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$(p, p; r)$-Convexity Preserving Infinite Matrices*

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Abstract

This paper deals with $(p, p; r)$-convexity of sequences. First, we give the necessary and sufficient conditions for a non-negative infinite matrix to preserve $(1, 1; 2)$-convexity of sequences. Using this result, it is shown that the Borel matrix and the Cešaro matrix do not preserve $(1, 1; 2)$-convexity of sequences, thus proving that the theorem pertaining to Cešaro matrix given in [10] is incorrect. Furthermore, we prove that for any $p \neq 1$, the Cešaro matrix does not preserve $(p, p; 1)$-convexity of sequences.

1 Introduction

If $p > 0$, $q > 0$, then the sequence $\{x_n\}_{n=0}^{\infty}$ of real numbers is said to be (p, q)-convex if

$$L_{p,q}(x_n) = x_n - (p + q)x_{n-1} + pqx_{n-2} \geq 0$$

for $n \geq 2$. This operator $L_{p,q}$ generates the second order difference Δ^2 when $p = q = 1$. Several authors [1, 2, 4, 5, 6, 8] have proved various results on the (p, q)- convex sequences. In [3, 7, 9], the authors discuss the matrix transformations of (p, q)-convex sequences in the case of lower triangular matrices. In [11], the authors give the necessary and sufficient condtions for a non-negative infinite matrix to transform a (p, q)-convex sequence into a (p, q)-convex sequence.

In [10], the author introduces the difference operator on a sequence $\{x_n\}$ as $L_{p;r}(x_n) = x_n - p^r x_{n+r}$ for a natural number r. We define alternate form of the operator as $L_{p;r}(x_n) = x_n - p^r x_{n-r}$ and $L_{p,q;r}(x_n) = L_{p;r}(x_n) - q^r L_{p;r}(x_{n-r})$. Thus

$$L_{p,q;r}(x_n) = x_n - (p^r + q^r)x_{n-r} + p^r q^r x_{n-2r} \quad \text{for } n \geq 2r.$$

Also, in [10], the author defines a sequence $\{x_n\}$ to be a $(p, q; r)$-convex sequence if $L_{p,q;r}(x_n) \geq 0$ for $n \geq 2r$. When $r = 1$, this operator generates (p, q)-convex sequences. Clearly $L_{p,q;r}$ is a linear operator. The main aim of the paper is to discuss the $(1, 1; 2)$-convex sequences. A sequence $\{x_n\}$ is $(1, 1; 2)$-convex if

$$L_{1,1;2}(x_n) = x_n - 2x_{n-2} + x_{n-4} \geq 0 \text{ for } n \geq 4.$$

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In this paper we give the necessary and sufficient conditions for a non-negative infinite matrix to preserve $(1,1;2)$-convexity of sequences. In Section 4 we give an example of such an infinite matrix. Also, we show that the Borel matrix and the Cešaro matrix do not transform every $(1,1;2)$-convex sequence into a $(1,1;2)$-convex sequence. In addition, we show that the Cešaro matrix fails to satisfy one of the conditions given in [11] to preserve $(p,p;1)$-convexity of sequences for $p \neq 1$.

2 Preliminaries

For any given sequence $\{x_n\}$, we can find a corresponding sequence $\{c_k\}_{k=0}^{\infty}$ such that

$$c_0 = x_0, \quad c_1 = x_1,$$

and for $k \geq 2$, c_k's are given by

$$c_k = \begin{cases} x_{2j} - \sum_{i=0}^{j-1} (j-i+1) c_{2i}, & \text{if } k = 2j, \\ x_{2j+1} - \sum_{i=0}^{j-1} (j-i+1) c_{2i+1}, & \text{if } k = 2j+1, \end{cases} \tag{1}$$

which implies that $\{x_n\}$ can be represented by

$$x_n = \begin{cases} \sum_{i=0}^{k} (k-i+1) c_{2i}, & \text{if } n = 2k, \\ \sum_{i=0}^{k} (k-i+1) c_{2i+1}, & \text{if } n = 2k+1, \end{cases} \tag{2}$$

for $n \geq 0$. As a consequence we get the following lemma.

LEMMA 2.1. If the sequence $\{x_n\}$ is given by the representation (2), then $L_{1,1;2}(x_n) = c_n$. Thus, the sequence $\{x_n\}$ is $(1,1;2)$-convex if and only if $c_n \geq 0$ for $n \geq 4$.

