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Existence Theory And Ulam-Hyers Stability To Anti-Periodic Integral Boundary Value Problem Of Implicit Fractional Differential Equations*

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Abstract

In this paper, we study the existence and stability results of Hyers-Ulam type to a class of implicit nonlinear fractional differential equations (FDEs) corresponding to implicit integral boundary condition. Using fixed point theory and non-linear functional analysis, we investigate the mentioned aspects of qualitative theory. At the end of our results, we provide an illustrative and interesting example to justify our obtained results.

1 Introduction

Fractional derivative is used as a global operator for modeling of various processes and physical systems which arise in subjects like physics, dynamics, fluid mechanics, control theory, chemistry and mathematical biology, etc., see [1, 2, 3, 4, 5, 6, 7]. For its importance and large number of applications, this area has been attracted the attention of many mathematicians and researchers in the last few decades. The keen interest of the researchers in the investigation of FDEs is due to the fact that these differential equations describe the properties of hereditary materials and processes more accurate and significantly. More specifically FDEs with integral boundary condition (IBC) are applicable in different fields of applied sciences including population dynamics, thermo-elasticity, problems concerning to blood flow, underground water flow, chemical engineering and so on, see [8, 9]. Various aspects of qualitative theory for FDEs including existence theory and approximate solutions have up-to-date investigated.

Now we want to discuss another aspect of qualitative theory which is stability analysis. In fields such as numerical analysis, optimization theory and nonlinear analysis, mostly we deal with approximate solutions and hence we need to check how closed are these solutions to the actual solution of the concerned system or systems. Many approaches can be used for this purpose but Hyers-Ulam stability (HUS) approach is a simple and easy one. Initially this concept was given by Ulam [10] in 1940, which was

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properly formulated by Hyers [11] in 1941 for problems of functional equations in Banach spaces, see [12, 13]. Later on this concept was extended by other mathematicians to differential and integral equations, see [14, 15, 16].

Very recently in [17], Ali et al. investigated the existence and Hyers-Ulam stability analysis to a class of implicit type integral BVP of FDEs. Similarly the aforesaid aspects have been studied in many other recent papers, such as [18, 19, 20, 21]. In this manuscript, we study the following class of implicit FDEs with implicit IBC

$$\begin{aligned} {}^{C}\mathscr{D}^{\omega}p(t) &= \mathscr{G}(t, p(t), {}^{C}\mathscr{D}^{\omega}p(t)),\ 0 < \omega \leq 1, \\ p(0) &= -\int_0^T \frac{(T-\xi)^{\omega-1}}{\Gamma(\omega)}\mathcal{F}(\xi, p(\xi), {}^{C}\mathscr{D}^{\omega}p(\xi))d\xi, \end{aligned} \tag{1}$$

where the notation ${}^{C}\mathscr{D}^{\omega}$ is used for Caputo fractional derivative of order $0 < \omega \leq 1$, $\mathscr{X} = [0, T]$ and $\mathscr{G}, \mathcal{F} : \mathscr{X} \times \mathscr{R}^2 \to \mathscr{R}$.

Using classical fixed point theory due to Banach and Schaefer, we derive necessary conditions for the existence, uniqueness and stability analysis to the concerned class of FDEs given in (1).

This paper is managed as: In section 2, we provide auxiliary results concerning to fractional derivative, integration and Hyers-Ulam stability. In section 3, we establish main results. In section 4, we provide a suitable example which illustrates the obtained results.

2 Preliminaries

We represent the space of all continuous functions, $C(\mathscr{X}, \mathcal{R})$ by $\mathscr{A}$. We define $\mathscr{A} = \{p : \mathscr{X} \to \mathcal{R} :\ p \in C(\mathscr{X})\}$. Clearly, $\mathscr{A}$ is Banach space with the norm defined by $\|p\| := \sup\{|p(t)|,\ t \in \mathscr{X}\}$.

DEFINITION 1 ([1, 2]). If $\omega > 0$, the Riemann-Liouville integral of function $\mathscr{G} \in L^1(\mathscr{X}, \mathscr{R})$, is defined by

$$I^{\omega}\mathscr{G}(t) = \frac{1}{\Gamma(\omega)}\int_0^t (t-\xi)^{\omega-1}\mathscr{G}(\xi)\, d\xi,$$

provided that integral on the right is point-wise defined on $(0, \infty)$.

DEFINITION 2 ([1, 2]). If $\omega > 0$, then the Caputo fractional derivative of function $\mathscr{G}$ on the interval $\mathscr{X}$, is defined by

$${}^{C}\mathscr{D}^{\omega}\mathscr{G}(t) = \frac{1}{\Gamma(n-\omega)}\int_0^t (t-\xi)^{n-\omega-1}\mathscr{G}^{(n)}(\xi)\, d\xi,$$

provided that integral on the right is point wise defined on $(0, \infty)$, where $n = [\omega] + 1$ in which $[\omega]$ represents the integer part of ω.

LEMMA 1 ([1, 2]). For $\omega > 0$, the following result holds

$${}^{C}\mathscr{D}^{\omega}\mathscr{G}(t) = \mathscr{G}(t) + \mathbf{r}_0 + \mathbf{r}_1 t + \mathbf{r}_2 t^2 + \cdots + \mathbf{r}_{i-1}t^{i-1},$$

where $\mathbf{r}_{i-1} \in \mathscr{R}$ and $i = 1, 2, \ldots, n$.

