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Some Fixed Point Theorems In Ordered Banach Spaces And Application*

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Abstract

In this paper we generalize some important results associated with nonlinear contractive mappings. Our principle results discuss the existence of fixed points for the operator $A.(B_1+B_2)$ where A, B_1 and B_2 are operators that satisfy some properties. An application is considered in the last section of this paper.

1 Introduction

Fixed point theory play an important role in many fields of sciences. For example, in economics [6, 7], physics [8], biology [1, 2], technology and more. In mathematics the most well known result in the theory of fixed points is the Banach contraction mapping principle, see [10], and the more general fixed point theorem is due to Boyd and Wong, see [3]; for more fixed point theorems see [4, 10] and the references therein. Mathematicians can solve a large number of problems by the fixed point theory and these problems can be described by some differential or integral or integro-differential equations.

In the present paper we consider the following nonlinear integral equation in the Banach algebra $C([0,1],\mathbb{R})$,

$$x(t)=f(t,x(t))\Big(\int_0^1 k_1(t,s)g_1(x(s))ds+\int_0^1 k_2(t,s)g_2(x(s))ds\Big), \tag{1.1}$$

where $f:[0,1]\times\mathbb{R}\to\mathbb{R},\ g_1,\ g_2:\mathbb{R}\to\mathbb{R}^+$ and $k_1,\ k_2:[0,1]\times[0,1]\to\mathbb{R}^+$ are continuous functions (here $\mathbb{R}$ denotes the set of all real numbers and $\mathbb{R}^+$ denotes the set of all nonnegative real numbers). We discuss the existence of solutions for (1.1) in the third section. In the second section we prove some auxiliary fixed point theorems concerned with the nonlinear contractions as a first step, then we prove the existence of solutions for the equation $B_1x+B_2x=x$ where B_1 is a nonlinear contraction and B_2 is a completely continuous map. This result is a generalization of the Krasnoselskii nonlinear alternative type for the sum of a contraction and a completely continuous

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map [9]. Finally, as a final step of the second section, we present a result which ensure the existence of solutions for the abstract equation $(A(B_1 + B_2))x = x$ in a Banach algebra space equipped with a cone K. In the following, we introduce some useful definitions and theorems.

Let X be a Banach space with a norm $\|.\|_X$, in short $\|.\|$. A Banach algebra space is a complex Banach space together with an associative and distributive multiplication such that:

$$\lambda(x.y) = (\lambda x).y = x.(\lambda y), \quad \|x.y\| \leq \|x\|\,\|y\| \quad \forall\,(x,y) \in X^2, \forall \lambda \in \mathbb{C}.$$

A nonempty and closed subset K of X is called a cone if $K+K \subset K$, $\lambda K \subset K$ for $\lambda \geq 0$ and $K \cap \{-K\} = \{0\}$. Each cone K induces a partial order $\leq$ on X by $x \leq y \Leftrightarrow y - x \in K$, $x < y$ will stand for $x \leq y$ and $x \neq y$. The pair $(X, \leq)$ or (X, K) is a partially ordered Banach space.

A mapping $A : X \to X$ is called α-contraction if there exists a positive real number $0 \leq \alpha < 1$ such that $\|Ax - Ay\| \leq \alpha\|x - y\|$ for all x, y in X, and it is called nonlinear contraction if there exists a nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for every pair of points x, y in X we have $\|Ax - Ay\| \leq \phi(\|x - y\|)$ with $\phi(t) < t$ for all $t > 0$ and $\phi(0) = 0$. An operator $A : X \to X$ is called compact if $\overline{A(X)}$ is a compact subset of X. Similarly, $A : X \to X$ is called totally bounded if A maps the bounded subsets of X into relatively compact subsets of X. Finally, $A : X \to X$ is called completely continuous operator if it is continuous and totally bounded.