PROOF. Since $L_{1,1;2}(x_n) = x_n - 2x_{n-2} + x_{n-4}$, it suffices to show that

$$x_n - 2x_{n-2} + x_{n-4} = c_n \text{ for } n \geq 4.$$

Using (2), we can write for $n = 4, 5, 6, \cdots$,

$$\begin{aligned} & x_n - 2x_{n-2} + x_{n-4} \\ = & \begin{cases} \sum_{i=0}^{k} (k-i+1) c_{2i} - 2 \sum_{i=0}^{k-1} (k-i) c_{2i} + \sum_{i=0}^{k-2} (k-i-1) c_{2i}, & \text{if } n = 2k, \\ \sum_{i=0}^{k} (k-i+1) c_{2i+1} - 2 \sum_{i=0}^{k-1} (k-i) c_{2i+1} + \sum_{i=0}^{k-2} (k-i-1) c_{2i+1}, & \text{if } n = 2k+1, \end{cases} \\ = & \begin{cases} c_{2k} + \sum_{i=0}^{k-2} (k-i+1-2(k-i)+k-i-1) c_{2i}, & \text{if } n = 2k, \\ c_{2k+1} + \sum_{i=0}^{k-2} (k-i+1-2(k-i)+k-i-1) c_{2i+1}, & \text{if } n = 2k+1, \end{cases} \end{aligned}$$

$$
\begin{aligned}
&= \begin{cases} c_{2k}, & \text{if } n = 2k, \\ c_{2k+1}, & \text{if } n = 2k+1, \end{cases} \\
&= c_n.
\end{aligned}
$$

Thus, for any sequence $\{x_n\}$,

$$L_{1,1,;2}(x_n) = c_n, \quad \text{for } n \geq 4.$$

Hence the lemma holds.

Now, we give below some definitions.

Let $A = [a_{n,k}]$ be a non-negative infinite matrix defining a sequence to sequence transformation by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{n,k} x_k.$$

Then, we define the matrices $[\alpha_{n,m}]$ and $[\beta_{n,i}]$ as

$$\alpha_{n,m} = \begin{cases} \sum_{j=l}^{\infty} a_{n,2j} = a_{n,2l} + a_{n,2l+2} + a_{n,2l+4} + \cdots, & \text{if } m = 2l, \\ \sum_{j=l}^{\infty} a_{n,2j+1} = a_{n,2l+1} + a_{n,2l+3} + a_{n,2l+5} + \cdots, & \text{if } m = 2l+1, \end{cases}$$

$$\beta_{n,i} = \begin{cases} \sum_{l=k}^{\infty} \alpha_{n,2l} = \alpha_{n,2k} + \alpha_{n,2k+2} + \alpha_{n,2k+4} + \cdots, & \text{if } i = 2k, \\ \sum_{l=k}^{\infty} \alpha_{n,2l+1} = \alpha_{n,2k+1} + \alpha_{n,2k+3} + \alpha_{n,2k+5} + \cdots, & \text{if } i = 2k+1. \end{cases}$$

Thus,

$$\beta_{n,i} = \begin{cases} \sum_{l=k}^{\infty} \left(\sum_{j=l}^{\infty} a_{n,2j} \right) \text{if } i = 2k, \\ \sum_{l=k}^{\infty} \left(\sum_{j=l}^{\infty} a_{n,2j+1} \right) \text{if } i = 2k+1. \end{cases}$$

Interchanging the order of summation, we get

$$\beta_{n,i} = \begin{cases} \sum_{j=k}^{\infty} \sum_{l=k}^{j} a_{n,2j}, \text{if } i = 2k, \\ \sum_{j=k}^{\infty} \sum_{l=k}^{j} a_{n,2j+1}, \text{if } i = 2k+1. \end{cases}$$

Therefore, we can write

$$\beta_{n,i} = \begin{cases} \sum_{j=k}^{\infty} (j-k+1) a_{n,2j}, & \text{if } i = 2k \\ \sum_{j=k}^{\infty} (j-k+1) a_{n,2j+1}, & \text{if } i = 2k+1. \end{cases} \tag{3}$$

