

Interval Oscillation Criteria For Conformable Fractional Differential Equations With Impulses*

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Abstract

This article is devoted to the study of interval oscillation criteria, for impulsive conformable fractional differential equations. Some new sufficient conditions are established, using the Riccati technique. The conditions obtained, extend some well known results, in the literature, on differential equations without impulses and generalize those on the classical integer order impulsive differential equations. Moreover, our results depart from the majority of results on this subject, since they are based on information on a sequence of subintervals of $[0, \infty)$, rather than on the whole linear interval. An example is given to illustrate our main results.

1 Introduction

The theory of fractional differential equations is considered as an important tool in modeling real life phenomena. The notion of fractional differential derivative first appeared in the late 17th century. It is well known that fractional differential equations are a more general form of the integer order differential equations, extending those equations to an arbitrary (non-integer) order. Many important mathematical models use fractional order derivatives. But the most frequently used definitions involve integration which is nonlocal: Riemann-Liouville derivative & Caputo derivative [3, 14, 21]. Those fractional derivatives in the fractional calculus have seemed complicated and lacked some basic properties, like the product rule and the chain rule. But in 2014, Khalil [6] et. al introduced a new fractional derivative called the conformable derivative which closely resembles the classical derivative. In recent years, many researchers have found that the fractional differential equations constitute a more accurate description of real world phenomena. Nowadays, they are extensively being used in physics, electro chemistry, control theory and electromagnetic fields [2, 7, 20].

The study of the qualitative behavior of the solutions of impulsive differential equations has rapidly expanded in the last few decades [1, 9, 12]. In particular the problem

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of the oscillation and non oscillation of integer order impulsive differential equations has extensively been studied by several authors, see [4, 5, 15, 16] and reference cited there in. However interval oscillation criteria for integer order impulsive differential equations have been investigated by few authors [8, 10, 11, 17, 18, 19].

To the best of our knowledge there seems that no work has been done on the interval oscillation of impulsive conformable fractional differential equations. Motivated by the above observation, we propose to initiate the following model of the form

$$\begin{cases} T_\alpha(r(t)g(T_\alpha(x(t)) + \mu(t)x(t))) + q(t)f(x(t - \tau)) = e(t), & t \geq t_0 \quad t \neq t_k, \\ x(t_k^+) = a_k x(t_k), \\ T_\alpha(x(t_k^+)) = b_k T_\alpha(x(t_k)), & k = 1, 2, \dots, \end{cases} \tag{1}$$

where T_α denotes the conformable fractional derivative of order $\alpha, 0 < \alpha \leq 1$. Next, we assume the following hypotheses (H) hold:

(H₁) $r(t) \in C^\alpha([t_0, \infty), (0, +\infty))$, $\mu(t) \in C^\alpha([t_0, \infty), \mathbb{R})$, $q(t), e(t) \in PC([t_0, \infty), \mathbb{R})$, where PC represents the class of functions which are piecewise continuous in t with discontinuities of first kind, only at $t = t_k, k = 1, 2, \dots$ and left continuous at $t = t_k$, while a_k, b_k are real constants satisfying $a_k > -1, a_k \leq b_k, k = 1, 2, \dots, t - \tau < t, \lim_{t \rightarrow \infty} t - \tau = \infty, 0 < t_0 < t_1 < \dots < t_k < \dots, \lim_{k \rightarrow \infty} t_k = \infty$.

(H₂) $f, g \in C(\mathbb{R}, \mathbb{R})$ are convex in $[0, \infty)$ with $xf(x) > 0$ and $\frac{f(x)}{x} \geq \epsilon > 0$, for $x \neq 0, xg(x) > 0, g(x) \leq \gamma x$ for $x \neq 0, g^{-1} \in C(\mathbb{R}, \mathbb{R})$ is a continuous functions with $xg^{-1}(x) > 0$ for $x \neq 0$ and there exist positive constant η such that $g^{-1}(xy) \leq \eta g^{-1}(x)g^{-1}(y)$ for $xy \neq 0$ and $\int_{t_0}^\infty s^{\alpha-1}g^{-1}\left(\frac{1}{r(s)}\right) ds = \infty$.

(H₃) For any $T \geq 0$ there exists intervals $[c_1, d_1]$ and $[c_2, d_2]$ contained in $[T, \infty)$ such that $c_1 < d_1 \leq d_1 + \tau \leq c_2 < d_2, c_j, d_j \notin \{t_k\}, j = 1, 2, k = 1, 2, \dots, r(t) > 0, q(t) \geq 0$, for $t \in [c_1 - \tau, d_1] \cup [c_2 - \tau, d_2]$ and $e(t)$ has different signs in $[c_1 - \tau, d_1]$ and $[c_2 - \tau, d_2]$, for instance, let

$$e(t) \leq 0 \quad \text{for } t \in [c_1 - \tau, d_1] \quad \text{and} \quad e(t) \geq 0 \quad \text{for } t \in [c_2 - \tau, d_2].$$

Denote

$$I(s) := \max\{j : t_0 < t_j < s\}, \quad r_j := \max\{r(t) : t \in [c_j, d_j]\}, \quad j = 1, 2,$$

$$E_p(c_j, d_j) = \{p \in C^\alpha[c_j, d_j], p(t) \not\equiv 0, p(c_j) = p(d_j) = 0, j = 1, 2\}.$$

For two constants $c, d \notin \{t_k\}$ with $c < d$ and a function $\varphi \in C([c, d], \mathbb{R})$, we define the operator $\Pi : C([c, d], \mathbb{R}) \rightarrow \mathbb{R}$ by

$$\Pi_c^d[\varphi] = \begin{cases} 0, & I(c) = I(d), \\ \varphi(t_{I(c)+1})\theta(c) + \sum_{i=I(c)+2}^{I(d)} \varphi(t_i)\varepsilon(t_i), & I(c) < I(d), \end{cases}$$

where

$$\theta(c) = \frac{a_{I(c)+1} - b_{I(c)+1}}{a_{I(c)+1}(t_{I(c)+1}^\alpha - c^\alpha)} \quad \text{and} \quad \varepsilon(t_i) = \frac{a_i - b_i}{a_i(t_i^\alpha - t_{i-1}^\alpha)}.$$

This paper is organized as follows: In Section 2, we present some definitions and results that will be needed later. In Section 3, we discuss the interval oscillation criteria for the problem in (1). In Section 4, we present an example to illustrate our main results.

