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Functional Equations Characterizing The Sine And Cosine Functions Over A Convex Polygon*

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Abstract

Two functional equations exhibiting functions with a constant sum over points lying in a hyperplane are solved. These functional equations are employed to characterize the sine and cosine functions.

1 Introduction

In [2], Benz solved the functional equation

$$f(x)f(y)f(z) = f(x) + f(y) + f(z) \qquad (x, y, z \in (0, \pi/2))$$

with

$$x + y + z = \pi$$

obtaining a general solution $f : (0, \pi/2) \to (0, \infty)$ of the form

$$f(x) = \tan\left(kx + (1-k)\frac{\pi}{3}\right) \qquad (x \in (0, \pi/2)),$$

with an arbitrary constant $k \in [-1/2, 1]$. This confirmed a question posed by Davison [1]. Such a result can be regarded as a functional equation characterizing the trigonometric tangent function over a triangle. In [3] and [4], Hengrawit et al extended this result by solving a generalized functional equation over a convex polygon, which can also be regarded as characterizing the tangent function. Analyzing the work in [3] and [4], in a recent paper [5], the following two functional equations, with a constant parameter sum over a hyperplane, which can be used to characterize the sine and cosine functions, respectively, are solved:

$$\sum_{M=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^M \sum_{1 \le i_1 < \cdots < i_{2M+1} \le n} \left(\left(\prod_{k=1}^{2M+1} \frac{f(x_{i_k})}{g(x_{i_k})} \right) \left(\prod_{j=1}^{n} g(x_j) \right) \right) = 0, \qquad (1)$$

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$$\prod_{j=1}^{n} g\left(x_j\right) + \sum_{M=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^M \sum_{1 \le i_1 < \cdots < i_{2M} \le n} \left(\left(\prod_{k=1}^{2M} \frac{f\left(x_{i_k}\right)}{g\left(x_{i_k}\right)} \right) \left(\prod_{j=1}^{n} g\left(x_j\right) \right) \right) = (-1)^n, \quad (2)$$

with $n \geq 3$. It is natural to ask whether there are other functional equations that can be used to characterize the sine and cosine functions which are different from the above equations (1) and (2). In this work, we affirmatively answer this question by solving two other functional equations differing from (1) and (2), which also characterize the sine and cosine functions. In the work of Benz the parameters involved are the three (corresponding to $n = 3$) angles in a triangle, while those in [3] and [4] are the angles of a convex polygon. The restriction $n \geq 3$ is still adopted in this work. In the final section, the possibilities of $n < 3$ are investigated to ensure that this condition is essential.

2 Preliminary Results

We start with a theorem and a lemma taken from [5] which are needed.

THEOREM 1 ([5, Theorem 1.2]). Let n be an integer ≥ 3, and let $I_1 := (a, b),\ I_2 := (c, d)$ be two non-empty open intervals. Then the function $\phi : I_1 \to I_2$ satisfies the constant sum functional equation

$$\sum_{i=1}^{n} \phi(x_i) = U_1,$$

where U_1 is a real constant, subject to the hyperplane condition

$$\sum_{i=1}^{n} x_i = U_2,$$

where U_2 is a real constant, if and only if,

$$\phi(x) = k\left(x - \frac{U_2}{n}\right) + \frac{U_1}{n}$$

for some fixed k lying in the range

$$\max\left\{\frac{nc - U_1}{nb - U_2}, \frac{nd - U_1}{na - U_2}\right\} < k < \min\left\{\frac{nc - U_1}{na - U_2}, \frac{nd - U_1}{nb - U_2}\right\}.$$

LEMMA 1 ([5, Lemma 4.2]). Let n be an integer ≥ 2. If $x_1, \ldots, x_n \in (0, \pi)$, then

$$\begin{aligned} &\sin(x_1 + \cdots + x_n) \\ = &\sum_{M=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^M \sum_{1 \le i_1 < \cdots < i_{2M+1} \le n} \left(\left(\prod_{k=1}^{2M+1} \frac{\sin x_{i_k}}{\cos x_{i_k}} \right) \left(\prod_{j=1}^{n} \cos x_j \right) \right), \end{aligned}$$

and

$$
\begin{aligned}
&\cos(x_1+\cdots+x_n)\\
&= \prod_{j=1}^{n}\cos x_j + \sum_{M=1}^{\lfloor \frac{n}{2} \rfloor}(-1)^M \sum_{1\le i_1<\cdots<i_{2M}\le n}\left(\left(\prod_{k=1}^{2M}\frac{\sin x_{i_k}}{\cos x_{i_k}}\right)\left(\prod_{j=1}^{n}\cos x_j\right)\right).
\end{aligned}
$$

3 Main Results

We now prove our main theorem.