Furthermore, for $n \geq 4$ and for each $i = 0, 1, 2, \ldots$ we obtain, by the linearity of the operator $L_{1,1;2}$,

$$L_{1,1;2}(\beta_{n,i}) = \begin{cases} \sum_{j=k}^{\infty} \big(j-k+1\big) L_{1,1;2}(a_{n,2j}), & \text{if } i = 2k, \\ \sum_{j=k}^{\infty} \big(j-k+1\big) L_{1,1;2}(a_{n,2j+1}), & \text{if } i = 2k+1. \end{cases} \tag{4}$$

Also, we need the matrix $[a_{n,k}]$ to satisfy the condition

$$\sum_{k=1}^{\infty} k a_{n,k} < \infty. \tag{5}$$

so that from (3) for $k = 0, 1, 2, \cdots$

$$\beta_{n,i} = \begin{cases} \sum\limits_{j=k}^{\infty}(j-k)a_{n,2j} + \sum\limits_{j=k}^{\infty} a_{n,2j}, & \text{if } i = 2k, \\ \sum\limits_{j=k}^{\infty}(j-k)a_{n,2j+1} + \sum\limits_{j=k}^{\infty} a_{n,2j+1}, & \text{if } i = 2k+1, \end{cases}$$
$$< \infty.$$

Thus, $\beta_{n,i}$ is well-defined.

3 Main Results

In this section we prove the necessary and sufficient conditions for a non-negative infinite matrix A to transform a $(1,1;2)$-convex sequence into a $(1,1;2)$-convex sequence showing that each column of the corresponding matrix $[\beta_{n,i}]$ is a $(1,1;2)$-convex sequence.

THEOREM 3.1. A non-negative infinite matrix $A = [a_{n,k}]$ satisfying

$$\sum_{k=1}^{\infty} k a_{n,k} < \infty,$$

preserves $(1,1;2)$-convexity of sequences if and only if for $n = 4, 5, 6 \ldots,$

(i) $L_{1,1;2}\left(\beta_{n,0}\right) = L_{1,1;2}\left(\beta_{n,1}\right) = L_{1,1;2}\left(\beta_{n,2}\right) = L_{1,1;2}\left(\beta_{n,3}\right) = 0.$

(ii) $L_{1,1;2}\left(\beta_{n,i})\right) \geq 0$ for $i \geq 4$, where the matrix $[\beta_{n,i}]$ is defined by

$$\beta_{n,i} = \begin{cases} \sum\limits_{j=k}^{\infty}(j-k+1)a_{n,2j}, & \text{if } i = 2k, \\ \sum\limits_{j=k}^{\infty}(j-k+1)a_{n,2j+1}, & \text{if } i = 2k+1. \end{cases}$$

First, we prove the following lemma.

LEMMA 3.1. If $\{x_n\}$ is any sequence, then the transformed sequence $\{(Ax)_n\}$ satisfies that for $n \geq 4$,

$$(Ax)_n = \sum_{i=0}^{\infty} c_i \beta_{n,i}$$

where c_i's are given by (1).

PROOF. From (2), we have

$$x_{2k} = \sum_{i=0}^{k}(k-i+1)c_{2i} \quad \text{and} \quad x_{2k+1} = \sum_{i=0}^{k}(k-i+1)c_{2i+1}.$$

Then the nth term of the transformed sequence is

$$\begin{aligned}(Ax)_n &= \sum_{k=0}^{\infty} a_{n,k}x_k \\ &= \sum_{k=0}^{\infty} a_{n,2k}x_{2k} + \sum_{k=0}^{\infty} a_{n,2k+1}x_{2k+1} \\ &= \sum_{k=0}^{\infty} a_{n,2k}\Big(\sum_{i=0}^{k}(k-i+1)c_{2i}\Big) + \sum_{k=0}^{\infty} a_{n,2k+1}\Big(\sum_{i=0}^{k}(k-i+1)c_{2i+1}\Big).\end{aligned}$$

Interchanging the order of summation,

$$(Ax)_n = \sum_{i=0}^{\infty} c_{2i}\Big(\sum_{k=i}^{\infty}(k-i+1)a_{n,2k}\Big) + \sum_{i=0}^{\infty} c_{2i+1}\Big(\sum_{k=i}^{\infty}(k-i+1)a_{n,2k+1}\Big).$$

Using (3), we can write

$$(Ax)_n = \sum_{i=0}^{\infty} c_{2i}\beta_{n,2i} + \sum_{i=0}^{\infty} c_{2i+1}\beta_{n,2i+1} = \sum_{i=0}^{\infty} c_i\beta_{n,i}.$$

Hence the lemma holds.