2 Preliminaries

In this section, we give Definitions 2.1–2.3 and Theorem 2.1.

DEFINITION 2.1. A solution of equation (1) is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory.

We use the following definition introduced by R. Khalil et al. [6].

DEFINITION 2.2. Given $f : [0, \infty) \rightarrow \mathbb{R}$. Then the conformable fractional derivative of f of order α is defined by

$$T_\alpha(f)(t) = \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}$$

for all $t > 0, \alpha \in (0, 1]$. If f is α -differentiable in some $(0, a), a > 0$ and $\lim_{t \rightarrow 0^+} f^{(\alpha)}(t)$ exists, then we define

$$f^{(\alpha)}(0) = \lim_{t \rightarrow 0^+} f^{(\alpha)}(t).$$

DEFINITION 2.3. $I_\alpha^a(f)(t) = I_1^a(t^{\alpha-1}f) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx$, where the integral is the usual Riemann improper integral and $\alpha \in (0, 1)$.

Conformable fractional derivatives have the following properties:

THEOREM 2.1. Let $\alpha \in (0, 1]$ and f, g be α -differentiable at some point $t > 0$. Then

- (i) $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, for all $a, b \in \mathbb{R}$.
- (ii) $T_\alpha(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$.
- (iii) $T_\alpha(\lambda) = 0$ for all constant functions $f(t) = \lambda$.
- (iv) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.
- (v) $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.
- (vi) If f is differentiable, then $T_\alpha(f)(t) = t^{1-\alpha} \frac{df}{dt}(t)$.

3 Main Results

In this section, we established some new interval oscillation criteria for equation (1), using the Riccati transformation and integral average method.

THEOREM 3.1. Assume that conditions (H_1) – (H_3) hold. Furthermore, for any $T \geq 0$ there exist c_j, d_j satisfying $T \leq c_1 < d_1, T \leq c_2 < d_2$ and $p(t) \in E_p(c_j, d_j)$ such that

$$\begin{aligned} & \int_{c_j}^{t_{I(c_j)+1}} \left[\gamma(p'(t))^2 t^{2-2\alpha} r(t) - Q(t)p^2(t)N_{I(c_j)}^j(t) \right] dt \\ & + \sum_{k=I(c_j)+1}^{I(d_j)-1} \int_{t_k}^{t_{k+1}} \left[\gamma(p'(t))^2 t^{2-2\alpha} r(t) - Q(t)p^2(t)N_k^j(t) \right] dt \\ & + \int_{t_{I(d_j)}}^{d_j} \left[\gamma(p'(t))^2 t^{2-2\alpha} r(t) - Q(t)p^2(t)N_{I(d_j)}^j(t) \right] dt \\ & + \int_{c_j}^{d_j} w(t)p^2(t)(1-\alpha)t^{-\alpha} dt \leq r_j \Pi_{c_j}^{d_j} [p^2(t)] \end{aligned} \tag{2}$$

for $I(c_j) < I(d_j), j = 1, 2$, where $Q(t) = \epsilon q(t)$ and

$$N_k^j(t) = \begin{cases} \frac{\tau\alpha}{\tau\alpha a_k + b_k(t^\alpha - t_k^\alpha)} \frac{(t-\tau)^\alpha - (t_k-\tau)^\alpha}{t_k^\alpha - (t_k-\tau)^\alpha}, & t \in (t_k, t_k + \tau), \\ \frac{(t-\tau)^\alpha - t_k^\alpha}{t^\alpha - t_k^\alpha}, & t \in [t_k + \tau, t_{k+1}), \end{cases}$$

then every solution of the problem (1) is oscillatory.

PROOF. Assume to the contrary that $x(t)$ is a nonoscillatory solution of (1). Without loss of generality we may assume that $x(t)$ is an eventually positive solution of (1). Then there exists $t_1 \geq t_0$ such that $x(t) > 0$ for $t \geq t_1$. Therefore, from (1), it follows that

$$T_\alpha (r(t)g(T_\alpha x(t) + \mu(t)x(t))) = e(t) - q(t)f(x(t-\tau)) \quad \text{for } t \in [t_1, \infty).$$

Thus $T_\alpha (r(t)g(T_\alpha x(t) + \mu(t)x(t))) \geq 0$ or $T_\alpha (r(t)g(T_\alpha x(t) + \mu(t)x(t))) < 0, t \geq t_1$ for some $t_1 \geq t_0$. We now claim that

$$T_\alpha (r(t)g(T_\alpha x(t) + \mu(t)x(t))) \geq 0 \quad \text{for } t \geq t_1. \tag{3}$$

Suppose the opposite of (3), namely, $T_\alpha (r(t)g(T_\alpha x(t) + \mu(t)x(t))) < 0$. Since this expression is strictly decreasing in $[t_1, \infty)$, there exists $t_2 \in [t_1, \infty)$ such that

$$T_\alpha (r(t_2)g(T_\alpha x(t_2) + \mu(t_2)x(t_2))) < 0.$$

It is clear that

$$r(t)g(T_\alpha x(t) + \mu(t)x(t)) < r(t_2)g(T_\alpha x(t_2) + \mu(t_2)x(t_2)) := -k,$$

where $k > 0$ is a constant. For $t \in [t_2, \infty)$, we have

$$r(t)g(T_\alpha x(t) + \mu(t)x(t)) < -k,$$

$$T_\alpha(x(t)) < g^{-1}\left(\frac{-k}{r(t)}\right) - \mu(t)x(t),$$

$$T_\alpha(x(t)) \leq -\gamma_1 g^{-1}\left(\frac{1}{r(t)}\right), \quad \text{where } \gamma_1 = \eta g^{-1}(k) \text{ for } t \in [t_2, \infty).$$

Integrating the above inequality from t_2 to t , we have

$$x(t) \leq x(t_2) - \gamma_1 \int_{t_2}^t s^{\alpha-1} g^{-1}\left(\frac{1}{r(s)}\right) ds.$$

Letting $t \rightarrow \infty$, we get $\lim_{t \rightarrow +\infty} x(t) = -\infty$ which by contradiction shows that (3) holds.