LEMMA 2 ([1, 2]). For $\omega > 0$, $n = [\omega] + 1$, the relation

$$I^{\omega C}\mathscr{D}^{\omega}\mathscr{G}(t) = \mathscr{G}(t) - \sum_{i=0}^{n-1} \frac{\mathscr{G}^i(0)}{i!} t^i \textit{ holds.}$$

Here we mention that in this paper the definitions concerning to stability have been adopted from [22].

DEFINITION 3. Problem (1) is HUS if there is a real number $C_{\mathscr{G}} > 0$ such that for each $\epsilon > 0$ and each solution $q \in \mathscr{A}$ of the inequality

$$|^C\mathscr{D}^{\omega}q(t) - \mathscr{G}(t, q(t), ^C\mathscr{D}^{\omega}q(t))| \leq \epsilon,\ t \in \mathscr{X}, \tag{2}$$

there exists a unique solution $p \in \mathscr{A}$ of problem (1) with

$$|q(t) - p(t)| \leq C_{\mathscr{G}}\epsilon, t \in \mathscr{X}.$$

DEFINITION 4. Problem (1) is GHUS if there is a function $\digamma_{\mathscr{G}} \in C(\mathscr{R}_+, \mathscr{R}_+)$, $\digamma_{\mathscr{G}}(0) = 0$ such that for each solution $q \in \mathscr{A}$ of the inequality (2) there exists a unique solution $p \in \mathscr{A}$ of problem (1) with

$$|q(t) - p(t)| \leq \digamma_{\mathscr{G}}(\epsilon),\ t \in \mathscr{X}.$$

DEFINITION 5. Problem (1) is HURS with respect to a function $\psi \in C(\mathscr{X}, \mathscr{R}_+)$ if there is a real number $C_{\mathscr{G},\psi} > 0$ such that for each solution $q \in \mathscr{A}$ of the inequality

$$|^C\mathscr{D}^{\omega}q(t) - \mathscr{G}(t, q(t), ^C\mathscr{D}^{\omega}q(t))| \leq \epsilon\psi(t),\ t \in \mathscr{X}, \tag{3}$$

there exists a unique solution $p \in \mathscr{A}$ of problem (1) with

$$|q(t) - p(t)| \leq C_{\mathscr{G}}\epsilon\psi(t),\ t \in \mathscr{X}.$$

DEFINITION 6. Problem (1) is GHURS with respect to $\psi \in C(\mathscr{X}, \mathscr{R}_+)$ if there is a real number $C_{\mathscr{G},\psi} > 0$ such that for each solution $q \in \mathscr{A}$ of the inequality

$$|^C\mathscr{D}^{\omega}q(t) - \mathscr{G}(t, q(t), ^C\mathscr{D}^{\omega}q(t))| \leq \psi(t),\ t \in \mathscr{X}, \tag{4}$$

there exists a unique solution $p \in \mathscr{A}$ of problem (1) with

$$|q(t) - p(t)| \leq C_{\mathscr{G},\psi}\psi(t),\ t \in \mathscr{X}.$$

REMARK 1. It is clear that

(i) Definition 3 implies Definition 4;

(ii) Definition 5 implies Definition 6.

REMARK 2. A function $q \in \mathscr{A}$ is a solution of the inequality (2) if and only if there exists a function $\Psi \in \mathscr{A}$ (dependent on p) such that

(i) $|\Psi(t)| \leq \epsilon$, for all $t \in \mathscr{X}$;

(ii) ${}^C\mathscr{D}^\omega q(t) = \mathscr{G}(t, q(t), {}^C\mathscr{D}^\omega q(t)) + \Psi(t)$, $t \in \mathscr{X}$.

REMARK 3. A function $q \in \mathscr{A}$ is a solution of the inequality (4) if and only if there exists a function $\Psi \in \mathscr{A}$ (dependent on p) such that

(i) $|\Psi(t)| \leq \epsilon\psi(t)$, for all $t \in \mathscr{X}$;

(ii) ${}^C\mathscr{D}^\omega q(t) = \mathscr{G}(t, q(t), {}^C\mathscr{D}^\omega q(t)) + \Psi(t)$, $t \in \mathscr{X}$.

THEOREM 1. (Schaefer's fixed point theorem [23]). Let $\mathscr{A}$ be a Banach space, $\mathscr{T} : \mathscr{A} \to \mathscr{A}$ is a completely continuous operator and the set $\mathscr{E} = \{p \in \mathscr{A} : p = \xi\mathscr{T}p, 0 < \xi < 1\}$ is bounded, then $\mathscr{T}$ has at least one fixed point in $\mathscr{A}$.