LEMMA 1 ([4]). Let $B(y_0, r)$ be a sphere in the complete metric space (X, d). Let also $T : B(y_0, r) \longrightarrow X$ be a contraction mapping which satisfies the Lipschitz condition

$$d(T(x_1), T(x_2)) \leq \gamma d\,(x_1, x_2)$$

for every pair points x_1, x_2 in $B(y_0, r)$, γ being a constant such that $0 \leq \gamma < 1$. Then if $d(y_0, T(y_0)) \leq r(1-\gamma)$, there exists a unique point $x_0 \in B\,(y_0, r)$ such that $T(x_0) = x_0$.

LEMMA 2 ([3]). (Boyd and Wong) Let X be a Banach space and let $T : X \longrightarrow X$ be a nonlinear contraction. Then T has a unique fixed point in X.

LEMMA 3 ([4]). (Nonlinear alternative) Let K be a convex subset of a normed linear space E, U an open subset of K, $N : \overline{U} \longrightarrow K$ a compact map and $0 \in U$. Then either

i. N has a fixed point in $\overline{U}$ or,

ii. there is a point $u \in \partial U$ and $\lambda \in (0, 1)$ such that $u = \lambda N u$.

THEOREM 1 ([9]). Let $(X, \|.\|)$ be a Banach space, B_1, B_2 be two operators from X into X such that B_1 is γ-contraction and B_2 is completely continuous. Assume also that

(H) there exists a sphere $B(0, r)$ in X with center 0 and radius r such that for every $y \in B(0, r)$:

$$r(1-\gamma) \geq \|B_1 0 + B_2 y\|.$$

Then either,

(a) the operator equations $x = (B_1 + B_2)x$ has a solution with $\|x\| \leq r$, or

(b) there exists a point $x_0 \in \partial B(0, r)$ and $\lambda \in (0, 1)$ such that $x_0 = \lambda B_1(\frac{x_0}{\lambda}) + \lambda B_2(x_0)$.

THEOREM 2 ([5]). Let $(X, \leq)$ be a partially ordered Banach space. Assume that X satisfies the following condition: if (x_n) is a nondecreasing sequence in X such that $x_n \to x$ then $x_n \leq x$ for all $n \in \mathbb{N}$. Let $F : X \longrightarrow X$ be a nondecreasing mapping such that

$$\|Fx - Fy\| \leq \|x - y\| - \psi(\|x - y\|) \text{ for } x \geq y,$$

where $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous and nondecreasing function such that ψ is positive in $(0, +\infty)$, $\psi(0) = 0$ and $\lim_{\xi \to +\infty} \psi(\xi) = +\infty$. If there exists $x_0 \in X$ with $x_0 \leq Fx_0$, then F has a fixed point.

If we consider the following condition

($*$) for any x and y in X there exists $\xi \in X$ which is comparable to x and y.

Then we have the following result.

THEOREM 3 ([5]). Adding condition ($*$) to the hypotheses of Theorem 2, we obtain the uniqueness of the fixed point.

2 Main Results

Let X be an ordered Banach Algebra equipped with the natural cone $K = \{u \in X, u \geq 0\}$. Assume that X satisfies condition ($*$) and the following: if (x_n) is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \leq x$ for all $n \in \mathbb{N}$. Also consider three operators $A, B_1, B_2 : X \to X$, such that $A(K) \subseteq K$, $B_i(X) \subseteq K$ for $i = 1, 2$, and assume that $A0 = 0$, $B_1 0 = 0$. The next main result is more general than Lemma 1.

PROPOSITION 1. Let $B(y_0, r)$ be a sphere in the complete metric space (X, d). Let also $T : B(y_0, r) \longrightarrow X$ be a nonlinear contraction mapping which satisfies the condition:

$$d(T(x_1), T(x_2)) \leq \phi(d(x_1, x_2))$$

for every pair of points x_1, x_2 in $B(y_0, r)$, where ϕ is an increasing function which satisfies the following conditions

$$\phi(t) < t, \forall t > 0, \text{ and } \frac{\phi(t)}{t} \leq \frac{\phi(t')}{t'}; \forall t \geq t' > 0.$$

Then if $d(y_0, T(y_0)) \leq r(1 - \frac{\phi(r)}{r})$, there exists a unique point $x_0 \in B(y_0, r)$ such that $T(x_0) = x_0$.