THEOREM 2. Let n be an integer ≥ 3 and let $I_1 := (a,b)$, $I_2 := (c,d)$ be two non-empty open intervals.

A) Let $\mathcal{F}(N_1,\ldots,N_n,R_1,\ldots,R_n)$ be a function of $2n$ variables and let $t\in I_1, T\in\mathbb{R}$. Suppose that $S,C : I_1\to I_2$ are two bijections satisfying

$$\mathcal{F}(S(x_1),\ldots,S(x_n),C(x_1),\ldots,C(x_n)) = S(x_1+\cdots+x_n). \tag{3}$$

Suppose also that the functions $U,V : I_1\to I_2$ satisfy

$$S^{-1}\circ U = C^{-1}\circ V \tag{4}$$

$$\mathcal{F}(U(\alpha_1),\ldots,U(\alpha_n),V(\alpha_1),\ldots,V(\alpha_n)) = S(t) \tag{5}$$

subject to the condition

$$\alpha_1+\cdots+\alpha_n = T \qquad (\alpha_1,\ldots,\alpha_n\in I_1).$$

Then

$$U(x) = S\left(k\left(x-\frac{T}{n}\right)+\frac{t}{n}\right),\quad V(x) = C\left(k\left(x-\frac{T}{n}\right)+\frac{t}{n}\right) \qquad (x\in I_1),$$

for some fixed $k\in\mathbb{R}$ lying in the range

$$\max\left\{\frac{nc-t}{nb-T},\frac{nd-t}{na-T}\right\} < k < \min\left\{\frac{nc-t}{na-T},\frac{nd-t}{nb-T}\right\}.$$

B) Let $\mathcal{H}(Y_1,\ldots,Y_{n-1},Z_1,\ldots,Z_{n-1})$ be a function of $2(n-1)$ variables and let $W,w\in\mathbb{R}$. Suppose that $S,C : I_1\to I_2$ are two bijections satisfying

$$\mathcal{H}(S(x_1),\ldots,S(x_{n-1}),C(x_1),\ldots,C(x_{n-1})) = C(x_1+\cdots+x_{n-1}). \tag{6}$$

Suppose also that the functions $U,V : I_1\to I_2$ satisfy

$$S^{-1}\circ U = C^{-1}\circ V \tag{7}$$

$$\mathcal{H}(U(\alpha_1),\ldots,U(\alpha_{n-1}),V(\alpha_1),\ldots,V(\alpha_{n-1})) = C\left(w-\left(S^{-1}\circ U\right)(\alpha_n)\right), \tag{8}$$

for some $\alpha_1, \ldots, \alpha_n \in I_1$ such that

$$w - \left(S^{-1} \circ U\right)(\alpha_n) \in I_1 \quad \text{and} \quad \alpha_1 + \cdots + \alpha_n = W.$$

Then

$$U(x) = S\left(\ell\left(x - \frac{W}{n}\right) + \frac{w}{n}\right), \quad V(x) = C\left(\ell\left(x - \frac{W}{n}\right) + \frac{w}{n}\right) \qquad (x \in I_1),$$

for some fixed $\ell \in \mathbb{R}$ lying in the range

$$\max\left\{\frac{nc - w}{nb - W}, \frac{nd - w}{na - W}\right\} < \ell < \min\left\{\frac{nc - w}{na - W}, \frac{nd - w}{nb - W}\right\}.$$

PROOF. A) By (4), there exists $\phi : I_1 \to I_1$ such that

$$U(x) = S(\phi(x)) \quad \text{and} \quad V(x) = C(\phi(x)) \qquad (x \in I_1).$$