We define the Riccati transformation

$$w(t) := \frac{r(t)g(T_\alpha(e^{I_\alpha \mu(t)}x(t)))}{e^{I_\alpha \mu(t)}x(t)}.$$

It follows from (1) that $w(t)$ satisfies

$$T_\alpha(w(t)) \leq \frac{e(t)}{x(t)} - q(t)\epsilon \frac{x(t-\tau)}{x(t)} - \frac{w^2(t)}{\gamma r(t)}.$$

By assumption (H_3) , we can choose $c_1, d_1 \geq t_0$ such that $r(t) > 0$, $q(t) \geq 0$ for $t \in [c_1 - \tau, d_1]$ and $e(t) \leq 0$ for $t \in [c_1 - \tau, d_1]$. From (1), we can easily see that

$$t^{1-\alpha}w'(t) \leq -Q(t)\frac{x(t-\tau)}{x(t)} - \frac{w^2(t)}{\gamma r(t)}. \quad (4)$$

For $t = t_k$, $k = 1, 2, \dots$, we have

$$w(t_k^+) := \frac{r(t_k^+)g(T_\alpha(e^{I_\alpha \mu(t_k^+)}x(t_k^+)))}{e^{I_\alpha \mu(t_k^+)}x(t_k^+)} \leq \frac{b_k}{a_k}w(t_k).$$

First, we consider the case that $I(c_1) < I(d_1)$. In this case, all the impulsive moments in $[c_1, d_1]$ are $t_{I(c_1)+1}, t_{I(c_1)+2}, \dots, t_{I(d_1)}$. Choose a $p(t) \in E_p(c_1, d_1)$. Multiplying both sides of (4) by $p^2(t)$ and integrating the resulting inequality, from c_1 to d_1 , we

obtain

$$\begin{aligned}
 & \int_{c_1}^{t_{I(c_1)+1}} p^2(t)t^{1-\alpha}w'(t)dt + \int_{t_{I(c_1)+1}}^{t_{I(c_1)+2}} p^2(t)t^{1-\alpha}w'(t)dt + \dots + \int_{t_{I(d_1)}}^{d_1} p^2(t)t^{1-\alpha}w'(t)dt \\
 \leq & - \int_{c_1}^{t_{I(c_1)+1}} p^2(t)\frac{w^2(t)}{\gamma r(t)}dt - \int_{t_{I(c_1)+1}}^{t_{I(c_1)+2}} p^2(t)\frac{w^2(t)}{\gamma r(t)}dt - \dots - \int_{t_{I(d_1)}}^{d_1} p^2(t)\frac{w^2(t)}{\gamma r(t)}dt \\
 & - \int_{c_1}^{t_{I(c_1)+1}} p^2(t)Q(t)\frac{x(t-\tau)}{x(t)}dt - \int_{t_{I(c_1)+1}}^{t_{I(c_1)+1+\tau}} p^2(t)Q(t)\frac{x(t-\tau)}{x(t)}dt \\
 & - \int_{t_{I(c_1)+1+\tau}}^{t_{I(c_1)+2}} p^2(t)Q(t)\frac{x(t-\tau)}{x(t)}dt - \dots - \int_{t_{I(d_1)-1+\tau}}^{t_{I(d_1)}} p^2(t)Q(t)\frac{x(t-\tau)}{x(t)}dt \\
 & - \int_{t_{I(d_1)}}^{d_1} p^2(t)Q(t)\frac{x(t-\tau)}{x(t)}dt.
 \end{aligned}$$

Using integration by parts on the left-hand side, and noting that $p(c_1) = p(d_1) = 0$, we get

$$\begin{aligned}
 & \sum_{k=I(c_1)+1}^{I(d_1)} p^2(t_k)t_k^{1-\alpha} [w(t_k) - w(t_k^+)] \\
 \leq & - \int_{c_1}^{d_1} \left[\frac{p(t)w(t)}{\sqrt{\gamma r(t)}} - p'(t)t^{1-\alpha}\sqrt{\gamma r(t)} \right]^2 dt \\
 & - \int_{c_1}^{t_{I(c_1)+1}} p^2(t)Q(t)\frac{x(t-\tau)}{x(t)}dt - \sum_{k=I(c_1)+1}^{I(d_1)-1} \left[\int_{t_k}^{t_k+\tau} p^2(t)Q(t)\frac{x(t-\tau)}{x(t)}dt \right. \\
 & \left. + \int_{t_k+\tau}^{t_{k+1}} p^2(t)Q(t)\frac{x(t-\tau)}{x(t)}dt \right] - \int_{t_{I(d_1)}}^{d_1} p^2(t)Q(t)\frac{x(t-\tau)}{x(t)}dt \\
 & + \int_{c_1}^{d_1} t^{2-2\alpha}\gamma r(t)(p'(t))^2 dt + \int_{c_1}^{d_1} (1-\alpha)t^{-\alpha}p^2(t)w(t)dt. \tag{5}
 \end{aligned}$$

We consider several cases to estimate $\frac{x(t-\tau)}{x(t)}$.

Case 1: For $t \in (t_k, t_{k+1}) \subset [c_1, d_1]$. If $t \in (t_k, t_{k+1}) \subset [c_1, d_1]$, since $t_{k+1} - t_k > \tau$, we consider two subcases:

Case 1.1: If $t \in [t_k + \tau, t_{k+1}]$, then $t - \tau \in [t_k, t_{k+1} - \tau]$ and there are no impulsive moments in $(t - \tau, t)$. Then, for any $t \in [t_k + \tau, t_{k+1}]$, we have

$$x(t) - x(t_k^+) = T_\alpha(x(\xi)) \left(\frac{t^\alpha - t_k^\alpha}{\alpha} \right), \quad \xi \in (t_k, t).$$

From this,

$$x(t) \geq T_\alpha(x(\xi)) \left(\frac{t^\alpha - t_k^\alpha}{\alpha} \right).$$

We obtain

$$\frac{T_\alpha(x(t))}{x(t)} < \frac{\alpha}{t^\alpha - t_k^\alpha}.$$

Integrating it from $t - \tau$ to t , we have

$$\frac{x(t - \tau)}{x(t)} > \frac{(t - \tau)^\alpha - t_k^\alpha}{t^\alpha - t_k^\alpha}.$$