PROOF. Choose $m < r$ such that $d(y_0, T(y_0)) \leq m(1 - \frac{\phi(m)}{m}) < r(1 - \frac{\phi(r)}{r})$. Let $M = \{y \in X, d(y, y_0) \leq m\}$ where $T : M \to X$ and we show that $T(M) \subseteq M$:

$$\begin{aligned} d(T(y), y_0) &\leq d(T(y), T(y_0)) + d(T(y_0), y_0) \\ &\leq \phi(d(y, y_0)) + m(1 - \frac{\phi(m)}{m}) \\ &= m. \end{aligned}$$

Since M is complete, by Boyd and Wong fixed point theorem; there exits a unique x_0 such that $x_0 = T(x_0)$.

The next result is more general than Theorem 1.

THEOREM 4. Let X be a Banach space, A, B be operators from X to X such that A is a nonlinear contraction with an increasing function ϕ which satisfies: $\forall \lambda \in (0, 1)$, $\forall t \in \mathbb{R}^+ : \lambda\phi(\frac{t}{\lambda}) \leq \phi(t)$ and $\frac{\phi(t)}{t}$ be a nonincreasing function on $\mathbb{R}^+$ and B be completely continuous. Assume also that

(H) there exists a sphere $B(0, r)$ in X such that for every $y \in B(0, r)$:

$$r(1 - \frac{\phi(r)}{r}) \geq \|A0 + By\|.$$

Then either,

i) $x = Ax + Bx$ has a solution x, with $\|x\| \leq r$, or

ii) there is a point $u \in X$, with $\|u\| = r$ and $\lambda \in (0, 1)$ such that $u = \lambda A(\frac{u}{\lambda}) + \lambda Bu$.

Before the proof of this theorem we need the next auxiliary result.

LEMMA 4. Let A be a nonlinear contraction from a Banach space X into itself with an increasing function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $\phi(t) < t$ for all $t > 0$ and $\phi(0) = 0$, then $(I - A)^{-1}$ exists and is continuous.

PROOF. For every y in X, define the function g on X by $g(x) = Ax + y$. Since A is a nonlinear contraction, then g is a nonlinear contraction with the same function ϕ,

$$\|g(x) - g(x')\| = \|Ax + y - Ax' - y\| \leq \phi(\|x - x'\|),$$

then by Boyd and Wong fixed point theorem there exists a unique solution of the equation $g(x) = x$ or equivalently,

$$\text{there exists a unique } x \in X : Ax + y = x$$

and so

$$\text{for all } y \in X, \text{ there exists a unique } x \in X \text{ such that } (I - A)x = y,$$

i.e., $(I - A)$ is bijective.

Now we show that $(I - A)^{-1}$ is continuous. Let $y,\ y' \in X$, then there exist $x,\ x' \in X$ such that

$$y = (I - A)x \text{ and } y' = (I - A)x',$$

and we have

$$\|y - y'\| = \|(I - A)x - (I - A)x'\| \geq \|x - x'\| - \phi(\|x - x'\|)$$

where

$$\begin{aligned} \|x - x'\| &= \|(I - A)^{-1}(I - A)x - (I - A)^{-1}(I - A)x'\| \\ &= \|(I - A)^{-1}y - (I - A)^{-1}y'\|. \end{aligned}$$

Since ϕ is increasing, $\|x - x'\| > \phi(\|x - x'\|)$ and $\phi(0) = 0$, then we have $\lim_{y \to y'} \|x - x'\| = 0$, this yields that $(I - A)^{-1}$ is continuous on X.

In the following, we prove Theorem 4.