Thus, (5) becomes

$$\mathcal{F}(S(\phi(\alpha_1)), \ldots, S(\phi(\alpha_n)), C(\phi(\alpha_1)), \ldots, C(\phi(\alpha_n))) = S(t).$$

By (3), we have

$$S(\phi(\alpha_1) + \cdots + \phi(\alpha_n)) = S(t).$$

Then

$$\phi(\alpha_1) + \cdots + \phi(\alpha_n) = t \quad \text{subject to} \quad \alpha_1 + \cdots + \alpha_n = T.$$

By Theorem 1 [5, Theorem 1.2], we have

$$\phi(x) = k\left(x - \frac{T}{n}\right) + \frac{t}{n} \qquad (x \in I_1),$$

where $k \in \mathbb{R}$ lying in the range

$$\max\left\{\frac{nc - t}{nb - T}, \frac{nd - t}{na - T}\right\} < k < \min\left\{\frac{nc - t}{na - T}, \frac{nd - t}{nb - T}\right\}.$$

B) By (7), there exists $\phi : I_1 \to I_1$ such that

$$U(x) = S(\phi(x)) \quad \text{and} \quad V(x) = C(\phi(x)) \qquad (x \in I_1).$$

Thus, (8) becomes

$$\mathcal{H}(S(\phi(\alpha_1)), \ldots, S(\phi(\alpha_{n-1})), C(\phi(\alpha_1)), \ldots, C(\phi(\alpha_{n-1}))) = C(w - \phi(\alpha_n)).$$

By (6), we have

$$C(\phi(\alpha_1) + \cdots + \phi(\alpha_{n-1})) = C(w - \phi(\alpha_n)).$$

Then

$$\phi(\alpha_1) + \cdots + \phi(\alpha_n) = w$$

subject to

$$\alpha_1 + \cdots + \alpha_n = W.$$

By Theorem 1 [5, Theorem 1.2], we have

$$\phi(x) = \ell\left(x - \frac{W}{n}\right) + \frac{w}{n} \qquad (x \in I_1),$$

where $\ell \in \mathbb{R}$ lying in the range

$$\max\left\{\frac{nc-w}{nb-W}, \frac{nd-w}{na-W}\right\} < \ell < \min\left\{\frac{nc-w}{na-W}, \frac{nd-w}{nb-W}\right\}.$$

The proof is complete.

EXAMPLE 1. Let $n \geq 3$, $I_1 := (0, \pi/2)$, $I_2 := (0,1), u \in I_1$. The trigonometric sine and cosine functions $\sin, \cos : I_1 \to I_2$ are two bijections satisfying

$$\begin{aligned} & \mathcal{F}\left(\sin x_1, \ldots, \sin x_n, \cos x_1, \ldots, \cos x_n\right) \\ \equiv & \sum_{M=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^M \sum_{1 \leq i_1 < \cdots < i_{2M+1} \leq n} \left(\left(\prod_{k=1}^{2M+1} \frac{\sin x_{i_k}}{\cos x_{i_k}}\right)\left(\prod_{j=1}^{n} \cos x_j\right)\right) \\ = & \sin(x_1 + \cdots + x_n). \end{aligned}$$

Suppose the functions $U, V : I_1 \to I_2$ satisfy

$$\mathcal{F}\left(U(\alpha_1), \ldots, U(\alpha_n), V(\alpha_1), \ldots, V(\alpha_n)\right) = \sin(u)$$

subject to the two conditions

$$\sin^{-1} \circ U = \cos^{-1} \circ V \quad \text{and} \quad \alpha_1 + \cdots + \alpha_n = \frac{(n-2)\pi}{2} \qquad (\alpha_1, \ldots, \alpha_n \in I_1).$$

Then

$$U(x) = \sin\left(m\left(x - \frac{(n-2)\pi}{2n}\right) + \frac{u}{n}\right), \; V(x) = \cos\left(m\left(x - \frac{(n-2)\pi}{2n}\right) + \frac{u}{n}\right) \; (x \in I_1),$$

for some fixed $m \in \mathbb{R}$ lying in the range

$$\max\left\{\frac{-u}{\pi}, \frac{-2(n-u)}{(n-2)\pi}\right\} < m < \min\left\{\frac{2u}{(n-2)\pi}, \frac{(n-u)}{\pi}\right\}.$$