Case 1.2: If $t \in (t_k, t_k + \tau)$, then $t - \tau \in (t_k - \tau, t_k)$ and there is an impulsive moment t_k in $(t - \tau, t)$. Similarly to Case 1.1, we obtain

$$x(t) - x(t_k - \tau) = T_\alpha(x(\xi_1)) \left(\frac{t^\alpha - (t_k - \tau)^\alpha}{\alpha} \right), \quad \xi_1 \in (t_k - \tau, t_k]$$

or

$$\frac{T_\alpha(x(t))}{x(t)} < \frac{\alpha}{\gamma} \frac{1}{t^\alpha - (t_k - \tau)^\alpha}.$$

Integrating it from $t - \tau$ to t and by using Definition 3, we get

$$\frac{x(t - \tau)}{x(t_k)} > \frac{(t - \tau)^\alpha - (t_k - \tau)^\alpha}{t_k^\alpha - (t_k - \tau)^\alpha} > 0, \quad t \in (t_k, t_k + \tau). \quad (6)$$

For any $t \in (t_k, t_k + \tau)$, we have

$$x(t) - x(t_k^+) \leq T_\alpha(x(t_k^+)) \left(\frac{t^\alpha - t_k^\alpha}{\alpha} \right).$$

Using the impulsive conditions in equation (1), we get

$$x(t) - a_k x(t_k) < b_k T_\alpha(x(t_k)) \left(\frac{t^\alpha - t_k^\alpha}{\alpha} \right)$$

$$\frac{x(t)}{x(t_k)} < b_k \frac{T_\alpha(x(t_k))}{x(t_k)} \left(\frac{t^\alpha - t_k^\alpha}{\alpha} \right) + a_k.$$

Using $\frac{T_\alpha(x(t_k))}{x(t_k)} < \frac{1}{\tau}$, we obtain

$$\frac{x(t)}{x(t_k)} < a_k + \frac{b_k}{\tau} \left(\frac{t^\alpha - t_k^\alpha}{\alpha} \right).$$

That is,

$$\frac{x(t_k)}{x(t)} > \frac{\tau \alpha}{\tau \alpha a_k + b_k (t^\alpha - t_k^\alpha)}. \quad (7)$$

From (6) and (7), we get

$$\frac{x(t - \tau)}{x(t)} > \frac{\tau \alpha}{\tau \alpha a_k + b_k (t^\alpha - t_k^\alpha)} \frac{(t - \tau)^\alpha - (t_k - \tau)^\alpha}{t_k^\alpha - (t_k - \tau)^\alpha} \geq 0.$$

Case 2: If $t \in [c_1, t_{I(c_1)+1}]$, we consider three subcases:

Case 2.1: If $t_{I(c_1)} > c_1 - \tau$ and $t \in [t_{I(c_1)} + \tau, t_{I(c_1)+1}]$, then $t - \tau \in [t_{I(c_1)}, t_{I(c_1)+1} - \tau]$ and there are no impulsive moments in $(t - \tau, t)$. Proceeding as in Case 1.1 and using the Mean-value Theorem on $(t_{I(c_1)}, t_{I(c_1)+1}]$, we get

$$\frac{x(t - \tau)}{x(t)} > \frac{(t - \tau)^\alpha - t_{I(c_1)}^\alpha}{t^\alpha - t_{I(c_1)}^\alpha} > 0, \quad t \in [t_{I(c_1)} + \tau, t_{I(c_1)+1}].$$

Case 2.2: If $t_{I(c_1)} > c_1 - \tau$ and $t \in [c_1, t_{I(c_1)} + \tau)$, then $t - \tau \in [c_1 - \tau, t_{I(c_1)})$ and there is an impulsive moment $t_{I(c_1)}$ in $(t - \tau, t)$. Making a similar analysis as in Case 1.2, we have

$$\frac{x(t - \tau)}{x(t)} > \frac{\tau\alpha}{\tau\alpha a_{I(c_1)} + b_{I(c_1)}(t^\alpha - t_{I(c_1)}^\alpha)} \frac{(t - \tau)^\alpha - (t_{I(c_1)} - \tau)^\alpha}{t_{I(c_1)}^\alpha - (t_{I(c_1)} - \tau)^\alpha} \geq 0, \quad t \in (c_1, t_{I(c_1)} + \tau).$$

Case 2.3: If $t_{I(c_1)} < c_1 - \tau$, then for any $t \in [c_1, t_{I(c_1)+1}]$, $t - \tau \in [c_1 - \tau, t_{I(c_1)+1} - \tau]$ and there are no impulsive moments in $(t - \tau, t)$. Working as in Case 1.1, we get

$$\frac{x(t - \tau)}{x(t)} > \frac{(t - \tau)^\alpha - t_{I(c_1)}^\alpha}{t^\alpha - t_{I(c_1)}^\alpha} > 0, \quad t \in [c_1, t_{I(c_1)+1}].$$

Case 3: For $t \in (t_{I(d_1)}, d_1]$, we consider three subcases:

Case 3.1: If $t_{I(d_1)} + \tau < d_1$ and $t \in [t_{I(d_1)} + \tau, d_1]$, then $t - \tau \in [t_{I(d_1)}, d_1 - \tau]$ and there are no impulsive moments in $(t - \tau, t)$. Using a similar analysis as in Case 2.1, we have

$$\frac{x(t - \tau)}{x(t)} > \frac{(t - \tau)^\alpha - t_{I(d_1)}^\alpha}{t^\alpha - t_{I(d_1)}^\alpha} > 0, \quad t \in [t_{I(d_1)} + \tau, d_1].$$

Case 3.2: If $t_{I(d_1)} + \tau < d_1$ and $t \in [t_{I(d_1)}, t_{I(d_1)} + \tau)$, then $t - \tau \in [t_{I(d_1)} - \tau, t_{I(d_1)})$ and there is an impulsive moment $t_{I(d_1)}$ in $(t - \tau, t)$. Using a similar analysis as in Case 2.2, we obtain

$$\frac{x(t - \tau)}{x(t)} > \frac{\tau\alpha}{\tau\alpha a_{I(d_1)} + b_{I(d_1)}(t^\alpha - t_{I(d_1)}^\alpha)} \frac{(t - \tau)^\alpha - (t_{I(d_1)} - \tau)^\alpha}{t_{I(d_1)}^\alpha - (t_{I(d_1)} - \tau)^\alpha} \geq 0.$$