PROOF. The operator $\lambda A\big(\frac{1}{\lambda}\ \cdot\big) : \overline{B(0, r)} \to X$ is a nonlinear contraction with the same function ϕ. Also, for every $y \in \overline{B}(0, r)$ the mapping $x \longmapsto \lambda A(\frac{x}{\lambda}) + \lambda By$ is a nonlinear contraction with the same function ϕ. Now by assumption (H) and Proposition 1, we have that there exists a unique solution x of the equation $\lambda A(\frac{x}{\lambda}) + \lambda By = x$ in $B(0, r)$. This yields that

$$\frac{x}{\lambda} = A(\frac{x}{\lambda}) + By$$

or

$$(I - A)\frac{x}{\lambda} = By$$

and

$$x = \lambda(I - A)^{-1}By.$$

From Lemma 4, the existence and the continuity of $(I - A)^{-1}$ are obtained. Moreover, since B is completely continuous, B is compact on $\overline{B}(0, r)$. Thus so is $(I - A)^{-1}B$. Therefore by the Nonlinear Alternative with ($K = X$ and $U = B(0, r)$) either $x = \lambda(I - A)^{-1}Bx$ has a solution in $\overline{B}(0, r)$ for $\lambda = 1$, or there exists $u \in \partial B(0, r)$ and $\lambda \in (0, 1)$ such that $u = \lambda A(\frac{u}{\lambda}) + \lambda Bu$, and this complete the proof.

Now we present the principal theorem of this paper.

THEOREM 5. Let X be an ordered Banach Algebra equipped with the natural cone K, and let the operators $A,\ B_1,\ B_2 : X \longrightarrow X$ defined as above. Assume that

(i) For all $x \geq x'$ we have $Ax \geq Ax'$ and

$$\|Ax - Ax'\| \leq \frac{1}{M}\Big(\|x - x'\| - \psi(\|x - x'\|)\Big),$$

where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous and an increasing function such that ψ is positive in $(0, \infty)$, $\psi(0) = 0$, $\lim\limits_{\xi \to \infty} \psi(\xi) = +\infty$, ψ^{-1} verify conditions of Theorem 4 and there exists $\theta > 0$ with $\alpha\theta < 1$ such that

$$\xi - \theta \leq \psi\left(\xi\right) < \xi \text{ and } \psi\left(\xi\right) > \alpha\theta\xi;\ \forall \xi > 0,$$

(ii) B_1 is α-contraction,

(iii) B_2 is a completely continuous operator,

(vi) there exists $r > 0$ such that:

$$\forall y \in B(0, r) :\ B_2 y \in B\big(0, \frac{M}{\theta}(r - \psi^{-1}(\theta\alpha r))\big),$$

(v) $\forall x \in \partial B(0, r),\ \forall \lambda \in (0, 1) : x \neq \lambda A(\frac{x}{\lambda}) B_1\big(\frac{x}{\lambda}\big) + \lambda A x B_2 x.$

Then the operator equation $AxB_1x + AxB_2x = x$ has a solution in $\overline{B}(0, r)$. Here,

$$M = \|B_1(\overline{B}(0, r))\| + \|B_2(\overline{B}(0, r))\|.$$

PROOF. Let $y \in X$ and define a mapping $A_{y,1} : X \to X$ by $A_{y,1}(x) = AxB_1y,\ x \in X$. Notice that, for all $x \leq x'$ then $Ax \leq Ax'$, we have; $AxB_1y \leq Ax'B_1y$, and notice that for any x and x' in X; $A_{y,1}$ satisfies the following property

$$\begin{aligned} \|A_{y,1}x - A_{y,1}x'\| &= \|AxB_1y - Ax'B_1y\| \\ &\leq \|Ax - Ax'\|.\|B_1y\| \\ &\leq \frac{1}{M}(\|x - x'\| - \psi(\|x - x'\|)).M \\ &\leq \|x - x'\| - \psi(\|x - x'\|), \end{aligned}$$

and so we have an element $x_0 = 0$ such that $0 \leq A_{y,1}(0) = A0B_1y \in X$ i.e., $x_0 \leq A_{y,1}x_0,\ \forall y \in X$. Now an application of the fixed point Theorem 3, yields that there is a unique fixed point

$$x^* = A_{y,1}(x^*),$$

or equivalently $x^* = Ax^*B_1y,\ x^* \in X$. Define a mapping $N_1 : X \to X$ by