EXAMPLE 2. Let $n \geq 3$, $I_1 := (0, \pi/2)$, $I_2 := (0,1), v \in I_1$. The trigonometric sine and cosine functions $\sin, \cos : I_1 \to I_2$ are two bijections satisfying

$$\begin{aligned} & \mathcal{G}\left(\cos x_1, \ldots, \cos x_n, \sin x_1, \ldots, \sin x_n\right) \\ & \equiv \prod_{j=1}^{n} \cos x_j + \sum_{M=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^M \sum_{1 \leq i_1 < \cdots < i_{2M} \leq n} \left(\left(\prod_{k=1}^{2M} \frac{\sin x_{i_k}}{\cos x_{i_k}}\right)\left(\prod_{j=1}^{n} \cos x_j\right)\right) \\ & = \cos(x_1 + \cdots + x_n). \end{aligned}$$

Suppose the functions $U, V : I_1 \to I_2$ satisfy

$$\mathcal{G}\left(V(\alpha_1), \ldots, V(\alpha_n), U(\alpha_1), \ldots, U(\alpha_n)\right) = \cos(v)$$

subject to the two conditions

$$\sin^{-1} \circ U = \cos^{-1} \circ V,$$

$$\alpha_1 + \cdots + \alpha_n = \frac{(n-2)\pi}{2} \qquad (\alpha_1, \ldots, \alpha_n \in I_1).$$

Then

$$U(x) = \sin\left(c\left(x - \frac{(n-2)\pi}{2n}\right) + \frac{v}{n}\right), V(x) = \cos\left(c\left(x - \frac{(n-2)\pi}{2n}\right) + \frac{v}{n}\right) \ (x \in I_1),$$

for some fixed $c \in \mathbb{R}$ lying in the range

$$\max\left\{\frac{-v}{\pi}, \frac{-2(n-v)}{(n-2)\pi}\right\} < c < \min\left\{\frac{2v}{(n-2)\pi}, \frac{(n-v)}{\pi}\right\}.$$

EXAMPLE 3. Let $n \geq 3,\ I_1 := (0, \pi/2),\ I_2 := (0,1), s \in \mathbb{R}$. The trigonometric sine and cosine functions $\sin, \cos : I_1 \to I_2$ are two bijections satisfying

$$\begin{aligned}
&\mathcal{H}\left(\sin x_1, \ldots, \sin x_{n-1}, \cos x_1, \ldots, \cos x_{n-1}\right) \\
\equiv\ & \prod_{j=1}^{n-1} \cos x_j + \sum_{M=1}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^M \sum_{1 \leq i_1 < \cdots < i_{2M} \leq n-1} \left(\left(\prod_{k=1}^{2M} \frac{\sin x_{i_k}}{\cos x_{i_k}}\right)\left(\prod_{j=1}^{n-1} \cos x_j\right)\right) \\
& \cos(x_1 + \cdots + x_{n-1}).
\end{aligned}$$

Suppose the functions $U, V : I_1 \to I_2$ satisfy

$$\sin^{-1} \circ U = \cos^{-1} \circ V,$$

$$\mathcal{H}\left(U(\alpha_1), \ldots, U(\alpha_{n-1}), V(\alpha_1), \ldots, V(\alpha_{n-1})\right) = \cos\left(s - \left(\sin^{-1} \circ U\right)(\alpha_n)\right),$$

for some $\alpha_1, \ldots, \alpha_n \in I_1$ such that

$$s - \left(\sin^{-1} \circ U\right)(\alpha_n) \in I_1 \quad \text{and} \quad \alpha_1 + \cdots + \alpha_n = \frac{(n-2)\pi}{2}.$$

Then

$$U(x) = \sin\left(y\left(x - \frac{(n-2)\pi}{2n}\right) + \frac{s}{n}\right), \quad V(x) = \cos\left(y\left(x - \frac{(n-2)\pi}{2n}\right) + \frac{s}{n}\right) \ (x \in I_1),$$

for some fixed $y \in \mathbb{R}$ lying in the range

$$\max\left\{\frac{-s}{\pi}, \frac{-2(n-s)}{(n-2)\pi}\right\} < y < \min\left\{\frac{2s}{(n-2)\pi}, \frac{(n-s)}{\pi}\right\}.$$