3 Existence of at Least One Solution and Hyers-Ulam Stability: Main Results

LEMMA 3. The antiperiodic integral boundary value problem

$$\begin{aligned} &{}^C\mathscr{D}^\omega p(t) = \mathscr{G}(t, p(t), {}^C\mathscr{D}^\omega p(t)),\ 0 < \omega \leq 1, \\ &p(0) = -\int_0^T \frac{(T-\xi)^{\omega-1}}{\Gamma(\omega)} \mathcal{F}(\xi, p(\xi), {}^C\mathscr{D}^\omega p(\xi)) d\xi, \end{aligned} \tag{5}$$

has a solution p, given by

$$p(t) = \frac{1}{\Gamma(\omega)} \int_0^t (t-\xi)^{\omega-1} \alpha(\xi) d\xi - \frac{1}{\Gamma(\omega)} \int_0^T (T-\xi)^{\omega-1} \mathcal{F}(\xi, p(\xi), {}^C\mathscr{D}^\omega p(\xi)) d\xi, \tag{6}$$

where $\alpha \in \mathscr{A}$ is given as

$$\alpha(t) = \mathscr{G}(t, p(t), {}^C\mathscr{D}^\omega p(t)).$$

PROOF. Let

$${}^C\mathscr{D}^\omega p(t) = \alpha(t).$$

By Lemma 1, we have

$$p(t) = \frac{1}{\Gamma(\omega)} \int_0^t (t-\xi)^{\omega-1} \alpha(t) d\xi + \mathbf{r}_0. \tag{7}$$

Applying the given condition, we obtain

$$\mathbf{r}_0 = -\frac{1}{\Gamma(\omega)} \int_0^T (T-\xi)^{\omega-1} \mathcal{F}(\xi, p(\xi), {}^C\mathscr{D}^{\omega} p(\xi)) d\xi. \tag{8}$$

By putting this value in (7), we get (6).

Here we list some assumptions upon which our results are based.

(H$_1$) $\mathscr{G} : \mathscr{X} \times \mathscr{R}^2 \to \mathscr{R}$ is continuous;

(H$_2$) $\mathcal{F} : \mathscr{X} \times \mathscr{R}^2 \to \mathscr{R}$ is continuous;

(H$_3$) there exist constants $\aleph_1 > 0$ and $0 < \aleph_2 < 1$ such that for each $t \in \mathscr{X}$ and for all $\sigma, \bar{\sigma}, \theta, \bar{\theta} \in \mathscr{R}$, the following relation holds

$$|\mathscr{G}(t, \sigma, \theta) - \mathscr{G}(t, \bar{\sigma}, \bar{\theta})| \leq \aleph_1 |\sigma - \bar{\sigma}| + \aleph_2 |\theta - \bar{\theta}|;$$

(H$_4$) there exist constants $\aleph_3 > 0$ and $0 < \aleph_4 < 1$ such that for each $t \in \mathscr{X}$ and for all $\sigma, \bar{\sigma}, \theta, \bar{\theta} \in \mathscr{R}$, the following relation holds

$$|\mathcal{F}(t, \sigma, \theta) - \mathcal{F}(t, \bar{\sigma}, \bar{\theta})| \leq \aleph_3 |\sigma - \bar{\sigma}| + \aleph_4 |\theta - \bar{\theta}|;$$

(H$_5$) there exist constants $l, m, n \in C(\mathscr{X}, \mathscr{R}^+)$, such that $|\mathscr{G}(t, \sigma(t), \theta(t))| \leq l(t) + m(t)|\sigma| + n(t)|\theta|$, with $n^* = \sup_{t \in \mathscr{X}} n(t) < 1$;

(H$_6$) there exist constants $b, c, e \in C(\mathscr{X}, \mathscr{R}^+)$, such that $|\mathcal{F}(t, \sigma(t), \theta(t))| \leq b(t) + c(t)|\sigma| + e(t)|\theta|$, with $e^* = \sup_{t \in \mathscr{X}} e(t) < 1$.

THEOREM 2. If the hypothesis (H_1)–(H_4) together with the inequality

$$\frac{T^{\omega}}{\Gamma(\omega+1)} \left(\frac{\aleph_1}{1-\aleph_2} + \frac{\aleph_3}{1-\aleph_4} \right) < 1, \tag{9}$$

are satisfied, then problem (5) has just one (unique) solution.

PROOF. We define an operator $\mathscr{T} : \mathscr{A} \to \mathscr{A}$ by

$$\mathscr{T} p(t) = \frac{1}{\Gamma(\omega)} \int_0^t (t-\xi)^{\omega-1} \alpha(\xi) d\xi - \frac{1}{\Gamma(\omega)} \int_0^T (T-\xi)^{\omega-1} \beta(\xi) d\xi,$$

where $\alpha, \beta \in \mathscr{A}$:

$$\alpha(t) = \mathscr{G}(t, p(t), \alpha(t)) \quad \text{and} \quad \beta(t) = \mathcal{F}(t, p(t), \beta(t)).$$

We use Banach Contraction Mapping Principle. Consider for $p, q \in \mathscr{A}$,

$$\begin{aligned} |\mathscr{T} p(t) - \mathscr{T} q(t)| &\leq \frac{1}{\Gamma(\omega)} \int_0^t (t-\xi)^{\omega-1} |\alpha(\xi) - \bar{\alpha}(\xi)| d\xi \\ &+ \frac{1}{\Gamma(\omega)} \int_0^T (T-\xi)^{\omega-1} |\beta(\xi) - \bar{\beta}(\xi)| d\xi, \end{aligned} \tag{10}$$

where $\bar{\alpha}, \bar{\beta} \in \mathscr{A}$:

$$\bar{\alpha}(t) = \mathscr{G}(t, q(t), \bar{\alpha}(t)) \text{ and } \bar{\beta}(t) = \mathcal{F}(t, q(t), \bar{\beta}(t)).$$

