Applied Mathematics E-Notes, 19(2019), 210-219 © ISSN 1607-2510
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Associated Surfaces Obtained Kinematically And Their Projection Areas*

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Received 15 April 2018

Abstract

In this study, we study special two-parameter homothetic motions obtained by the frame of a regular surface with non-umbilical points. This frame has been constructed by taking the unit principal directions and the unit normal vector of the surface. By using the homothetic motion of this frame, we project the orbit surface of a fixed point onto an arbitrary plane and compute its oriented projection area depending on the geodesic curvatures of parameter curves and the principal curvatures of the surface. Also, we reobtain a Holditch-type result for the projection areas.

1 Introduction

Kinematic describes the motion of a point or a point system depending on time. If a point moves with respect to one parameter, then it traces its one-dimensional path, orbit curve. If a point moves with respect to two parameters, then it traces its two-dimensional path, orbit surface. H.R. Müller obtained the area of the region enclosed by the closed orbit curve in planar kinematics [5] and the area of the planar region enclosed by the projection of the closed orbit curve and orbit surface in spatial kinematics [7]. By using a special metric, Müller has shown that the classical Holditch theorem[1] can also be transferred to projection areas in Euclidean 3-space [7, 9] (see also [3, 10] for the generalizations in spatial homothetic motions). The volumes of the trajectory surfaces of points have been studied under three-parameter motions [6, 8] and three-parameter homothetic motions in Euclidean 3-space [4]. Furthermore, by considering a special two-parameter motion (the motion of the orthonormal frame along a regular surface whose parameter curves are lines of curvature), Urban obtained the volume of the region traced by a line segment [11].

In this study, we study special two-parameter homothetic motions obtained by the frame of a regular surface with non-umbilical points. This frame has been constructed by taking the unit principal directions and the unit normal vector of the surface as

*Mathematics Subject Classifications: 53A17, 53A05.

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[1]If a chord of a closed curve, of constant length $a + b$, be divided into two parts of lengths a, b, respectively, the difference between the areas of the closed curve, and of the locus of the dividing point, will be πab [1, 2].

in [11]. By using the homothetic motion of this special frame, we project the orbit surface of a fixed point onto an arbitrary plane and compute its oriented projection area depending on the geodesic curvatures of parameter curves and the principal curvatures of the surface. Also, we reobtain the Holditch-type result for the projection areas.

2 Preliminaries

Let $D \subset E^2$ be an open set and $\mathbb{M}$ be a regular surface given by its parametric equation $X(u,v)$. We assume that $\mathbb{M}$ does not have any umbilical and flat points. We also suppose that the parameter curves of $\mathbb{M}$ are lines of curvature. Let $\mathbf{X_u}$ and $\mathbf{X_v}$ be the tangent vectors of the parameter curves and $\mathbf{n}$ be the unit normal vector of the surface. Then $\mathbf{n}$ is obtained by

$$\mathbf{n} = \frac{\mathbf{X_u} \times \mathbf{X_v}}{\|\mathbf{X_u} \times \mathbf{X_v}\|}.$$

The coefficients of the first and second fundamental forms are given by

$$E = \langle \mathbf{X_u},\ \mathbf{X_u} \rangle,\ \ F = \langle \mathbf{X_u}, \mathbf{X_v} \rangle,\ \ G = \langle \mathbf{X_v}, \mathbf{X_v} \rangle,$$

$$L = \langle \mathbf{X_{uu}}, \mathbf{n} \rangle,\ \ M = \langle \mathbf{X_{uv}}, \mathbf{n} \rangle,\ \ N = \langle \mathbf{X_{vv}}, \mathbf{n} \rangle. \tag{1}$$

Since the parameter curves of the surface are lines of curvature, we have

$$F = M = 0, \tag{2}$$

and the principal curvatures are obtained as

$$\kappa_1 = \frac{L}{E}, \quad \kappa_2 = \frac{N}{G}. \tag{3}$$

Let

$$\mathbf{r_1} = \frac{\mathbf{X_u}}{\| \mathbf{X_u} \|}, \quad \mathbf{r_2} = \frac{\mathbf{X_v}}{\| \mathbf{X_v} \|}.$$