Case 3.3: If $t_{I(d_1)} + \tau \geq d_1$, then for any $t \in (t_{I(d_1)}, d_1]$, we get $t - \tau \in (t_{I(d_1)} - \tau, d_1 - \tau]$ and there is an impulsive moment $t_{I(d_1)}$ in $(t - \tau, t)$. Proceeding as in Case 3.2, we get

$$\frac{x(t - \tau)}{x(t)} > \frac{\tau\alpha}{\tau\alpha a_{I(d_1)} + b_{I(d_1)}(t^\alpha - t_{I(d_1)}^\alpha)} \frac{(t - \tau)^\alpha - (t_{I(d_1)} - \tau)^\alpha}{t_{I(d_1)}^\alpha - (t_{I(d_1)} - \tau)^\alpha} \geq 0.$$

Combining all these cases, we have

$$\frac{x(t - \tau)}{x(t)} > \begin{cases} N_{I(c_1)}^1(t) & \text{for } t \in [c_1, t_{I(c_1)+1}], \\ N_k^1(t) & \text{for } t \in (t_k, t_{k+1}], \quad k = I(c_1) + 1, \dots, I(d_1) - 1, \\ N_{I(d_1)}^1(t) & \text{for } t \in (t_{I(d_1)+1}, d_1]. \end{cases}$$

Hence, by (5) and from $r(t)g(T_\alpha(x(t)) + \mu(t)x(t))$ being non-increasing in $(c_1, t_{I(c_1)+1}]$, we have

$$\begin{aligned}
& \sum_{k=I(c_1)+1}^{I(d_1)} p^2(t_k)t_k^{1-\alpha} [w(t_k) - w(t_k^+)] \\
\leq & \int_{c_1}^{t_{I(c_1)+1}} \left[(p'(t))^2 t^{2-2\alpha} \gamma r(t) - p^2(t)Q(t)N_{I(c_1)}^1(t) \right] dt \\
& + \sum_{k=I(c_1)+1}^{I(d_1)-1} \int_{t_k}^{t_{k+1}} \left[(p'(t))^2 t^{2-2\alpha} \gamma r(t) - p^2(t)Q(t)N_k^1(t) \right] dt \\
& + \int_{t_{I(d_1)}}^{d_1} \left[(p'(t))^2 t^{2-2\alpha} \gamma r(t) - p^2(t)Q(t)N_{I(d_1)}^1(t) \right] dt \\
& + \int_{c_1}^{d_1} (1-\alpha)t^{-\alpha} p^2(t)w(t)dt. \tag{8}
\end{aligned}$$

Thus

$$\begin{aligned}
x(t) & > x(t) - x(c_1) = T_\alpha(x(\xi_2)) \left(\frac{t^\alpha - c_1^\alpha}{\alpha} \right) \\
& \geq \frac{r(t)(T_\alpha(x(t)) + \mu(t)x(t))}{r(\xi_2)} \left(\frac{t^\alpha - c_1^\alpha}{\alpha} \right), \quad \xi_2 \in (c_1, t).
\end{aligned}$$

Letting $t \rightarrow t_{I(c_1)+1}^-$, it follows that

$$w(t_{I(c_1)+1}) < \frac{r_1}{t_{I(c_1)+1}^\alpha - c_1^\alpha}. \tag{9}$$

Similarly we can prove that on $(t_{k-1}, t_k]$, $k = I(c_1) + 2, \dots, I(d_1)$,

$$w(t_k) < \frac{r_1}{t_k^\alpha - t_{k-1}^\alpha}. \tag{10}$$

Hence, from (9) and (10), we have

$$\begin{aligned}
& \sum_{k=I(c_1)+1}^{I(d_1)} p^2(t_k)t_k^{1-\alpha} w(t_k) \left[\frac{a_k - b_k}{a_k} \right] \\
\geq & r_1 \left[p^2(t_{I(c_1)+1})t_{I(c_1)+1}^{1-\alpha} \frac{a_{I(c_1)+1} - b_{I(c_1)+1}}{a_{I(c_1)+1}} \frac{1}{t_{I(c_1)+1}^\alpha - c_1^\alpha} \right. \\
& \left. + \sum_{k=I(c_1)+1}^{I(d_1)} p^2(t_k)t_k^{1-\alpha} \frac{a_k - b_k}{a_k} \frac{1}{t_k^\alpha - t_{k-1}^\alpha} \right] \\
\geq & r_1 \Pi_{c_1}^{d_1} [p^2(t)]. \tag{11}
\end{aligned}$$

Thus we have

$$\sum_{k=I(c_1)+1}^{I(d_1)} p^2(t_k)t_k^{1-\alpha}w(t_k) \left[\frac{a_k - b_k}{a_k} \right] \geq r_1 \Pi_{c_1}^{d_1}[p^2(t)].$$

Therefore, using (8), we get

$$\begin{aligned} & \int_{c_1}^{t_{I(c_1)+1}} \left[\gamma(p'(t))^2 t^{2-2\alpha} r(t) - p^2(t)Q(t)N_{I(c_1)}^1(t) \right] dt \\ & + \sum_{k=I(c_1)+1}^{I(d_1)-1} \int_{t_k}^{t_{k+1}} \left[\gamma(p'(t))^2 t^{2-2\alpha} r(t) - p^2(t)Q(t)N_k^1(t) \right] dt \\ & + \int_{t_{I(d_1)}}^{d_1} \left[\gamma(p'(t))^2 t^{2-2\alpha} r(t) - p^2(t)Q(t)N_{I(d_1)}^1(t) \right] dt \\ & + \int_{c_1}^{d_1} (1 - \alpha)t^{-\alpha} p^2(t)w(t) dt > r_1 \Pi_{c_1}^{d_1}[p^2(t)], \end{aligned}$$

which contradicts (2).