Hence by (H_3) and (H_4), we have

$$\begin{aligned} |\alpha(t) - \bar{\alpha}(t)| &= |\mathscr{G}(t, p(t), \alpha(t)) - \mathscr{G}(t, q(t), \bar{\alpha}(t))| \\ &\leq \aleph_1 |p(t) - q(t) + \aleph_2 |\alpha(t) - \bar{\alpha}(t)|. \end{aligned}$$

Thus

$$|\alpha(t) - \bar{\alpha}(t)| \leq \frac{\aleph_1}{1 - \aleph_2} |p(t) - q(t)|. \tag{11}$$

Similarly

$$|\beta(t) - \bar{\beta}(t)| \leq \frac{\aleph_3}{1 - \aleph_4} |p(t) - q(t)|. \tag{12}$$

Using (11) and (12) in (10), we have

$$\begin{aligned} |\mathscr{T} p(t) - \mathscr{T} q(t)| &\leq \frac{\aleph_1}{(1 - \aleph_2)\Gamma(\omega)} \int_0^t (t - \xi)^{\omega - 1} |p(\xi) - q(\xi)| d\xi \\ &+ \frac{\aleph_3}{(1 - \aleph_4)(\Gamma(\omega)} \int_0^T (T - \xi)^{\omega - 1} |p(\xi) - q(\xi)| d\xi \\ &\leq \frac{T^\omega}{\Gamma(\omega + 1)} \left(\frac{\aleph_1}{1 - \aleph_2} + \frac{\aleph_3}{1 - \aleph_4} \right) |p(t) - q(t)|. \end{aligned}$$

Thus

$$\|\mathscr{T} p - \mathscr{T} q\| \leq \frac{T^\omega}{\Gamma(\omega + 1)} \left(\frac{\aleph_1}{1 - \aleph_2} + \frac{\aleph_3}{1 - \aleph_4} \right) \|p - q\|.$$

Since

$$\frac{T^\omega}{\Gamma(\omega + 1)} \left(\frac{\aleph_1}{1 - \aleph_2} + \frac{\aleph_3}{1 - \aleph_4} \right) < 1,$$

therefore, by Banach Contraction Mapping Principle, $\mathscr{T}$ has unique fixed point. Thus problem (5) has just one (unique) solution.

THEOREM 3. Under the hypothesis (H_1)–(H_6), problem (5) has at least one solution.

PROOF. In this result we recall the Schaefer's fixed point theorem and consider the predefined operator $\mathscr{T}$. The proof accomplishes in four steps.

Step 1: We claim that $\mathscr{T}$ is continuous. Take a sequence $\{p_n\}$ in $\mathscr{A}$ such that $p_n \to p \in \mathscr{A}$. For $t \in \mathscr{X}$, consider

$$\begin{aligned} |\mathscr{T} p_n(t) - \mathscr{T} p(t)| &\leq \frac{1}{\Gamma(\omega)} \int_0^t (t - \xi)^{\omega - 1} |\alpha_n(\xi) - \alpha(\xi)| d\xi \\ &+ \frac{1}{\Gamma(\omega)} \int_0^T (T - \xi)^{\omega - 1} |\beta_n(\xi) - \beta(\xi)| d\xi, \end{aligned}$$

where $\alpha_n, \beta_n \in \mathscr{A}$:

$$\alpha_n(t) = \mathscr{G}(t, p_n(t), \alpha_n(t)) \text{ and } \beta_n(t) = \mathcal{F}(t, p_n(t), \beta_n(t)).$$

Hence by (H_3) and (H_4), we have

$$\begin{aligned} |\alpha_n(t) - \alpha(t)| &= |\mathscr{G}(t, p_n(t), \alpha_n(t)) - \mathscr{G}(t, p(t), \alpha(t))| \\ &\leq \aleph_1 |p_n(t) - p(t)| + \aleph_2 |\alpha_n(t) - \alpha(t)|. \end{aligned}$$

Thus

$$|\alpha_n(t) - \alpha(t)| \leq \frac{\aleph_1}{1 - \aleph_2} |p_n(t) - p(t)|.$$

Similarly

$$|\beta_n(t) - \beta(t)| \leq \frac{\aleph_3}{1 - \aleph_4} |p_n(t) - p(t)|.$$

Thus

$$\begin{aligned} |\mathscr{T} p_n(t) - \mathscr{T} p(t)| &\leq \frac{\aleph_1}{(1 - \aleph_2)\Gamma(\omega)} \int_0^t (t - \xi)^{\omega - 1} |p_n(\xi) - p(\xi)| d\xi \\ &\quad + \frac{\aleph_3}{(1 - \aleph_4)\Gamma(\omega)} \int_0^T (T - \xi)^{\omega - 1} |p_n(\xi) - p(\xi)| d\xi. \end{aligned}$$

Since for each $t \in \mathscr{X}$ the sequence $p_n \to p$ as $n \to \infty$, hence Lebesgue dominated convergent theorem yields that

$$|\mathscr{T} p_n(t) - \mathscr{T} p(t)| \to 0 \text{ as } n \to \infty$$

or

$$\|\mathscr{T} p_n - \mathscr{T} p\| \to 0 \text{ as } n \to \infty.$$

Thus $\mathscr{T}$ is continuous on $\mathscr{X}$.