EXAMPLE 4. Let $n \geq 3$, $I_1 := (0, \pi/2)$, $I_2 := (0,1), r \in \mathbb{R}$. The trigonometric sine and cosine functions $\sin, \cos : I_1 \to I_2$ are two bijections satisfying

$$\begin{aligned} & \mathcal{E}\left(\cos x_1, \dots, \cos x_{n-1}, \sin x_1, \dots, \sin x_{n-1}\right) \\ \equiv & \sum_{M=0}^{\lfloor \frac{n-2}{2} \rfloor} (-1)^M \sum_{1 \leq i_1 < \cdots < i_{2M+1} \leq n-1} \left(\left(\prod_{k=1}^{2M+1} \frac{\sin x_{i_k}}{\cos x_{i_k}} \right) \left(\prod_{j=1}^{n-1} \cos x_j \right) \right) \\ = & \sin(x_1 + \cdots + x_{n-1}). \end{aligned}$$

Suppose the functions $U, V : I_1 \to I_2$ satisfy

$$\sin^{-1} \circ U = \cos^{-1} \circ V,$$

$$\mathcal{E}\left(V(\alpha_1), \dots, V(\alpha_{n-1}), U(\alpha_1), \dots, U(\alpha_{n-1})\right) = \sin\left(r - \left(\cos^{-1} \circ V\right)(\alpha_n)\right),$$

for some $\alpha_1, \dots, \alpha_n \in I_1$ such that

$$r - \left(\cos^{-1} \circ V\right)(\alpha_n) \in I_1 \quad \text{and} \quad \alpha_1 + \cdots + \alpha_n = \frac{(n-2)\pi}{2}.$$

Then

$$U(x) = \sin\left(z\left(x - \frac{(n-2)\pi}{2n}\right) + \frac{r}{n}\right), \quad V(x) = \cos\left(z\left(x - \frac{(n-2)\pi}{2n}\right) + \frac{r}{n}\right) \ (x \in I_1),$$

for some fixed $z \in \mathbb{R}$ lying in the range

$$\max\left\{\frac{-r}{\pi}, \frac{-2(n-r)}{(n-2)\pi}\right\} < z < \min\left\{\frac{2r}{(n-2)\pi}, \frac{(n-r)}{\pi}\right\}.$$

4 The cases $n = 1, 2$

In this section, we investigate the results of Theorem 2 when $n = 1, 2$. We illustrate by examples that the (implicit) uniqueness of solution is lost in the case $n = 2$, while the existence of solution is lost in the case $n = 1$.

PROPOSITION 1. Let $\mathcal{F}(N_1, N_2, R_1, R_2)$ be a function of 4 variables, let $I_1 := (a,b)$, $I_2 := (c,d)$ be two non-empty open intervals and let $t \in I_1, T \in \mathbb{R}$. Suppose that $S, C : I_1 \to I_2$ are two bijections satisfying

$$\mathcal{F}\left(S(x_1), S(x_2), C(x_1), C(x_2)\right) = S\left(x_1 + x_2\right). \tag{9}$$

Suppose the functions $U, V : I_1 \to I_2$ satisfy

$$\mathcal{F}\left(U(\alpha_1), U(\alpha_2), V(\alpha_1), V(\alpha_2)\right) = S(t) \tag{10}$$

subject to the two conditions

$$S^{-1} \circ U = C^{-1} \circ V \tag{11}$$

$$\alpha_1 + \alpha_2 = T \qquad (\alpha_1, \alpha_2 \in I_1). \tag{12}$$

Then one pair of solutions to (10) is given by

$$U(x) = S\left(A\left(x - \frac{T}{2}\right) + \frac{t}{2}\right), \quad V(x) = C\left(A\left(x - \frac{T}{2}\right) + \frac{t}{2}\right) \qquad (x \in I_1),$$