If $I(c_1) = I(d_1)$, then $r_1 \Pi_{c_1}^{d_1}[p^2(t)] = 0$ and there are no impulsive moments in $[c_1, d_1]$. Similarly to the proof of (8), we obtain

$$\int_{c_1}^{d_1} \left[\gamma(p'(t))^2 t^{2-2\alpha} r(t) - p^2(t)Q(t)N_{I(c_1)}^1(t) + p^2(t)(1 - \alpha)t^{-\alpha}w(t) \right] dt > 0.$$

This again contradicts our assumption. Finally, if $x(t)$ is eventually negative, we can consider $[c_2, d_2]$ and reach a similar contradiction. The proof of the theorem is complete.

Next, we establish new oscillation criteria for (1), using the integral average method [13]. Let $D = \{(t, s) : t_0 \leq s \leq t\}$, then the functions $H_1, H_2 \in C(D, \mathbb{R})$ are said to belong to the class \mathcal{H} if

(H₄) $H_1(t, t) = H_2(t, t) = 0, H_1(t, s) > 0, H_2(t, s) > 0$ for $t > s$ and

(H₅) H_1 and H_2 have partial derivatives $\frac{\partial H_1}{\partial t}$ and $\frac{\partial H_2}{\partial s}$ on D such that

$$\frac{\partial H_1}{\partial t} = h_1(t, s)H_1(t, s), \quad \frac{\partial H_2}{\partial s} = -h_2(t, s)H_2(t, s)$$

where $h_1, h_2 \in L_{loc}(D, \mathbb{R})$.

$$\begin{aligned} \Gamma_{1,j} &= \int_{c_j}^{t_{I(c_j)+1}} H_1(t, c_j)Q(t)N_{I(c_j)}^j(t)dt \\ &+ \sum_{k=I(c_j)+1}^{I(\lambda_j)-1} \int_{t_k}^{t_{k+1}} H_1(t, c_j)Q(t)N_k^j(t)dt \\ &+ \int_{t_{I(\lambda_j)}}^{\lambda_j} H_1(t, c_j)Q(t)N_{I(d_j)}^j(t)dt \\ &+ \int_{c_j}^{\lambda_j} H_1(t, c_j) \left[\frac{w(t)}{\gamma r(t)} - t^{1-\alpha}h_1(t, c_j) - (1 - \alpha)t^{-\alpha} \right] w(t)dt \end{aligned}$$

and

$$\begin{aligned} \Gamma_{2,j} &= \int_{\lambda_j}^{t_{I(\lambda_j)+1}} H_2(d_j, t) Q(t) N_{I(\lambda_j)}^j(t) dt \\ &+ \sum_{k=I(\lambda_j)+1}^{I(d_j)-1} \int_{t_k}^{t_{k+1}} H_2(d_j, t) Q(t) N_k^j(t) dt \\ &+ \int_{t_{I(d_j)}}^{d_j} H_2(d_j, t) Q(t) N_{I(d_j)}^j(t) dt \\ &+ \int_{\lambda_j}^{d_j} H_2(d_j, t) \left[\frac{w(t)}{\gamma r(t)} + t^{1-\alpha} h_2(d_j, t) - (1-\alpha)t^{-\alpha} \right] w(t) dt. \end{aligned}$$

THEOREM 3.2. Assume that conditions $(H_1) - (H_3)$ hold. Furthermore, for any $T \geq 0$ there exist c_j, d_j satisfying $(H_4), (H_5)$ with $c_1 < \lambda_1 < d_1 \leq c_2 < \lambda_2 < d_2$. If there exist $H_1, H_2 \in \mathcal{H}$ such that

$$\frac{1}{H_1(\lambda_1, c_1)} \Gamma_{1,1} + \frac{1}{H_2(d_1, \lambda_1)} \Gamma_{2,1} > \Lambda(H_1, H_2; c_j, d_j), \quad (12)$$

where

$$\Lambda(H_1, H_2; c_j, d_j) = - \left\{ \frac{r_j}{H_1(\lambda_j, c_j)} \Pi_{c_j}^{\lambda_j} [H_1(\cdot, c_j)] + \frac{r_j}{H_2(d_j, \lambda_j)} \Pi_{\lambda_j}^{d_j} [H_2(d_j, \cdot)] \right\},$$

then every solution of (1) is oscillatory.

PROOF. Suppose to the contrary that there is a nonoscillatory solution $x(t)$ of the problem (1). Notice whether or not there are impulsive moments in $[c_1, \lambda_1]$ and $[\lambda_1, d_1]$, we should consider the following cases $I(c_1) < I(\lambda_1) < I(d_1)$, $I(c_1) = I(\lambda_1) < I(d_1)$, $I(c_1) < I(\lambda_1) = I(d_1)$ and $I(c_1) = I(\lambda_1) = I(d_1)$. Moreover, the impulsive moments of $x(t - \tau)$ involve the following two cases $t_{I(\lambda_j)} + \tau > \lambda_j$ and $t_{I(\lambda_j)} + \tau \leq \lambda_j$. Consider the case $I(c_1) < I(\lambda_1) < I(d_1)$, with $t_{I(\lambda_j)} + \tau > \lambda_j$. For this case, the impulsive moments are $t_{I(\lambda_1)+1}, t_{I(\lambda_1)+2}, \dots, t_{I(d_1)}$ in $[\lambda_1, d_1]$. Multiplying both sides of (4) by $H_1(t, c_1)$ and integrating from c_1 to λ_1 , we obtain

$$\int_{c_1}^{\lambda_1} H_1(t, c_1) t^{1-\alpha} w'(t) dt \leq - \int_{c_1}^{\lambda_1} H_1(t, c_1) Q(t) \frac{x(t-\tau)}{x(t)} dt - \int_{c_1}^{\lambda_1} H_1(t, c_1) \frac{w^2(t)}{\gamma r(t)} dt.$$

Applying integration by parts on the R.H.S of first integral, we get

$$\begin{aligned} & \int_{c_1}^{\lambda_1} H_1(t, c_1) Q(t) \frac{x(t-\tau)}{x(t)} dt \\ & + \int_{c_1}^{\lambda_1} \left(\frac{w(t)}{\gamma r(t)} - t^{1-\alpha} h_1(t, c_1) - (1-\alpha)t^{-\alpha} \right) w(t) H_1(t, c_1) dt \\ & \leq - \sum_{k=I(c_1)+1}^{I(\lambda_1)} H_1(t_k, c_1) t_k^{1-\alpha} [w(t_k) - w(t_k^+)] - H_1(\lambda_1, c_1) \lambda_1^{1-\alpha} w(\lambda_1). \quad (13) \end{aligned}$$