Step 2: In this step we claim that bounded sets are mapped into bounded sets by $\mathscr{T}$ in $\mathscr{A}$. Next for each $p \in \mathscr{E}_{\Bbbk} = \{p \in \mathscr{A} : \|p\| \leq \Bbbk\}$, we have to prove $\|\mathscr{T}(p)\| \leq N$, with $N > 0$.

Consider for $t \in \mathscr{X}$ such that

$$\begin{aligned} |\mathscr{T} p(t)| &= \left| \frac{1}{\Gamma(\omega)} \int_0^t (t - \xi)^{\omega - 1} \alpha(\xi) d\xi - \frac{1}{\Gamma(\omega)} \int_0^T (T - \xi)^{\omega - 1} \beta(\xi) d\xi \right| \\ &\leq \frac{1}{\Gamma(\omega)} \int_0^t (t - \xi)^{\omega - 1} |\alpha(\xi)| d\xi + \frac{1}{\Gamma(\omega)} \int_0^T (T - \xi)^{\omega - 1} |\beta(\xi)| d\xi, \end{aligned} \tag{13}$$

where $\alpha, \beta \in \mathscr{A}$:

$$\alpha(t) = \mathscr{G}(t, p(t), \alpha(t)) \text{ and } \beta(t) = \mathcal{F}(t, p(t), \beta(t)).$$

By (H_5), we have

$$\begin{aligned} \alpha(t) &= \mathscr{G}(t, p(t), \alpha(t)) \\ &\leq l(t) + m(t)\|p\| + n(t)|\alpha(t)| \\ &\leq l(t) + m(t)\mathbb{k} + n(t)|\alpha(t)| \\ &\leq l^* + m^*\|p\| + n^*|\alpha(t)|, \end{aligned}$$

where $l^* = \sup_{t\in\mathscr{X}} l(t),\ m^* = \sup_{t\in\mathscr{X}} m(t)\ and\ n^* = \sup_{t\in\mathscr{X}} n(t) < 1$. Then

$$|\alpha(t)| \leq \frac{l^* + m^*\mathbb{k}}{1 - n^*} =: \hbar.$$

Similarly by (H_6), we obtain

$$|\beta(t)| \leq \frac{b^* + c^*\mathbb{k}}{1 - e^*} =: \hbar^*,$$

where $\hbar$ and $\hbar^*$ are positive constants. Thus from (13), we have

$$\|\mathscr{T}p\| = \frac{T^\omega}{\Gamma(\omega+1)}\left(\hbar + \hbar^*\right) =: N.$$

Step 3: We claim that a bounded set is mapped into equi-continuous set of $\mathscr{A}$ by $\mathscr{T}$. We take $t_1, t_2 \in \mathscr{X}$ such that $t_1 < t_2$ and assume that $\mathscr{E}_{\mathbb{k}}$ is a bounded set as in the previous step. Then for $p \in \mathscr{E}_{\mathbb{k}}$, consider

$$|\mathscr{T}p(t_2) - \mathscr{T}p(t_1)| = \left|\frac{1}{\Gamma(\omega)}\int_0^{t_2}(t_2-\xi)^{\omega-1}\alpha(\xi)d\xi - \frac{1}{\Gamma(\omega)}\int_0^{t_1}(t_1-\xi)^{\omega-1}\alpha(\xi)d\xi\right|.$$

In **Step 2**, we have obtained that

$$|\alpha(t)| \leq \frac{l^* + m^*\mathbb{k}}{1 - n^*} =: \hbar.$$

Thus

$$|\mathscr{T}p(t_2) - \mathscr{T}p(t_1)| \leq \hbar\left|\frac{1}{\Gamma(\omega)}\int_0^{t_2}(t_2-\xi)^{\omega-1}d\xi - \frac{1}{\Gamma(\omega)}\int_0^{t_1}(t_1-\xi)^{\omega-1}d\xi\right|. \tag{14}$$

We see that the right hand side of (14) tends to zero as $t_1 \to t_2$. Therefore, as a conclusion of **Step 1**–**Step 3** and Arzela-Ascoli theorem, $\mathscr{T} : \mathscr{A} \to \mathscr{A}$ is a completely continuous mapping.

Step 4: A priori bound. Define a set

$$\mathscr{L} = \{p \in \mathscr{A} : p = \varpi(\mathscr{T}p)\ for\ some\ 0 < \varpi < 1\}.$$

We need to show that $\mathscr{L}$ is bounded. Let $p \in \mathscr{L}$, then for some $0 < \mathscr{L} < 1$, $p = \mathscr{L}(\mathscr{T}p)$. Therefore, for $t \in \mathscr{X}$, we have

$$\begin{aligned} |p(t)| &= \left|\frac{\varpi}{\Gamma(\omega)}\int_0^t(t-\xi)^{\omega-1}\alpha(\xi)d\xi - \frac{\varpi}{\Gamma(\omega)}\int_0^T(T-\xi)^{\omega-1}\beta(\xi)d\xi\right| \\ &\leq \left|\frac{1}{\Gamma(\omega)}\int_0^t(t-\xi)^{\omega-1}\alpha(\xi)d\xi\right| + \left|\frac{1}{\Gamma(\omega)}\int_0^T(T-\xi)^{\omega-1}\beta(\xi)d\xi\right| \end{aligned} \tag{15}$$

or

$$|p(t)| \leq \frac{1}{\Gamma(\omega)}\int_0^t (t-\xi)^{\omega-1}|\alpha(\xi)|d\xi + \frac{1}{\Gamma(\omega)}\int_0^T (T-\xi)^{\omega-1}|\beta(\xi)|d\xi.$$