PROOF OF THEOREM 3.1. To prove the sufficiency of the conditions given in the theorem, assume that conditions (i) and (ii) are true. For any $(1,1;2)$-convex sequence $\{x_n\}$, by Lemma 2.1, $c_i \geq 0$ for $i \geq 4$. Using Lemma 3.1 and the linearity of the operator $L_{1,1;2}$, we can write for $n \geq 4$,

$$L_{1,1;2}(Ax)_n = \sum_{i=0}^{\infty} c_i L_{1,1;2}(\beta_{n,i}) \geq 0. \tag{6}$$

Thus, the sequence $\{(Ax)_n\}$ is also $(1,1;2)$-convex. Conversely, assume that the matrix A preserves $(1,1;2)$-convexity of sequences. Suppose that condition (i) fails to hold. Then for some $i = 0,1,2,3$,

$$L_{1,1;2}(\beta_{n,i}) \neq 0 \quad \text{for some} \quad n \geq 4.$$

In particular, if

$$L_{1,1;2}(\beta_{n,0}) \neq 0, \text{for some } n \geq 4,$$

then there exists an $N \geq 4$ such that

$$L_{1.1;2}(\beta_{N,0}) = L \neq 0.$$

Consider the sequence $\{u_n\}$ given by

$$u = \left\{ \begin{array}{ccccccccc} u_0 & u_1 & u_2 & u_3 & u_4 & \ldots, & u_{2k} & u_{2k+1} & \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \\ -L, & 0, & -2L, & 0, & -3L, & & -(k+1)L, & 0, & \end{array} \right\}.$$

Then, $\{u_n\}$ is a $(1,1;2)$-convex sequence because using equation (1) and Lemma 2.1 we see that $c_0 = u_0 = -L, c_1 = u_1 = 0$, $c_2 = u_2 - 2c_0 = 0$, $c_3 = u_3 - 2c_1 = 0$ and for $k \geq 2$,

$$\begin{aligned} c_{2k} &= u_{2k} - 2u_{2k-2} + u_{2k-4} \\ &= -(k+1)L - 2(-kL) + (-(k-1)L) = 0, \\ c_{2k+1} &= u_{2k+1} - 2u_{2k-1} + u_{2k-3} = 0. \end{aligned}$$

Then for the transformed sequence $\{(Au)_n\}$, we have from (6)

$$\begin{aligned} L_{1,1;2}(Au)_N &= c_0 L_{1,1;2}(\beta_{N,0}) + \sum_{i=1}^{\infty} c_{2i} L_{1,1;2}(\beta_{N,2i}) + \sum_{i=0}^{\infty} c_{2i+1} L_{1,1;2}(\beta_{N,2i+1}) \\ &= c_0 L_{1,1;2}(\beta_{N,0}) \\ &= -L^2 < 0, \end{aligned}$$

which contradicts that the transformed sequence $\{(Au)_n\}$ must be $(1,1;2)$-convex. Similarly, if $L_{1,1;2}(\beta_{N,1})$ or $L_{1,1;2}(\beta_{N,2})$ or $L_{1,1;2}(\beta_{N,3}) = L \neq 0$ for some $N \geq 4$, then consider the sequences

$$v = \left\{ \begin{array}{ccccccccc} v_0 & v_1 & v_2 & v_3 & v_4 & \ldots, & v_{2k+1} & v_{2k} & \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \\ 0, & -L, & 0, & -2L, & 0, & & -(k+1)L, & 0, & \end{array} \right\}$$

$$w = \left\{ \begin{array}{ccccccccc} w_0 & w_1 & w_2 & w_3 & w_4 & \ldots, & w_{2k} & w_{2k+1} & \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \\ 0, & 0, & -L, & 0, & -2L, & & -kL, & 0, & \end{array} \right\}$$

and

$$t = \left\{ \begin{array}{ccccccccccc} t_0 & t_1 & t_2 & t_3 & t_4 & t_5 & \ldots & t_{2k} & t_{2k+1} & \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \\ 0, & 0, & 0, & -L, & 0, & -2L, & & 0, & -kL, & \end{array} \right\}$$

respectively. It is obvious that $\{v_n\}$, $\{w_n\}$ and $\{t_n\}$ are $(1,1;2)$-convex sequences with the corresponding

$$c_1 = -L \text{ and } c_i = 0 \text{ for } i \neq 1 \text{ for the sequence} \{v_n\},$$

$$c_2 = -L \text{ and } c_i = 0 \text{ for } i \neq 2 \text{ for the sequence } \{w_n\},$$

and

$$c_3 = -L \text{ and } c_i = 0 \text{ for } i \neq 3 \text{ for the sequence}\{t_n\}.$$