By Theorem 3.1, we divide the interval $[c_1, \lambda_1]$ into several and calculating the function $\frac{x(t - \tau)}{x(t)}$, we obtain

$$\begin{aligned} \int_{c_1}^{\lambda_1} H_1(t, c_1)Q(t) \frac{x(t - \tau)}{x(t)} dt &\geq \int_{c_1}^{t_{I(c_1)+1}} H_1(t, c_1)Q(t)N_{I(c_1)}^1(t)dt \\ &+ \sum_{k=I(c_1)+1}^{I(\lambda_1)-1} \int_{t_k}^{t_{k+1}} H_1(t, c_1)Q(t)N_k^1(t)dt \\ &+ \int_{t_{I(\lambda_1)}}^{\lambda_1} H_1(t, c_1)Q(t)N_{I(\lambda_1)}^1(t)dt. \end{aligned} \tag{14}$$

From (13) and (14), we obtain

$$\begin{aligned} &\int_{c_1}^{t_{I(c_1)+1}} H_1(t, c_1)Q(t)N_{I(c_1)}^1(t)dt + \sum_{k=I(c_1)+1}^{I(\lambda_1)-1} \int_{t_k}^{t_{k+1}} H_1(t_k, c_1)Q(t)N_k^1(t)dt \\ &+ \int_{t_{I(\lambda_1)}}^{\lambda_1} H_1(t, c_1)Q(t)N_{I(\lambda_1)}^1(t)dt \\ &+ \int_{c_1}^{\lambda_1} \left[\frac{w(t)}{\gamma r(t)} - t^{1-\alpha}h_1(t, c_1) - (1 - \alpha)t^{-\alpha} \right] w(t)H_1(t, c_1)dt \\ \leq &- \sum_{k=I(c_1)+1}^{I(\lambda_1)} H_1(t_k, c_1)t_k^{1-\alpha} \left[\frac{a_k - b_k}{a_k} \right] w(t_k) - H_1(\lambda_1, c_1)\lambda_1^{1-\alpha}w(\lambda_1). \end{aligned} \tag{15}$$

On the other hand multiplying both sides of (4) by $H_2(d_1, t)$, integrating from λ_1 to d_1 and following a similar procedure as above, we get

$$\begin{aligned} &\int_{\lambda_1}^{t_{I(\lambda_1)+1}} H_2(d_1, t)Q(t)N_{I(\lambda_1)}^1(t)dt + \sum_{k=I(\lambda_1)+1}^{I(d_1)-1} \int_{t_k}^{t_{k+1}} H_2(d_1, t_k)Q(t)N_k^1(t)dt \\ &+ \int_{t_{I(d_1)}}^{d_1} H_2(d_1, t)Q(t)N_{I(d_1)}^1(t)dt \\ &+ \int_{\lambda_1}^{d_1} \left[\frac{w(t)}{\gamma r(t)} + t^{1-\alpha}h_2(d_1, t) - (1 - \alpha)t^{-\alpha} \right] w(t)H_2(d_1, t)dt \\ \leq &- \sum_{k=I(\lambda_1)+1}^{I(d_1)} H_2(d_1, t_k) \left[\frac{a_k - b_k}{a_k} \right] w(t_k) + H_2(d_1, \lambda_1)\lambda_1^{1-\alpha}w(\lambda_1). \end{aligned} \tag{16}$$

Dividing (15) and (16) by $H_1(\lambda_1, c_1)$ and $H_2(d_1, \lambda_1)$ respectively and summing the

resulting inequalities, we get

$$\begin{aligned} & \frac{1}{H_1(\lambda_1, c_1)}\Gamma_{1,1} + \frac{1}{H_2(d_1, \lambda_1)}\Gamma_{2,1} \\ & \leq - \left[\frac{1}{H_1(\lambda_1, c_1)} \sum_{k=I(c_1)+1}^{I(\lambda_1)} H_1(t_k, c_1) \left[\frac{a_k - b_k}{a_k} \right] w(t_k) \right. \\ & \quad \left. + \frac{1}{H_2(d_1, \lambda_1)} \sum_{k=I(\lambda_1)+1}^{I(d_1)} H_2(d_1, t_k) \left[\frac{a_k - b_k}{a_k} \right] w(t_k) \right]. \end{aligned} \tag{17}$$

Using a similar method as in (10), we obtain

$$\begin{cases} - \sum_{k=I(c_1)+1}^{I(\lambda_1)} H_1(t_k, c_1) \left[\frac{a_k - b_k}{a_k} \right] w(t_k) \leq -r_1 \Pi_{c_1}^{\lambda_1} [H_1(\cdot, c_1)], \\ - \sum_{k=I(\lambda_1)+1}^{I(d_1)} H_2(d_1, t_k) \left[\frac{a_k - b_k}{a_k} \right] w(t_k) \leq -r_2 \Pi_{\lambda_1}^{d_1} [H_2(d_1, \cdot)]. \end{cases} \tag{18}$$

From (17) and (18), we obtain

$$\begin{aligned} \frac{1}{H_1(\lambda_1, c_1)}\Gamma_{1,1} + \frac{1}{H_2(d_1, \lambda_1)}\Gamma_{2,1} & \leq - \left\{ r_1 \Pi_{c_1}^{\lambda_1} [H_1(\cdot, c_1)] + r_2 \Pi_{\lambda_1}^{d_1} [H_2(d_1, \cdot)] \right\} \\ & \leq \Lambda(H_1, H_2; c_j, d_j) \end{aligned} \tag{19}$$

which is a contradiction to condition (12). Suppose $x(t) < 0$, we take the interval $[c_2, d_2]$ for equation (1). The proof is similar and hence omitted.

4 Example

In this section, we present an example to illustrate the results established in Section 3.