$$N_1 y = z,$$

where $z \in X$ is the unique solution of the equation $z = AzB_1y,\ y \in X$. Now, we show that N_1 is a nonlinear contraction; we have by definition:

$$\begin{aligned}
\|N_1y - N_1y'\| &= \|z - z'\| = \|AzB_1y - AzB_1y'\| \\
&= \|AN_1yB_1y - AN_1y'B_1y'\| \\
&= \|AN_1yB_1y - AN_1y'B_1y + AN_1y'B_1y - AN_1y'B_1y'\| \\
&\le \|AN_1yB_1y - AN_1y'B_1y\| + \|AN_1y'B_1y - AN_1y'B_1y'\| \\
&\le \|AN_1y - AN_1y'\|\|B_1y\| + \|AN_1y'\|\|B_1y - B_1y'\| \\
&\le \|N_1y - N_1y'\| - \psi(\|N_1y - N_1y'\|) \\
&\quad + \|AN_1y'\|\|B_1y - B_1y'\|,
\end{aligned}$$

and this implies that

$$\psi(\|N_1y - N_1y'\|) \le \|AN_1y'\|\|B_1y - B_1y'\|.$$

Since $\xi - \theta \le \psi(\xi)$, we have

$$\psi(\|N_1y - N_1y'\|) \le \theta\|B_1y - B_1y'\|,$$

and so

$$\|N_1y - N_1y'\| \le \psi^{-1}(\theta\alpha\|y - y'\|).$$

Then, for every $t \ge 0$ we put $\phi(t) = \psi^{-1}(\theta\alpha t)$, and note that the function ϕ is increasing on $\mathbb{R}^+$. By this, we conclude that N_1 is a nonlinear contraction map with a function ϕ. Now, let $y \in X$ and define another mapping $A_{y,2} : X \longrightarrow X$ by

$$A_{y,2}(x) = AxB_2y,\ x \in X.$$

Notice that, for all $x \le x'$ then $Ax \le Ax'$, since $B_2(X) \subset K$ we have $AxB_2y \le Ax'B_2y$, and for any x and x' in X, $A_{y,2}$ satisfies the following property

$$\begin{aligned}
\|A_{y,2}x - A_{y,2}x'\| &= \|AxB_2y - Ax'B_2y\| \\
&\le \frac{1}{M}(\|x - x'\| - \psi(\|x - x'\|)).M \\
&\le \|x - x'\| - \psi(\|x - x'\|),
\end{aligned}$$

and so we have an element $x_0 = 0$ such that $0 \le A_{y,2}(0) = A0B_2y \in X$ i.e. $x_0 \le A_{y,2}x_0,\ \forall y \in X$. Similarly as above, an application of the fixed point Theorem 3 yields that there is a unique fixed point

$$x^* = A_{y,2}(x^*),$$

or equivalently $x^* = Ax^*B_2y$, we have that $x^* \in X$.

Next, define a mapping $N_2 : X \to X$ by $N_2y = z$, where $z \in X$ is the unique solution of the equation $z = AzB_2y,\ y \in X$. Then, we show that N_2 is continuous and compact. Let $\{y_n\}_n$ be a convergent sequence in X to a point $y \in X$, then we have

$$\begin{aligned}
\|N_2 y_n - N_2 y\| &= \|z_n - z\| = \|A z_n B_2 y - A z B_2 y\| \\
&= \|A N_2 y_n B_2 y_n - A N_2 y B_2 y\| \\
&\leq \|A N_2 y_n B_2 y_n - A N_2 y B_2 y_n\| + \|A N_2 y B_2 y_n - A N_2 y B_2 y\| \\
&\leq \|A N_2 y_n - A N_2 y\|.\|B_2 y_n\| + \|A N_2 y\|.\|B_2 y_n - B_2 y\| \\
&\leq \frac{1}{M}(\|N_2 y_n - N_2 y\| - \psi(\|N_2 y_n - A N_2\|))M \\
&\quad +\|A N_2 y\|.\|B_2 y_n - B_2 y\|.
\end{aligned}$$