By the similar argument as in the case of the sequence $\{u_n\}$, we see that the transformed sequences $\{(Av)_n\}$, $\{(Aw)_n\}$, and $\{(At)_n\}$, are not $(1,1;2)$-convex sequences, which is a contradiction.

Next, suppose that condition (ii) is not true. First, assume that $L_{1,1;2}(\beta_{n,2i})$ for $i \geq 2$, fails to satisfy the condition. Then there exists an integer $j = 2k \geq 4$ such that the j-th column-sequence $\{\beta_{n,2k}\}_{n=0}^{\infty}$ of the matrix $[\beta_{n,i}]$ is not $(1,1;2)$-convex. i.e., for some $N \geq 4$, $L_{1,1,;2}(\beta_{N,2k}) = L < 0$. Consider the sequence

$$x = \left\{ \begin{matrix} x_0 & \ldots, & x_{2k-1} & x_{2k} & x_{2k+1} & x_{2k+2} & x_{2k+3} & x_{2k+4} & x_{2k+5} & \ldots \\ \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 0, & & 0, & 1, & 0, & 2, & 0, & 3, & 0, & \end{matrix} \right\}.$$

Then $\{x_n\}$ is a $(1,1;2)$-convex sequence because using equation (1) and Lemma 2.1 we see that

$$c_i = 0, \quad 0 < i < 2k,$$

$$c_{2k} = 1,$$

$$c_i = 0 \text{ for } i \geq 2k+1.$$

Thus, the sequence $\{x_n\}$ is $(1,1;2)$-convex. But from (6),

$$\begin{aligned} L_{1,1;2}(Ax)_N &= \sum_{i=0}^{\infty} c_{2i} L_{1,1;2}(\beta_{N,2i}) + \sum_{i=0}^{\infty} c_{2i+1} L_{1,1;2}(\beta_{N,2i+1}) \\ &= c_{2k} L_{1,1;2}(\beta_{N,2k}) = L < 0, \end{aligned}$$

which contradicts that the sequence $\{(Ax)_n\}$ is a $(1,1;2)$-convex sequence.
Next, assume that $L_{1,1:2}(\beta_{n,2i+1})$ for $i \geq 2$, fails to satisfy condition (ii).

Then there exists and integer $l = 2k+1 \geq 5$ such that the l-th column-sequence $\{\beta_{n,2k+1}\}_{n=0}^{\infty}$ of the matrix $[\beta_{n,i}]$ is not $(1,1;2)$-convex. That is, for some $N \geq 4$, $L_{1,1;2}(\beta_{N,2k+1}) = L < 0$. This case can be settled by a similar argument by considering the sequence

$$y = \left\{ \begin{matrix} y_0 & \ldots, & y_{2k} & y_{2k+1} & y_{2k+2} & y_{2k+3} & y_{2k+4} & y_{2k+5} & \ldots \\ \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 0, & & 0, & 1, & 0, & 2, & 0, & 3, & \end{matrix} \right\},$$

which implies that $c_{2k+1} = 1$ and all other c_i's are zero. This yields that

$$L_{1,1;2}(Ay)_N = c_{2k+1} L_{1,1;2}(\beta_{N,2k+1}) = L < 0,$$

a contradiction. This completes the proof.

4 Examples

We give below an example of $(1,1;2)$-convexity preserving matrix. Let the matrix $A=[a_{n,k}]$ be defined by

$$a_{n,k}=\frac{n+1}{(k+1)^3}.$$

Then for each n,

$$\sum_{k=1}^{\infty} k\frac{(n+1)}{(k+1)^3}=(n+1)\sum_{k=1}^{\infty}\Big[\frac{1}{(k+1)^2}-\frac{1}{(k+1)^3}\Big]<\infty.$$