EXAMPLE 4.1. Consider the following impulsive partial differential equation

$$\begin{cases} T_{\frac{1}{2}} \left(t \left(T_{\frac{1}{2}}(\sin(t)) + mt^{\frac{1}{2}} \cos \frac{\pi}{8} \sin(t) \right) \right) - \frac{3}{2}mt \sin(t - \frac{\pi}{8}) = e(t), \quad t \neq 2k\pi \pm \frac{\pi}{4}, \\ x(t_k^+) = \frac{1}{2}x(t_k), \\ T_{\frac{1}{2}}(x(t_k^+)) = \frac{3}{2}T_{\frac{1}{2}}(x(t_k)), \quad k = 1, 2, \dots \end{cases} \tag{20}$$

Here $\alpha = \frac{1}{2}$, $r(t) = t$, $\mu(t) = mt^{\frac{1}{2}} \cos(\pi/8)$, $q(t) = -3/2mt$, $f(u) = 2u$, $e(t) = t(\cos t - t \sin t) + mt \cos t(t \cos \frac{\pi}{8} + \frac{3}{2} \sin \frac{\pi}{8})$, $a_k = 1/2$, $b_k = 3/2$. Let $\tau = \frac{\pi}{8}$, $t_{k+1} - t_k = \frac{\pi}{2} > \frac{\pi}{8}$. Also for any $T > 0$, we choose k large enough such that $T < c_1 = 4k\pi - \frac{\pi}{2} < d_1 = 4k\pi$ and $c_2 = 4k\pi + \frac{\pi}{8} < d_2 = 4k\pi + \frac{\pi}{2}$, $k = 1, 2, 3, \dots$. Then there is an impulsive movement $t_k = 4k\pi - \frac{\pi}{4}$ in $[c_1, d_1]$ and an impulsive moment $t_{k+1} = 4k\pi + \frac{\pi}{4}$ in $[c_2, d_2]$.

For $\epsilon = 1$, we have $Q(t) = -3/4mt$, and we take $p(t) = \sin 4t$, $t_{I(c_1)} = 4k\pi - \frac{7\pi}{4}$, $t_{I(d_1)} = 4k\pi - \frac{\pi}{4}$, then by using simple calculation, the left side of Equation (2) is the

following:

$$\begin{aligned}
 & \int_{c_j}^{t_{I(c_j)+1}} \left[(p'(t))^2 t^{2-2\alpha} \gamma r(t) - Q(t) p^2(t) N_{I(c_j)}^j(t) \right] dt \\
 & + \sum_{k=I(c_j)+1}^{I(d_j)-1} \int_{t_k}^{t_{k+1}} \left[(p'(t))^2 t^{2-2\alpha} \gamma r(t) - Q(t) p^2(t) N_{I(c_j)}^j(t) \right] dt \\
 & + \int_{t_{I(d_j)}}^{d_j} \left[(p'(t))^2 t^{2-2\alpha} \gamma r(t) - Q(t) p^2(t) N_{I(d_j)}^j(t) \right] dt + \int_{c_j}^{d_j} w(t) p^2(t) (1-\alpha) t^{-\alpha} dt \\
 \leq & \int_{4k\pi - \frac{\pi}{2}}^{4k\pi - \frac{\pi}{4}} \left[32t^2 \cos^2 4t + \frac{3}{4} m t \sin^2 4t \left(\frac{\left(t - \frac{\pi}{8}\right)^{\frac{1}{2}} - \left(4k\pi - \frac{7\pi}{4}\right)^{\frac{1}{2}}}{t^{\frac{1}{2}} - \left(4k\pi - \frac{7\pi}{4}\right)^{\frac{1}{2}}} \right) \right] dt \\
 & + \int_{4k\pi - \frac{\pi}{4}}^{4k\pi - \frac{\pi}{8}} \left[32t^2 \cos^2 4t + \frac{3}{4} m t \sin^2 4t \left(\frac{\frac{\pi}{16}}{\frac{\pi}{32} + \frac{3}{2} \left[t^{\frac{1}{2}} - \left(4k\pi - \frac{\pi}{4}\right)^{\frac{1}{2}}\right]} \right) \right. \\
 & \quad \left. \times \left(\frac{\left(t - \frac{\pi}{8}\right)^{\frac{1}{2}} - \left(4k\pi - \frac{3\pi}{8}\right)^{\frac{1}{2}}}{\left(4k\pi - \frac{\pi}{4}\right)^{\frac{1}{2}} - \left(4k\pi - \frac{3\pi}{8}\right)^{\frac{1}{2}}} \right) \right] dt \\
 & + \int_{4k\pi - \frac{\pi}{8}}^{4k\pi} \left[32t^2 \cos^2 4t + \frac{3}{4} m t \sin^2 4t \left(\frac{\left(t - \frac{\pi}{8}\right)^{\frac{1}{2}} - \left(4k\pi - \frac{\pi}{4}\right)^{\frac{1}{2}}}{t^{\frac{1}{2}} - \left(4k\pi - \frac{7\pi}{4}\right)^{\frac{1}{2}}} \right) \right] dt \\
 & + \frac{1}{2} \int_{4k\pi - \frac{\pi}{2}}^{4k\pi} t \sin^2 4t \left(\frac{\cos t + m \cos \frac{\pi}{8} \sin t}{\sin t} \right) dt \\
 \simeq & 3486.0599 + m 8.9095.
 \end{aligned}$$

Since $I(c_1) = k - 1, I(d_1) = k, r_1 = 2$, we have

$$r_1 \Pi_{c_1}^{d_1} [p^2(t)] = 2 \left[\frac{a_{I(c)+1} - b_{I(c)+1}}{a_{I(c)+1} (t_{I(c_1)+1}^\alpha - c^\alpha)} \sin^2(4t_{I(c_1)+1}) \right] = 0.$$

Note that condition (2) is satisfied in $[c_1, d_1]$ if

$$3486.0599 < -m 8.9095, \tag{21}$$

we can choose the constant m very small enough so that (21) holds. Therefore, condition (6) is satisfied in $[c_1, d_1]$. We can work similarly, for $t \in [c_2, d_2]$. Hence by Theorem 3.1, every solution of (20) is oscillatory. In fact $x(t) = \sin t$ is one such solution.

Conclusion: In this article, we have presented new sufficient conditions to check for the interval oscillation of equation (1). To establish those conditions we have used the Riccati transformation and an integral averaging method. Our results are original,

complementing and generalizing on already existing results in the literature of integer order equations. We have provided an example to illustrate the use of our results.

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