Then

$$\psi(\|N_2 y_n - N_2 y\|) \leq \|A N_2 y\|.\|B_2 y_n - B_2 y\|,$$

and hence

$$\limsup_n \|\psi(\|N_2 y_n - N_2 y\|)\| \leq \limsup_n \|A N_2 y\|.\|B_2 y_n - B_2 y\|.$$

Since $\psi\,(0) = 0$, ψ positive and continuous we conclude that

$$\psi(\lim_n \|N_2 y_n - N_2 y\|) \leq 0.$$

This implies that

$$\lim_n \|N_2 y_n - N_2 y\| = 0.$$

Consequently, N_2 is a continuous map on X. In the following, we prove that N_2 is a compact operator on $\overline{B}(0, r)$. Then for any $z \in \overline{B}(0, r)$ we can write

$$\begin{aligned}
\|Az\| &\leq \|Aa\| + \|Az - Aa\| \\
&\leq \|Aa\| + \frac{1}{M}(\|z - a\| - \psi(\|z - a\|)) \\
&\leq \|Aa\| + \frac{r}{M} = c \text{ for some fixed } a \in X.
\end{aligned}$$

Let $\epsilon > 0$ be given; since B_2 is completely continuous, $B_2(\overline{B}(0, r))$ is totally bounded. Then, there is a set $Y = \{y_1, y_2, ..., y_n\}$ in $\overline{B}(0, r)$ such that $B_2(\overline{B}(0, r)) \subseteq \cup_{i=1}^{n} B_\delta(\omega_i)$ such that $\omega_i = B_2(y_i)$, and $\delta = \frac{1}{c}\psi(\epsilon) > 0$. Therefore, for any $y \in \overline{B}(0, r)$ we have an $y_k \in Y$ such that $\|B_2 y - B_2 y_k\| < \frac{1}{c}\psi(\epsilon)$. Also we have

$$\begin{aligned}
\|N_2 y_k - N_2 y\| &= \|A z_k B_2 y_k - A z B_2 y\| \\
&\leq \|A z_k B_2 y_k - A z_k B_2 y\| + \|A z_k B_2 y - A z B_2 y\| \\
&\leq \|A z_k - A z\|.\|B_2 y\| + \|A z_k\|.\|B_2 y_k - B_2 y\| \\
&\leq M \frac{1}{M}(\|z - z_k\| - \psi(\|z - z_k\|)) + \|A z_k\|.\|B_2 y_k - B_2 y\|.
\end{aligned}$$

Then

$$\psi(\|z - z_k\|) \leq \|A z_k\|\|B_2 y_k - B_2 y\|,$$

i.e.,

$$\psi(\|z - z_k\|) \leq c\|B_2y_k - B_2y\|.$$

Then since ψ is bijective, we write

$$\|z - z_k\| = \|N_2y_k - N_2y\| \leq \psi^{-1}(c\frac{1}{c}\psi(\epsilon)) = \epsilon.$$

This is true for every $y \in \overline{B}(0,r)$ and hence: $N_2(\overline{B}(0,r)) \subseteq \cup_{i=1}^{n} B_\epsilon(z_i)$ such that $z_i = N_2(y_i)$. As a result $N_2(\overline{B}(0,r))$ is totally bounded. Since N_2 is continuous, it is a compact operator on $\overline{B}(0,r)$. Since hypothesis (vi) holds, we have

$$\|N_10 + N_2y\| \leq \frac{Mr}{\theta}(1 - \frac{\phi(r)}{r}),\ \forall y \in B(0,r).$$

By

$$x \neq \lambda A(\frac{x}{\lambda})B_1(\frac{x}{\lambda}) + \lambda AxB_2x \text{ for all } x \in \partial B(0,r),\ \lambda \in (0,1),$$

we can apply Theorem 4 to yield that the operator equation $Ax(B_1 + B_2)x = x$ has a solution in $\overline{B}(0,r)$

3 Application

Consider the nonlinear integral equation (1.1) in the Banach algebra $C([0,1],\mathbb{R})$ equipped with the supremum norm and the natural cone $K = \{u \in X : u \geq 0\}$, where $f,\ g_i,\ k_i$ are continuous functions. Assume the following assertions:

