\text{or, } \Phi(z) = (z+1) \left\{ a'_m(z) \right\}^{\frac{1}{2}}.$$

Thus, solution of the differential equation (5) is given by

$$\Phi(z)Q(z) = C \int \Phi(t) \frac{b_n(t)}{a'_m(t)(t-1)} dt$$

$$\text{or, } (z+1) \left\{ a'_m(z) \right\}^{\frac{1}{2}} Q(z) = C \int (t+1) \left\{ a'_m(t) \right\}^{\frac{1}{2}} \frac{b_n(t)}{a'_m(t)(t-1)} dt$$

$$\text{or, } (z+1) \left\{ a'_m(z) \right\}^{\frac{1}{2}} Q(z) = C \int (t+1) \frac{b_n(t)}{\left\{ a'_m(t) \right\}^{\frac{1}{2}} (t-1)} dt$$

$$\text{or, } \left. C(t+1) \frac{b_n(t)}{\left\{ a'_m(t) \right\}^{\frac{1}{2}} (t-1)} \right|_{t=1} = 0 \Rightarrow C = 0.$$

Hence $Q(z) \equiv 0$.

THEOREM 3.2. $(0,2)$ Pál type interpolation problem on $\left\{ (z-1)a_m(z), \frac{b_n(z)}{(z+1)} \right\}$ for $1 \notin A_m(z)$ and $-1 \in B_n(z)$ is regular.

PROOF: The proof follows as that of Theorem 3.1.

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