Then $\{\mathbf{r_1}, \mathbf{r_2}, \mathbf{n}\}$ constitutes an orthonormal frame along the surface $\mathbb{M}$. Thus, if we take the partial derivatives of

$$\mathbf{X_u} = \sqrt{E}\,\mathbf{r_1},\ \mathbf{X_v} = \sqrt{G}\,\mathbf{r_2} \tag{4}$$

with respect to u and v, respectively, then we get, [11]

$$\begin{cases} \mathbf{X_{uu}} = \frac{E_u}{2\sqrt{E}}\mathbf{r_1} + \sqrt{E}\frac{\partial \mathbf{r_1}}{\partial u}, \\ \\ \mathbf{X_{uv}} = \frac{E_v}{2\sqrt{E}}\mathbf{r}_1 + \sqrt{E}\frac{\partial \mathbf{r_1}}{\partial v} = \frac{G_u}{2\sqrt{G}}\mathbf{r}_2 + \sqrt{G}\frac{\partial \mathbf{r_2}}{\partial u}, \\ \\ \mathbf{X_{vv}} = \frac{Gv}{2\sqrt{G}}\mathbf{r_2} + \sqrt{G}\frac{\partial \mathbf{r_2}}{\partial v}. \end{cases} \tag{5}$$

With (1), (2) and (3) it follows from (5):

$$\left\langle \frac{\partial \mathbf{r}_1}{\partial u}, \mathbf{n} \right\rangle = \frac{L}{\sqrt{E}} = \kappa_1 \sqrt{E}, \qquad \left\langle \frac{\partial \mathbf{r}_2}{\partial v}, \mathbf{n} \right\rangle = \frac{N}{\sqrt{G}} = \kappa_2 \sqrt{G}, \tag{6}$$

$$\left\langle \frac{\partial \mathbf{r_1}}{\partial v}, \mathbf{n} \right\rangle = -\left\langle \mathbf{r_1}, \frac{\partial \mathbf{n}}{\partial v} \right\rangle = 0, \quad \left\langle \frac{\partial \mathbf{r_2}}{\partial u}, \mathbf{n} \right\rangle = -\left\langle \mathbf{r}_2, \frac{\partial \mathbf{n}}{\partial u} \right\rangle = 0.$$

Let κ_{ig} be the oriented geodesic curvature of the i-th parameter curve of $\mathbb{M}$ ($i = 1, 2$), and let ds_i be the arc element of the i-th parameter curve. Then we may write, [11]

$$\dot{s}_1 = \frac{ds_1}{du} = \sqrt{\langle \mathbf{X_u}, \mathbf{X_u} \rangle} = \sqrt{E}, \;\; \dot{s}_2 = \frac{ds_2}{dv} = \sqrt{\langle \mathbf{X_v}, \mathbf{X_v} \rangle} = \sqrt{G}, \tag{7}$$

$$\kappa_{1g} = \frac{1}{(\dot{s}_1)^3} \det\{\mathbf{X_u}, \mathbf{X_{uu}}, \mathbf{n}\}, \quad \kappa_{2g} = \frac{1}{(\dot{s}_2)^3} \det\{\mathbf{X_v}, \, \mathbf{X_{vv}}, \, \mathbf{n}\}. \tag{8}$$

If we use (4), (5) and (7), the representations of κ_{ig} are obtained as, [11]

$$\begin{cases} \kappa_{1g} = \frac{1}{\sqrt{E}} \det\left\{\mathbf{r}_1, \frac{\partial \mathbf{r_1}}{\partial u}, \mathbf{n}\right\} = \frac{1}{\sqrt{E}} \left\langle \frac{\partial \mathbf{r_1}}{\partial u}, \mathbf{r}_2 \right\rangle = -\frac{1}{\sqrt{E}} \left\langle \frac{\partial \mathbf{r_2}}{\partial u}, \mathbf{r_1} \right\rangle, \\ \\ \kappa_{2g} = \frac{1}{\sqrt{G}} \det\left\{\mathbf{r_2}, \frac{\partial \mathbf{r_2}}{\partial v}, \mathbf{n}\right\} = -\frac{1}{\sqrt{G}} \left\langle \frac{\partial \mathbf{r_2}}{\partial v}, \mathbf{r_1} \right\rangle = \frac{1}{\sqrt{G}} \left\langle \frac{\partial \mathbf{r_1}}{\partial v}, \mathbf{r}_2 \right\rangle. \end{cases}$$