(1) There exists a positive function $h : [0,1] \longrightarrow \mathbb{R}_+$ such that

$$0 \leq f(t,u) - f(t,v) \leq h(t)(\frac{u-v}{H^* + u - v}) \quad \text{for } u \geq v,$$

where $H^* = \sup_{t\in[0,1]} h(t)$ and $f(t,0) = 0, \forall\, t \in [0,1]$.
The mapping f has the following property

$$f(t,u) \leq f(t,v); \forall t \in [0,1],\ \forall u,v \in \mathbb{R},\ u \leq v,$$

(2) there exists $\beta_1 \in [0,1[$ such that

$$|g_1(u) - g_1(v)| \leq \beta_1|u - v|;\ \forall u,v \in \mathbb{R},\ \forall t \in [0,1],$$

with $g_1(0) = 0$, where $\min(\beta_1 k_1^*, H^*\beta_1 k_1^*) < 1$ and $H^* > 1 + \beta_1 k_1^*(H^*)^2$,

(3) there exists $r_0 > 0$ such that

$$\sup_{t\in[0,1],\ u,v\in\overline{B}(0,r)} \Big| \int_0^1 k_i(t,s)g_i(u(s))ds - \int_0^1 k_i(t,s)g_i(v(s))ds \Big| = 2k_i^* g_i^*,$$

$$\frac{k_2^* g_2^* H^*}{2(k_1^* g_1^* + k_2^* g_2^*)} \leq r_0 - \psi^{-1}(H^*\beta_1 k_1^* r_0),$$

and

$$H^*k_2^*\psi^{-1}(H^*\beta_1k_1^*r_0)+(1-H^*\beta_1k_1^*+H^*k_2^*)r_0>(H^*)^2\beta_1(k_1^*)^2-H^*,$$

where $k_i^* = \sup_{t\in[0,1]}|\int_0^t k_i(t,s)ds| < \infty$ and $g_i^* = \sup_{|y|\le r_0}|g_i(y)| < \infty$ for $i=1,2$, and ψ^{-1} is the inverse of the increasing function $t\mapsto\psi(t)=t-\dfrac{t}{H^*+t}$,

(4) $2H^*(k_1^*g_1^*+k_2^*g_2^*)\le 1$.

Then we obtain the following result.

THEOREM 6. Assume that the assertions (1)–(4) hold true, then the integral equation (1.1) has a solution in $X=C([0,1],\mathbb{R})$.

PROOF. Define the operators A, B_1 and B_2 as follows:

$$Au(t)=f(t,u(t)),\ B_iu(t)=\int_0^t k_i(t,s)g_i(x(s))ds, i=1,2;\ \forall u\in X$$

We are going to verify that the operators A, B_1 and B_2 satisfy all conditions of Theorem 5, and we confirm this in the next claims.

Claim 1: A satisfies condition (i). For any $u,v\in X$ and $0\le t\le 1$ we have

$$u\le v\Longrightarrow Au(t)=f(t,u(t))\le f(t,v(t))=Av(t)$$

and

$$\begin{aligned}
|Au(t)-Av(t)| &= f(t,u(t))-f(t,v(t))\\
&\le h(t).\Big(\frac{u(t)-v(t)}{H^*+u(t)-v(t)}\Big)\\
&\le H^*\Big(\frac{\|u-v\|}{H^*+\|u-v\|}\Big)\\
&= H^*\Big(\|u-v\|-\|u-v\|+\frac{\|u-v\|}{H^*+\|u-v\|}\Big)\\
&\le H^*\Big(\|u-v\|-\Big(\|u-v\|-\frac{\|u-v\|}{H^*+\|u-v\|}\Big)\Big)\\
&\le H^*\Big(\|u-v\|-\psi(\|u-v\|)\Big)\\
&\le \frac{1}{2(k_1^*g_1^*+k_2^*g_2^*)}\Big(\|u-v\|-\psi(\|u-v\|)\Big)\\
&= \frac{1}{M}\Big(\|u-v\|-\psi(\|u-v\|)\Big)
\end{aligned}$$

where

$$M=\|B_1(\overline{B}(0,r_0))\|+\|B_2(\overline{B}(0,r_0))\|=2(k_1^*g_1^*+k_2^*g_2^*),$$