Thus, by (5), $\beta_{n,i}$ is well-defined for each $n=0,1,2,\cdots$ and $i=0,1,2,\cdots$. The matrix A satisfies all three conditions of Theorem 3.1, because for $n\geq 4$, using (4),

$$\begin{aligned}
L_{1,1;2}(\beta_{n,i}) &= \begin{cases}\sum\limits_{j=k}^{\infty}(j-k+1)[a_{n,2j}-2a_{n-2,2j}+a_{n-4,2j}], & \text{if } i=2k,\\ \sum\limits_{j=k}^{\infty}(j-k+1)[a_{n,2j+1}-2a_{n-2,2j+1}+a_{n-4,2j+1}], & \text{if } i=2k+1,\end{cases}\\
&= \begin{cases}\sum\limits_{j=k}^{\infty}\frac{(j-k+1)}{(2j+1)^3}[(n+1)-2(n-1)+(n-3)], & \text{if } i=2k,\\ \sum\limits_{j=k}^{\infty}\frac{(j-k+1)}{(2j+2)^3}[(n+1)-2(n-1)+(n-3)], & \text{if } i=2k+1,\end{cases}\\
&= 0.
\end{aligned}$$

Now, we show that the classical matrices Borel matrix and Cesàro matrix do not preserve $(1,1;2)$-convexity of sequences. The Borel matrix $B=[b_{n,k}]$ is given by

$$b_{n,k}=\frac{n^k}{e^n k!}.$$

We will show that the matrix B does not preserve $(1,1;2)$-convexity of sequences by proving that $L_{1,1;2}(\beta_{n,1})\neq 0$, which violates one of the conditions given in Theorem 3.1. For $n\geq 4$, using (4) with $k=0$,

$$\begin{aligned}
L_{1,1;2}(\beta_{n,1}) &= \sum_{j=0}^{\infty}(j+1)L_{1,1;2}(b_{n,2j+1})\\
&= \sum_{j=0}^{\infty}\frac{(j+1)}{(2j+1)!}\Big[\frac{n^{2j+1}}{e^n}-\frac{2(n-2)^{2j+1}}{e^{n-2}}+\frac{(n-4)^{2j+1}}{e^{n-4}}\Big]\\
&= \frac{1}{e^n}\sum_{j=0}^{\infty}\frac{(j+1)}{(2j+1)!}\Big[n^{2j+1}-2e^2(n-2)^{2j+1}+e^4(n-4)^{2j+1}\Big]. \qquad (7)
\end{aligned}$$

Since

$$\sum_{j=0}^{\infty}\frac{(j+1)}{(2j+1)!}x^{2j+1}=\frac{1}{2}(xe^x+e^x-1),$$

(7) reduces to

$$L_{1,1;2}(\beta_{n,1}) = \frac{1}{2e^n}\Big[ne^n + e^n - 1 - 2e^2\Big((n-2)e^{n-2} + e^{n-2} - 1\Big) + e^4\Big((n-4)e^{n-4} + e^{n-4} - 1\Big)\Big]$$
$$= -\frac{(e^2-1)^2}{e^n} < 0.$$

Thus, the Borel matrix does not preserve $(1, 1; 2)$-convexity.

Next, we will consider the Cesàro matrix which is given by

$$a_{n,k} = \begin{cases} \frac{1}{n+1}, & \text{if } k \le n, \\ 0, & \text{if } k > n, \end{cases}$$

and prove below that it does not preserve $(1, 1; 2)$-convexity of sequences by showing that the condition (i) of Theorem 3.1 does not hold.
Consider the corresponding matrix $[\beta_{n,i}]$ which is also lower triangular. When n is an even integer, assuming $n = 2m$ where $m \ge 2$, we get from (3)

$$\beta_{2m,0} = \sum_{j=0}^{m}(j+1)a_{2m,2j} = \frac{1}{2m+1}\sum_{j=0}^{m}(j+1),$$

$$\beta_{2m-2,0} = \sum_{j=0}^{m-1}(j+1)a_{2m-2,2j} = \frac{1}{2m-1}\sum_{j=0}^{m-1}(j+1),$$

and

$$\beta_{2m-4,0} = \sum_{j=0}^{m-2}(j+1)a_{2m-4,2j} = \frac{1}{2m-3}\sum_{j=0}^{m-2}(j+1).$$

Therefore,

$$L_{1,1;2}(\beta_{2m,0}) = \beta_{2m,0} - 2\beta_{2m-2,0} + \beta_{2m-4,0}$$
$$= \frac{(m+1)(m+2)}{2(2m+1)} - \frac{m(m+1)}{2m-1} + \frac{m(m-1)}{2(2m-3)}$$
$$= \frac{3}{(2m+1)(2m-1)(2m-3)} > 0.$$

Thus, condition (i) of Theorem 3.1 fails showing that the Cesàro matrix does not preserve $(1, 1; 2)$-convexity of sequences, which asserts that the theorem given in [10, p.40] is incorrect.