where $A : J \to (c - t/2, d - t/2)$ is an odd function on $J := (a - T/2, b - T/2)$. Moreover, another pair of solutions to (10) is given by

$$U(x) = S\left(k\left(x - \frac{T}{2}\right) + \frac{t}{2}\right), \quad V(x) = C\left(k\left(x - \frac{T}{2}\right) + \frac{t}{2}\right) \qquad (x \in I_1),$$

for some fixed $k \in \mathbb{R}$ lying in the range

$$\max\left\{\frac{2c - t}{2b - T}, \frac{2d - t}{2a - T}\right\} < k < \min\left\{\frac{2c - t}{2a - T}, \frac{2d - t}{2b - T}\right\}.$$

PROOF. By (11), there exists $\phi : I_1 \to I_1$ such that

$$U(x) = S(\phi(x)), \quad V(x) = C(\phi(x)) \qquad (x \in I_1).$$

Thus, (10) becomes

$$\mathcal{F}(S(\phi(\alpha_1)), S(\phi(\alpha_2)), C(\phi(\alpha_1)), C(\phi(\alpha_2))) = S(t).$$

By (9), we have

$$S(\phi(\alpha_1) + \phi(\alpha_2)) = S(t).$$

Then,

$$\phi(\alpha_1) + \phi(\alpha_2) = t \quad \text{subject to} \quad \alpha_1 + \alpha_2 = T. \tag{13}$$

From the condition (12), we have $2a < T < 2b$. Let $J := (a - T/2, b - T/2)$ and define $\psi : J \to I_2$ by

$$\psi(y) = \phi\left(y + \frac{T}{2}\right) \qquad (y \in J).$$

Thus, the relation (13) becomes

$$\psi(y_1) + \psi(y_2) = t \quad \text{subject to} \quad y_1 + y_2 = 0 \quad (y_i \in J).$$

Let $A : J \to (c - t/2, d - t/2)$ be an odd function. We claim that $\psi(y) = A(y) + t/2$ is a solution of (13). Since

$$\psi(y) + \psi(-y) = \left(A(y) + \frac{t}{2}\right) + \left(A(-y) + \frac{t}{2}\right) = t.$$

From the definition of ψ, we get

$$\phi(x) = A\left(x - \frac{T}{2}\right) + \frac{t}{2} \quad (x \in I_1).$$

We show that another pair solutions to (10) is given

$$U(x) = S\left(k\left(x - \frac{T}{2}\right) + \frac{t}{2}\right), \quad V(x) = C\left(k\left(x - \frac{T}{2}\right) + \frac{t}{2}\right) \qquad (x \in I_1).$$

Since

$$\begin{aligned}&\mathcal{F}\left(U(\alpha_1), U(\alpha_2), V(\alpha_1), V(\alpha_2)\right)\\ &= \mathcal{F}\left(S\left(k\bar{\alpha_1} + \frac{t}{2}\right), S\left(k\bar{\alpha_2} + \frac{t}{2}\right), C\left(k\bar{\alpha_1} + \frac{t}{2}\right), C\left(k\bar{\alpha_2} + \frac{t}{2}\right)\right)\end{aligned}$$

where $\bar{\alpha_i} = \alpha_i - T/2 \ \ (i = 1, 2)$, using (9) and (12), we have

$$\begin{aligned}&\mathcal{F}\left(S\left(k\bar{\alpha_1} + \frac{t}{2}\right), S\left(k\bar{\alpha_2} + \frac{t}{2}\right), C\left(k\bar{\alpha_1} + \frac{t}{2}\right), C\left(k\bar{\alpha_2} + \frac{t}{2}\right)\right)\\ &= S\left(\left(k\bar{\alpha_1} + \frac{t}{2}\right) + \left(k\bar{\alpha_2} + \frac{t}{2}\right)\right) = S(t).\end{aligned}$$

EXAMPLE 5. Let $I_1 := (0, \pi/2),\ I_2 := (0, 1);\ p \in I_1, T \in \mathbb{R}$. The trigonometric sine and cosine functions $\sin, \cos : I_1 \to I_2$ are bijections satisfying

$$\mathcal{F}\left(\sin(x_1), \sin(x_2), \cos(x_1), \cos(x_2)\right) := \sin(x_1)\cos(x_2) + \cos(x_1)\sin(x_2) = \sin\left(x_1 + x_2\right).$$