By (H_5) and (H_6) we have obtained that

$$|\alpha(t)| \leq \frac{l^* + m^*\mathbb{k}}{1-n^*} =: \hbar \;\text{ and }\; |\beta(t)| \leq \frac{b^* + c^*\mathbb{k}}{1-e^*} =: \hbar^*.$$

Hence we have

$$\begin{aligned} |p(t)| &\leq \frac{\hbar}{\Gamma(\omega)}\int_0^t (t-\xi)^{\omega-1}d\xi + \frac{\hbar^*}{\Gamma(\omega)}\int_0^T (T-\xi)^{\omega-1}d\xi \\ &\leq \frac{T^\omega}{\Gamma(\omega+1)}\Big(\hbar + \hbar^*\Big) =: N. \end{aligned}$$

Or $\|p\| \leq N$. This shows that the set $\mathscr{L}$ is bounded. Therefore, by Schaefer's fixed point theorem, $\mathscr{T}$ has at least one fixed point. Which confirms at least one exact solution of problem (5).

THEOREM 4. If the hypothesis (H_1)–(H_4) along with the inequality (9) are satisfied, then problem (5) is HUS as well as GHUS stable.

PROOF. Let q be an approximate solution of the inequality (2) and let p be the unique exact solution of the following problem

$$\begin{aligned} {}^C\mathscr{D}^\omega p(t) &= \mathscr{G}(t, p(t), {}^C\mathscr{D}^\omega p(t)),\; 0 < \omega \leq 1, \\ p(0) &= -\int_0^T \frac{(T-\xi)^{\omega-1}}{\Gamma(\omega)}\mathcal{F}(\xi, p(\xi), {}^C\mathscr{D}^\omega p(\xi))d\xi. \end{aligned}$$

By Lemma 3, we have

$$p(t) = \frac{1}{\Gamma(\omega)}\int_0^t (t-\xi)^{\omega-1}\alpha(\xi)d\xi - \frac{1}{\Gamma(\omega)}\int_0^T (T-\xi)^{\omega-1}\beta(\xi)d\xi,$$

where $\alpha, \beta \in \mathscr{A}$ is given as

$$\alpha(t) = \mathscr{G}(t, p(t), \alpha(t)) \;\text{ and }\; \beta(t) = \mathcal{F}(t, p(t), \beta(t)).$$

Since we have assumed that q is a solution of (2), hence we have by Remark 2.

$$\begin{aligned} {}^C\mathscr{D}^\omega q(t) &= \mathscr{G}(t, q(t), {}^C\mathscr{D}^\omega q(t)) + \Psi(t),\; 0 < \omega \leq 1, \\ q(0) &= -\int_0^T \frac{(T-\xi)^{\omega-1}}{\Gamma(\omega)}\mathcal{F}(\xi, q(\xi), {}^C\mathscr{D}^\omega q(\xi))d\xi. \end{aligned} \tag{16}$$

Clearly the solution of (16) will be

$$q(t) = \frac{1}{\Gamma(\omega)}\int_0^t (t-\xi)^{\omega-1}\bar{\alpha}(\xi)d\xi + \frac{1}{\Gamma(\omega)}\int_0^t (t-\xi)^{\omega-1}\Psi(\xi)d\xi - \frac{1}{\Gamma(\omega)}\int_0^T (T-\xi)^{\omega-1}\bar{\beta}(\xi)d\xi,$$

where $\bar{\alpha}, \bar{\beta} \in \mathscr{A}$ is given as

$$\bar{\alpha}(t) = \mathscr{G}(t, q(t), \bar{\alpha}(t)) \text{ and } \bar{\beta}(t) = \mathcal{F}(t, q(t), \bar{\beta}(t)).$$

For each $t \in \mathscr{X}$ consider

$$\begin{aligned} |q(t) - p(t)| &\leq \frac{1}{\Gamma(\omega)} \int_0^t (t-\xi)^{\omega-1} |\bar{\alpha}(\xi) - \alpha(\xi)| d\xi + \frac{1}{\Gamma(\omega)} \int_0^t (t-\xi)^{\omega-1} |\Psi(\xi)| d\xi \\ &+ \frac{1}{\Gamma(\omega)} \int_0^T (T-\xi)^{\omega-1} |\bar{\beta}(\xi) - \beta(\xi)| d\xi. \end{aligned} \tag{17}$$

By (H_3) and (H_4) we get

$$|\bar{\alpha}(t) - \alpha(t)| \leq \frac{\aleph_1}{1-\aleph_2} |q(t) - p(t)|$$

and

$$|\bar{\beta}(t) - \beta(t)| \leq \frac{\aleph_3}{1-\aleph_4} |q(t) - p(t)|.$$

Hence using part (i) of Remark 2, we have from (17)

$$\begin{aligned} |q(t) - p(t)| &\leq \frac{\aleph_1}{(1-\aleph_2)\Gamma(\omega)} \int_0^t (t-\xi)^{\omega-1} |q(\xi) - p(\xi)| d\xi + \frac{\epsilon}{\Gamma(\omega)} \int_0^t (t-\xi)^{\omega-1} d\xi \\ &+ \frac{\aleph_3}{(1-\aleph_4)\Gamma(\omega)} \int_0^T (T-\xi)^{\omega-1} |q(\xi) - p(\xi)| d\xi \\ &\leq \frac{T^\omega}{\Gamma(\omega+1)} \left(\frac{\aleph_1}{1-\aleph_2} + \frac{\aleph_3}{1-\aleph_4} \right) |q(t) - p(t)| + \frac{\epsilon T^\omega}{\Gamma(\omega+1)}. \end{aligned}$$