In fact, we give below a simple example of a $(1, 1; 2)$-convex sequence which is not transformed into a $(1, 1; 2)$-convex sequence by the Cesàro matrix.

$$x = \left\{ \begin{matrix} x_0 & x_1 & x_2 & x_3 & \ldots, & x_{2k} & x_{2k+1} & \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \\ -1, & 0, & -2, & 0, & & -(k+1), & 0, & \end{matrix} \right\}.$$

Obviously $\{x_n\}$ is $(1,1;2)$-convex sequence. But for the transformed sequence

$$(Ax)_n = \sum_{k=0}^{n} \frac{1}{n+1} x_k,$$

we see that for $k \geq 2$,

$$\begin{aligned} L_{1,1;2}(Ax)_{2k} &= (Ax)_{2k} - 2(Ax)_{2k-2} + (Ax)_{2k-4} \\ &= -\left(\frac{1+2+\cdots+(k+1)}{2k+1}\right) + 2\left(\frac{1+2+\cdots k}{2k-1}\right) - \left(\frac{1+2+\cdots+(k-1)}{2k-3}\right), \end{aligned}$$

which simplifies to $-\frac{3}{(4k^2-1)(2k-3)} < 0$.

We conclude this paper by showing that the Cesàro matrix does not preserve $(p,p;1)$-convexity when $p \neq 1$. In [11], the authors proved the following theorem giving the necessary and sufficient conditions for any matrix to preserve $(p,p,1)$-convexity of sequences.

THEOREM. A non-negative matrix A satisfying $\sum\limits_{k=1}^{\infty} kp^k a_{n,k} < \infty$ for $p \neq 1$ preserves $(p,p;1)$-convexity of sequences if and only if, for $n = 2,3,\cdots$,

$$\begin{aligned} &\text{(i)} \quad \Delta_{p,p}(\beta_{n,0}) = \Delta_{p,p}(\beta_{n,1}) = 0 \\ &\text{(ii)} \quad \Delta_{p,p}(\beta_{n,i}) \geq 0 \quad \text{for } i = 2,3\cdots, \end{aligned}$$

where $\beta_{n,i} = \sum\limits_{j=i}^{\infty} (j-i+1)p^{j-i} a_{n,k}$ and $\Delta_{p,p}(\beta_{n,i}) = \beta_{n,i} - 2p\beta_{n-1,i} + p^2\beta_{n-2,i}$.

We will now show that the Cesàro matrix $[a_{n,k}]$ does not satisfy one of the conditions given in the above theorem.

$$\begin{aligned} \Delta_{p,p}(\beta_{n,0}) &= \beta_{n,0} - 2p\beta_{n-1,0} + p^2\beta_{n-2,0} \\ &= \frac{1}{n+1}\sum_{j=0}^{n}(j+1)p^j - \frac{2p}{n}\sum_{j=0}^{n-1}(j+1)p^j + \frac{p^2}{n-1}\sum_{j=0}^{n-2}(j+1)p^j. \end{aligned}$$

Combining the terms containing similar powers of p, we get

$$\begin{aligned} \Delta_{p,p}(\beta_{n,0}) &= \sum_{j=2}^{n+1} p^{j-1}\Big(\frac{j}{n+1} - \frac{2(j-1)}{n} + \frac{j-2}{n-1}\Big) + \frac{1}{n+1} \\ &= \frac{-2}{n(n+1)(n-1)}\Big(p^{n-1} + 2p^{n-2} + 3p^{n-3} + \cdots + (n-1)p - \frac{n(n-1)}{2}\Big) \\ &= \frac{-2}{n(n+1)(n-1)}\Big[(p^{n-1}-1) + 2(p^{n-2}-1) + 3(p^{n-3}-1) + \cdots + (n-1)(p-1)\Big] \\ &\neq 0, \text{ when } p \neq 1. \end{aligned}$$

Hence, the Cesàro matrix fails to preserve $(p,p;1)$-convexity of sequences, when $p \neq 1$.

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