From (5), (6) and (8) the derivation equations are finally obtained as, [11]

$$\begin{cases} \frac{\partial \mathbf{r_1}}{\partial u} = (\kappa_{1g} \mathbf{r_2} + \kappa_1 \mathbf{n}) \sqrt{E}, \\ \\ \frac{\partial \mathbf{r_2}}{\partial u} = -\kappa_{1g} \sqrt{E} \mathbf{r_1}, \\ \\ \frac{\partial \mathbf{n}}{\partial u} = -\kappa_1 \sqrt{E} \mathbf{r}_1, \end{cases} , \qquad \begin{cases} \frac{\partial \mathbf{r_1}}{\partial v} = \kappa_{2g} \sqrt{G} \mathbf{r_2}, \\ \\ \frac{\partial \mathbf{r_2}}{\partial v} = (-\kappa_{2g} \mathbf{r_1} + \kappa_2 \mathbf{n}) \sqrt{G}, \\ \\ \frac{\partial \mathbf{n}}{\partial v} = -\kappa_2 \sqrt{G} \mathbf{r_2}. \end{cases} \tag{9}$$

DEFINITION. Let $\mathbb{M}$ be a regular surface and D be the region of the parameters of this surface.

$$d\mathcal{F}_{\mathbb{M}} = \|\mathbf{X_u} \times \mathbf{X_v}\| \, dudv \tag{10}$$

is called the scalar surface element of $\mathbb{M}$ and

$$d\mathbf{F}_{\mathbb{M}} = \mathbf{n} \, d\mathcal{F}_{\mathbb{M}} \tag{11}$$

is called the vectorial surface element of $\mathbb{M}$.

$$\mathbf{F}_{\mathbb{M}} = \iint\limits_D d\mathbf{F}_{\mathbb{M}}$$

is called the area vector, [9].

Using (10) and (11), the vectorial surface element of $X(u, v)$ is [9]

$$d\mathbf{F}_{\mathbb{M}} = \mathbf{X_u} \times \mathbf{X_v} dudv$$

and the area vector is

$$\mathbf{F}_{\mathbb{M}} = \iint\limits_D \mathbf{X_u} \times \mathbf{X_v} dudv.$$

3 Associated Surfaces Obtained Kinematically

Let us consider the above orthonormal frame $\{\mathbf{r}_1, \mathbf{r_2}, \mathbf{n}\}$ constructed along a surface $\mathbb{M}$ given by the parametric equation $X(u,v)$ with non-umbilical points. Let $h(u,v)$ be a differentiable function on D.

Let us consider the two-parameter homothetic motion in which the frame $\{\mathbf{r_1}, \mathbf{r_2}, \mathbf{n}\}$ is considered as the moving frame along the surface. Thus, the fixed point $\varphi(u,v) = \alpha_1\mathbf{r_1} + \alpha_2\mathbf{r_2} + \alpha_3\mathbf{n}$ with respect to this frame traces its orbit surface under such homothetic motions. For its orbit surface $\psi(u,v)$, we may then write

$$\psi(u,v) = X(u,v) + h(u,v)\varphi(u,v), \tag{12}$$

where h is the homothetic scale.

If we take partial derivatives of (12) according to u and v, we get

$$\begin{cases} \boldsymbol{\psi}_{\mathbf{u}}(u,v) = \mathbf{X_u}(u,v) + h_u(u,v)\varphi(u,v) + h(u,v)\varphi_{\mathbf{u}}(u,v), \\ \\ \boldsymbol{\psi}_{\mathbf{v}}(u,v) = \mathbf{X_v}(u,v) + h_v(u,v)\varphi(u,v) + h(u,v)\varphi_{\mathbf{v}}(u,v). \end{cases} \tag{13}$$

Also, since $\alpha_1,\ \alpha_2,\ \alpha_3$ are constants, we have

$$\varphi_{\mathbf{u}}(u,v) = \alpha_1\frac{\partial \mathbf{r_1}}{\partial u} + \alpha_2\frac{\partial \mathbf{r_2}}{\partial u} + \alpha_3\frac{\partial \mathbf{n}}{\partial u}, \tag{14}$$

$$\varphi_{\mathbf{v}}(u,v) = \alpha_1\frac{\partial \mathbf{r_1}}{\partial v} + \alpha_2\frac{\partial \mathbf{r_2}}{\partial v} + \alpha_3\frac{\partial \mathbf{n}}{\partial v}. \tag{15}$$