ψ is a positive increasing function, $\lim_{\xi\to\infty} \psi(\xi) = +\infty$, $\psi(0) = 0$, and for all $\xi > 0$; we have

$$\xi - H^* \leq \psi(\xi) < \xi \text{ and } \psi(\xi) > k_1^*\beta_1 H^*\xi.$$

Claim 2: The operator B_1 is a contraction. Let $u, v \in X, t \in [0,1]$. Then

$$\begin{aligned} |B_1u(t) - B_1v(t)| &\leq \int_0^t |k_1(t,s)|.|g_1(u(s)) - g_1(v(s))|ds \\ &\leq \beta_1\|u-v\| \int_0^t |k_1(t,s)|ds \\ &\leq k_1^*\beta_1\|u-v\|, \end{aligned}$$

i.e B_1 is a $k_1^*\beta_1$-contraction.

Claim 3: The operator B_2 is completely continuous. Since k_2 is continuous on $[0,1]^2$ and g_2 is continuous on X, then B_2 is completely continuous.

Claim 4: Condition (iv) holds. Let $y \in [-r_0, r_0]$, from hypothesis (3) we get

$$\begin{aligned} |B_2y(t)| &= |\int_0^t k_2(t,s)g_2(y(s))ds| \\ &\leq g_2^*k_2^* \\ &\leq \frac{2(k_1^*g_1^* + k_2^*g_2^*)}{H^*}\Big(r_0 - \psi^{-1}(H^*\beta_1k_1^*r_0)\Big), \end{aligned}$$

i.e.,

$$\|B_2y\| \leq \frac{2(k_1^*g_1^* + k_2^*g_2^*)}{H^*}\Big(r_0 - \psi^{-1}(H^*\beta_1k_1^*r_0)\Big) = \frac{M}{H^*}\Big(r_0 - \psi^{-1}(H^*\beta_1k_1^*r_0)\Big),$$

for all $y \in [-r_0, r_0]$.

Claim 5: Condition (v) holds. By contradiction, assume that there exists $u \in X$ such that $\|u\| = r_0$, $\lambda \in (0,1)$ and

$$u(t) = \lambda A\left(\frac{u}{\lambda}\right)(t)B_1\left(\frac{u}{\lambda}\right)(t) + \lambda Au(t)B_2u(t).$$

Then, we have

$$\begin{aligned} |u(t)| &\leq \frac{\lambda H^*\|u\|}{\lambda H^* + \|u\|}\frac{\beta_1k_1^*\|u\|}{\lambda} \\ &+ \frac{\lambda H^*\|u\|}{H^* + \|u\|}k_2^*\frac{2(k_1^*g_1^* + k_2^*g_2^*)}{H^*}(r_0 - \psi^{-1}(H^*\beta_1k_1^*r_0)) \end{aligned}$$

since $\|u\| < \lambda H^* + \|u\|$, $\forall \lambda \in (0,1)$; we have

$$|u(t)| < H^* \beta_1 k_1^* \|u\| + \frac{H^* \|u\|}{H^* + \|u\|} k_2^* \frac{2(k_1^* g_1^* + k_2^* g_2^*)}{H^*} (r_0 - \psi^{-1}(H^* \beta_1 k_1^* r_0)).$$

Hence

$$r_0 = \|u\| \leq H^* \beta_1 k_1^* r_0 + \frac{H^* k_2^* r_0}{H^* + r_0} \frac{2(k_1^* g_1^* + k_2^* g_2^*)}{H^*} (r_0 - \psi^{-1}(H^* \beta_1 k_1^* r_0))$$

and by a simple computation we have a contradiction with assumption (3). This yields that all the assertions of Theorem 5 are verified. We then conclude that the equation (1.1) has a solution $x \in C([0,1], \mathbb{R})$, with $\|x\| \leq r_0$.

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