Suppose the functions $U, V : I_1 \to I_2$ satisfy

$$\mathcal{F}\left(U(\alpha_1), U(\alpha_2), V(\alpha_1), V(\alpha_2)\right) = \sin(p) \tag{14}$$

subject to the two conditions

$$\sin^{-1} \circ U = \cos^{-1} \circ V,$$

$$\alpha_1 + \alpha_2 = T \qquad (\alpha_1, \alpha_2 \in I_1).$$

Then one pair of solutions to (14) is given

$$U(x) = \sin\left(A\left(x - \frac{T}{2}\right) + \frac{p}{2}\right), \quad V(x) = \cos\left(A\left(x - \frac{T}{2}\right) + \frac{p}{2}\right) \qquad (x \in I_1),$$

where $A : J \to (-p/2, 1-p/2)$ is an odd function on $J := (-T/2, \pi/2 - T/2)$. Moreover, another pair of solutions to (14) is given by

$$U(x) = \sin\left(m\left(x - \frac{T}{2}\right) + \frac{p}{2}\right), \quad V(x) = \cos\left(m\left(x - \frac{T}{2}\right) + \frac{p}{2}\right) \qquad (x \in I_1),$$

for some fixed $m \in \mathbb{R}$ lying in the range

$$\max\left\{\frac{-p}{\pi - T}, \frac{-(2-p)}{T}\right\} < m < \min\left\{\frac{p}{T}, \frac{2-p}{\pi - T}\right\}.$$

EXAMPLE 6. Let $I_1 := (0, \pi/2)$, $I_2 := (0, 1)$; $q \in I_1, T \in \mathbb{R}$. The trigonometric sine and cosine functions $\sin, \cos : I_1 \to I_2$ are bijections satisfying

$$\mathcal{F}(\cos(x_1), \cos(x_2), \sin(x_1), \sin(x_2)) := \cos(x_1)\cos(x_2) - \sin(x_1)\sin(x_2) = \cos(x_1 + x_2).$$

Suppose the functions $U, V : I_1 \to I_2$ satisfy

$$\mathcal{F}(V(\alpha_1), V(\alpha_2), U(\alpha_1), U(\alpha_2)) = \cos(q) \tag{15}$$

subject to the two conditions

$$\sin^{-1} \circ U = \cos^{-1} \circ V \quad \text{and} \quad \alpha_1 + \alpha_2 = T \qquad (\alpha_1, \alpha_2 \in I_1).$$

Then one pair of solutions to (15) is given by

$$U(x) = \sin\left(A\left(x - \frac{T}{2}\right) + \frac{q}{2}\right), \quad V(x) = \cos\left(A\left(x - \frac{T}{2}\right) + \frac{q}{2}\right) \qquad (x \in I_1),$$

where $A : J \to (-q/2, 1-q/2)$ is an odd function on $J := (-T/2, \pi/2 - T/2)$. Moreover, another pair of solutions to (15) is given by

$$U(x) = \sin\left(m\left(x - \frac{T}{2}\right) + \frac{q}{2}\right), \quad V(x) = \cos\left(m\left(x - \frac{T}{2}\right) + \frac{q}{2}\right) \qquad (x \in I_1),$$

for some fixed $m \in \mathbb{R}$ lying in the range

$$\max\left\{\frac{-q}{\pi - T}, \frac{-(2-q)}{T}\right\} < m < \min\left\{\frac{q}{T}, \frac{2-q}{\pi - T}\right\}.$$

Regarding the case $n = 1$, we make the following two remarks.

I) If $n = 1$, then the equation (3) in Theorem 2 becomes

$$\mathcal{F}(S(x_1), C(x_1)) = S(x_1),$$

showing C is constant function, contradicting the fact that C is a bijection.

II) If $n = 1$, then there is no valid equation (6) in Theorem 2, while if $n = 2$, then the equation (6) in Theorem 2 becomes

$$\mathcal{H}(S(x_1), C(x_1)) = C(x_1),$$

yielding C to be a constant function, which again contradicts its being bijective.

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