Thus

$$\|q - p\| \leq \frac{\frac{\epsilon T^\omega}{\Gamma(\omega+1)}}{1 - \frac{T^\omega}{\Gamma(\omega+1)} \left(\frac{\aleph_1}{1-\aleph_2} + \frac{\aleph_3}{1-\aleph_4} \right)}$$

or

$$\|q - p\| \leq \epsilon C_{\mathscr{G},\omega},$$

where

$$C_{\mathscr{G},\omega} = \frac{\frac{T^\omega}{\Gamma(\omega+1)}}{1 - \frac{T^\omega}{\Gamma(\omega+1)} \left(\frac{\aleph_1}{1-\aleph_2} + \frac{\aleph_3}{1-\aleph_4} \right)}.$$

Therefore, problem (5) is HUS. Further if we manage $\digamma_{\mathscr{G}}(\epsilon) = C_{\mathscr{G}}(\epsilon)$; $\digamma(0) = 0$, then problem (5) is generalized HUS.

For the proof of our next result we assume that

(H_7) There exist a nondecreasing function $\psi \in C(\mathscr{X}, \mathscr{R}_+)$ and a constant $\pounds_\psi > 0$ such that

$$I^{\omega}\psi(t) \leq \pounds_{\psi}\psi(t), \ for\ all\ t \in \mathscr{X}.$$

THEOREM 5. Assume that (H_1)–(H_7) along with the inequality (9) are satisfied, then the problem (5) HURS stable and consequently the problem (5) is GHURS.

PROOF. Let q be an approximate solution of the inequality (4) and p be the unique exact solution of the following problem

$$^{C}\mathscr{D}^{\omega}p(t) = \mathscr{G}(t, p(t), {}^{C}\mathscr{D}^{\omega}p(t)), \quad 0 < \omega \leq 1,$$
$$p(0) = -\int_0^T \frac{(T-\xi)^{\omega-1}}{\Gamma(\omega)}\mathcal{F}(\xi, p(\xi), {}^{C}\mathscr{D}^{\omega}p(\xi))d\xi.$$

By Lemma 3, we have

$$p(t) = \frac{1}{\Gamma(\omega)}\int_0^t (t-\xi)^{\omega-1}\alpha(\xi)d\xi - \frac{1}{\Gamma(\omega)}\int_0^T (T-\xi)^{\omega-1}\beta(\xi)d\xi,$$

where $\alpha, \beta \in \mathscr{A}$ is given as

$$\alpha(t) = \mathscr{G}(t, p(t), \alpha(t)) \ \text{ and } \ \beta(t) = \mathcal{F}(t, p(t), \beta(t)).$$

From the proof of Theorem 4, for each $t \in \mathscr{X}$, we have

$$\begin{aligned} |q(t)-p(t)| \leq \ & \frac{1}{\Gamma(\omega)}\int_0^t (t-\xi)^{\omega-1}|\bar{\alpha}(\xi)-\alpha(\xi)|d\xi + \frac{1}{\Gamma(\omega)}\int_0^t (t-\xi)^{\omega-1}|\Psi(\xi)|d\xi \\ & + \frac{1}{\Gamma(\omega)}\int_0^T (T-\xi)^{\omega-1}|\bar{\beta}(\xi)-\beta(\xi)|d\xi. \end{aligned} \tag{18}$$

By (H_3) and (H_4), we get

$$|\bar{\alpha}(t)-\alpha(t)| \leq \frac{\aleph_1}{1-\aleph_2}|q(t)-p(t)|$$

and

$$|\bar{\beta}(t)-\beta(t)| \leq \frac{\aleph_3}{1-\aleph_4}|q(t)-p(t)|.$$

Thus using the last two inequalities and part (i) of Remark 3, we have from (18)

$$\begin{aligned} |q(t)-p(t)| \leq \ & \frac{\aleph_1}{(1-\aleph_2)\Gamma(\omega)}\int_0^t (t-\xi)^{\omega-1}|q(\xi)-p(\xi)|d\xi + \frac{\epsilon}{\Gamma(\omega)}\int_0^t (t-\xi)^{\omega-1}\psi(t)d\xi \\ & + \frac{\aleph_3}{(1-\aleph_4)\Gamma(\omega)}\int_0^T (T-\xi)^{\omega-1}|q(\xi)-p(\xi)|d\xi \\ \leq \ & \frac{T^{\omega}}{\Gamma(\omega+1)}\left(\frac{\aleph_1}{1-\aleph_2}+\frac{\aleph_3}{1-\aleph_4}\right)|q(t)-p(t)| + \epsilon\pounds_{\psi}\psi(t) \quad (\because \ using\ (H_7)). \end{aligned}$$

Thus

$$\|q-p\| \le \frac{\epsilon \pounds_{\psi}\psi(t)}{1-\frac{T^{\omega}}{\Gamma(\omega+1)}\left(\frac{\aleph_1}{1-\aleph_2}+\frac{\aleph_3}{1-\aleph_4}\right)}.$$

Or

$$\|q-p\| \le C_{\mathscr{G},\omega}\epsilon \pounds_{\psi}\psi(t),$$

where

$$C_{\mathscr{G},\omega} = \frac{\pounds_{\psi}}{1-\frac{T^{\omega}}{\Gamma(\omega+1)}\left(\frac{\aleph_1}{1-\aleph_2}+\frac{\aleph_3}{1-\aleph_4}\right)}.$$

Therefore, problem (5) is HURS. In same line it is easy to check that the problem under consideration is GHURS.