If we substitute (9) into (14) and (15), we get

$$\begin{cases} \varphi_{\mathbf{u}}(u,v) = \{(-\alpha_2\kappa_{1g} - \alpha_3\kappa_1)\mathbf{r_1} + \alpha_1\kappa_{1g}\mathbf{r_2} + \alpha_1\kappa_1\mathbf{n}\}\sqrt{E}, \\ \varphi_{\mathbf{v}}(u,v) = \{-\alpha_2\kappa_{2g}\mathbf{r_1} + (\alpha_1\kappa_{2g} - \alpha_3\kappa_2)\mathbf{r}_2 + \alpha_2\kappa_2\mathbf{n}\}\sqrt{G}. \end{cases} \tag{16}$$

Finally, if we substitute (16) into (13),

$$\begin{cases} \boldsymbol{\psi}_{\mathbf{u}}(u,v) = \left(\sqrt{E} + h_u\alpha_1 - h\sqrt{E}\alpha_2\kappa_{1g} - h\sqrt{E}\alpha_3\kappa_1\right)\mathbf{r_1} \\ \qquad\qquad + \left(h_u\alpha_2 + h\sqrt{E}\alpha_1\kappa_{1g}\right)\mathbf{r_2} + \left(h_u\alpha_3 + h\sqrt{E}\alpha_1\kappa_1\right)\mathbf{n}, \\ \\ \boldsymbol{\psi}_{\mathbf{v}}(u,v) = \left(h_v\alpha_1 - h\sqrt{G}\alpha_2\kappa_{2g}\right)\mathbf{r_1} + \left(h_v\alpha_3 + h\sqrt{G}\alpha_2\kappa_2\right)\mathbf{n} \\ \qquad\qquad + \left(\sqrt{G} + h_v\alpha_2 + h\sqrt{G}\alpha_1\kappa_{2g} - h\sqrt{G}\alpha_3\kappa_2\right)\mathbf{r_2}, \end{cases}$$

are obtained. Therefore, we calculate the normal vector of the orbit surface as

$$\begin{aligned}
\psi_{\mathbf{u}} \times \psi_{\mathbf{v}} = &\Big\{h^2\sqrt{EG}\left(\alpha_1\alpha_2\kappa_{1g}\kappa_2 - \alpha_1^2\kappa_1\kappa_{2g} + \alpha_1\alpha_3\kappa_1\kappa_2\right) \\
&+\sqrt{G}hh_u\left(\alpha_2^2\kappa_2 - \alpha_1\alpha_3\kappa_{2g} + \alpha_3^2\kappa_2\right) \\
&+hh_v\sqrt{E}\left(\alpha_1\alpha_3\kappa_{1g} - \alpha_1\alpha_2\kappa_1\right) - h_u\alpha_3\sqrt{G} - h\sqrt{EG}\alpha_1\kappa_1\Big\}\mathbf{r_1} \\
&+\Big\{h^2\sqrt{EG}\left(-\alpha_1\alpha_2\kappa_1\kappa_{2g} + \alpha_2^2\kappa_{1g}\kappa_2 + \alpha_2\alpha_3\kappa_1\kappa_2\right) \\
&+\sqrt{G}hh_u\left(-\alpha_2\alpha_3\kappa_{2g} - \alpha_1\alpha_2\kappa_2\right) \\
&+hh_v\sqrt{E}\left({\alpha_1}^2\kappa_1 + \alpha_2\alpha_3\kappa_{1g} + \alpha_3^2\kappa_1\right) - h_v\alpha_3\sqrt{E} - h\sqrt{EG}\alpha_2\kappa_2\Big\}\mathbf{r_2} \\
&+\Big\{h^2\sqrt{EG}\left(\alpha_2\alpha_3\kappa_{1g}\kappa_2 + \alpha_3^2\kappa_1\kappa_2 - \alpha_1\alpha_3\kappa_1\kappa_{2g}\right) \\
&+\sqrt{G}hh_u\left(\alpha_1^2\kappa_{2g} - \alpha_1\alpha_3\kappa_2 + \alpha_2^2\kappa_{2g}\right) \\
&+hh_v\sqrt{E}\left(-{\alpha_2}^2\kappa_{1g} - \alpha_2\alpha_3\kappa_1 - \alpha_1^2\kappa_{1g}\right) \\
&+h\sqrt{EG}\left(\alpha_1\kappa_{2g} - \alpha_3\kappa_2 - \alpha_2\kappa_{1g} - \alpha_3\kappa_1\right) \\
&+h_v\sqrt{E}\alpha_2 + h_u\alpha_1\sqrt{G} + \sqrt{EG}\Big\}\mathbf{n}. \qquad (17)
\end{aligned}$$

Let's now calculate the oriented area of the region obtained by projecting $\psi(u,v)$ onto the plane with unit normal vector $\mathbf{e} = \sigma_1\mathbf{r_1} + \sigma_2\mathbf{r_2} + \sigma_3\mathbf{n}$.