4 Example

To demonstrate the aforesaid established theory, we provide the following problem.

EXAMPLE 1.

$$\begin{cases} {}^{C}\mathscr{D}^{\frac{1}{2}}p(t) = \dfrac{7+|p(t)|+|{}^{C}\mathscr{D}^{\frac{1}{2}}p(t)|}{105e^{t+3}\left(1+|p(t)|+|{}^{C}\mathscr{D}^{\frac{1}{2}}p(t)|\right)}, \ t\in\mathscr{X}=[0,1], \\ p(0) = \dfrac{1}{\Gamma(\frac{1}{2})}\displaystyle\int_0^1 (t-\xi)^{\frac{1}{2}}\left(\dfrac{\xi\sin|p(\xi)|+\sin|{}^{C}\mathcal{D}^{\frac{1}{2}}p(\xi)|}{50}\right)d\xi, \quad t\in\mathscr{X}=[0,1], \end{cases} \tag{19}$$

where $\omega = \frac{1}{2}$, $T = 1$. Set

$$\mathscr{G}(t,\sigma,\theta) = \frac{7+|\sigma|+|\theta|}{105e^{t+3}\left(1+|\sigma|+|\theta|\right)}, \quad \sigma\in C(\mathscr{X},\mathscr{R}),\ \theta\in\mathscr{R}.$$

Clearly both the functions $\mathscr{G}$ and $\mathcal{F}$ are continuous. For each $\sigma,\bar{\sigma}\in\mathscr{A},\ \theta,\bar{\theta}\in\mathscr{R}$ and $t\in[0,1]$, we have

$$|\mathscr{G}(t,\sigma,\theta)-\mathscr{G}(t,\bar{\sigma},\bar{\theta})| \le \frac{1}{105e^3}(|\sigma-\bar{\sigma}|+|\theta-\bar{\theta}|).$$

Which satisfies condition (H_3) with $\aleph_1 = \aleph_2 = \frac{1}{105e^3}$.

Similarly by setting

$$\mathcal{F}(t,\sigma,\theta) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^1 (t-\xi)^{\frac{1}{2}}\left(\frac{\xi\sin|\sigma|+\sin|\theta|}{50}\right)d\xi,$$

we have for each $t\in[0,1]$ and $\sigma,\bar{\sigma}\in\mathscr{A}$ *and* $\theta,\bar{\theta}\in\mathscr{R}$

$$|\mathcal{F}(t,\sigma,\theta)-\mathcal{F}(t,\bar{\sigma},\bar{\theta})| \le \frac{1}{25\sqrt{\pi}}\left(|\sigma-\bar{\sigma}|+|\theta-\bar{\theta}|\right).$$

Which satisfies (H_4) with $\aleph_3 = \aleph_4 = \frac{1}{25\sqrt{\pi}}$. Also

$$\begin{aligned}\frac{T^\omega}{\Gamma(\omega+1)}\left(\frac{\aleph_1}{1-\aleph_2}+\frac{\aleph_3}{1-\aleph_4}\right) &= \frac{T^\omega}{\Gamma(\omega+1)}\left(\frac{\frac{1}{105e^3}}{1-\frac{1}{105e^3}}+\frac{\frac{1}{25\sqrt{\pi}}}{1-\frac{1}{25\sqrt{\pi}}}\right)\\ &= \frac{2}{\sqrt{\pi}}\left(\frac{1}{105e^3-1}+\frac{1}{25\sqrt{\pi}-1}\right) < 1.\end{aligned}$$

We see, all the required conditions of Theorem 2, are fulfilled hence problem (19) has unique solution. Similarly we can show that by Theorem 3, the problem (19) has at least one solution. Also by letting $\psi(t) = t \in \mathscr{X}$, we have

$$I^{\frac{1}{2}}\psi(t) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-\xi)^{\frac{1}{2}-1}\xi d\xi \leq \frac{2t}{\sqrt{\pi}}.$$

Hence (H_7) satisfies with $\mathscr{L}_\psi = \frac{2}{\sqrt{\pi}}$. Therefore, by Theorem 5, the given problem is HURS and consequently it is generalized HURS.

5 Conclusion

FDEs describe the properties of hereditary materials and processes more accurate and significantly. Especially the nonlinear implicit type FDEs with integral boundary conditions are very important because the differential equations with integral boundary conditions have enormous number of applications in different fields of applied sciences. In this paper, we have successfully obtained some appropriate and sufficient conditions which guarantee the uniqueness, existence of at least one solution and its HU stability analysis to a class of nonlinear implicit FDEs with implicit anti-periodic integral boundary conditions. The obtained results can be applied in fields like numerical analysis and managerial sciences including business mathematics and economics etc.

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