The projection area $\mathcal{F}_\psi^n$ in the direction of the unit vector $\mathbf{e}$ of the orbit surface is

$$\mathcal{F}_\psi^n = \langle \mathbf{e}, \mathbf{F}_\psi\rangle = \left\langle \mathbf{e}, \iint_D d\mathbf{F}_\psi \right\rangle = \left\langle \mathbf{e}, \iint_D \psi_{\mathbf{u}} \times \psi_{\mathbf{v}} du dv \right\rangle. \qquad (18)$$

If we substitute (17) into (18), we obtain the oriented area of the region obtained by projecting the orbit surface as

$$\begin{aligned}
\mathcal{F}_\psi^n = &\iint_D \Big\{\alpha_1\alpha_2\Big(\sigma_1h^2\sqrt{EG}\kappa_{1g}\kappa_2 - \sigma_1hh_v\sqrt{E}\kappa_1 - \sigma_2h^2\sqrt{EG}\kappa_1\kappa_{2g} - \sigma_2hh_u\sqrt{G}\kappa_2\Big) \\
&+\alpha_1\alpha_3\Big(\sigma_1h^2\sqrt{EG}\kappa_1\kappa_2 - \sigma_1hh_u\sqrt{G}\kappa_{2g} - \sigma_3h^2\sqrt{EG}\kappa_1\kappa_{2g} + \sigma_1hh_v\sqrt{E}\kappa_{1g} \\
&-\sigma_3hh_u\sqrt{G}\kappa_2\Big) + \alpha_2\alpha_3\Big(\sigma_2h^2\sqrt{EG}\kappa_1\kappa_2 - \sigma_2hh_u\sqrt{G}\kappa_{2g} + \sigma_3h^2\sqrt{EG}\kappa_{1g}\kappa_2 \\
&-\sigma_3hh_v\sqrt{E}\kappa_1 + \sigma_2hh_v\sqrt{E}\kappa_{1g}\Big) + {\alpha_1}^2\Big(-\sigma_1h^2\sqrt{EG}\kappa_1\kappa_{2g} + \sigma_2hh_v\sqrt{E}\kappa_1 \\
&+\sigma_3hh_u\sqrt{G}\kappa_{2g} - \sigma_3hh_v\sqrt{E}\kappa_{1g}\Big) + {\alpha_2}^2\Big(\sigma_1hh_u\sqrt{G}\kappa_2 + \sigma_2h^2\sqrt{EG}\kappa_{1g}\kappa_2 \\
&+\sigma_3hh_u\sqrt{G}\kappa_{2g} - \sigma_3hh_v\sqrt{E}\kappa_{1g}\Big) + {\alpha_3}^2\Big(\sigma_1hh_u\sqrt{G}\kappa_2 + \sigma_3h^2\sqrt{EG}\kappa_1\kappa_2 \\
&+\sigma_2hh_v\sqrt{E}\kappa_1\Big) + \alpha_1\Big(-h\sigma_1\sqrt{EG}\kappa_1 + h\sigma_3\sqrt{EG}\kappa_{2g} + \sigma_3h_u\sqrt{G}\Big) \\
&+\alpha_2\Big(-h\sigma_2\sqrt{EG}\kappa_2 - h\sigma_3\sqrt{EG}\kappa_{1g} + \sigma_3h_v\sqrt{E}\Big) \\
&+\alpha_3\Big(-h_u\sigma_1\sqrt{G} - h_v\sigma_2\sqrt{E} - \sigma_3h\sqrt{EG}\kappa_2 - \sigma_3h\sqrt{EG}\kappa_1\Big) + \sigma_3\sqrt{EG}\Big\}du dv
\end{aligned}$$

or

$$\mathcal{F}_{\psi}^{n} = \mathcal{F}_{X}^{n} + \sum_{i=1}^{3} \mathbb{B}_i \alpha_i^2 + \sum_{1=i<j}^{3} \mathbb{B}_{ij} \alpha_i \alpha_j + \sum_{i=1}^{3} \mathbb{C}_i \alpha_i, \tag{19}$$

where

$$\mathbb{B}_1 = \iint_D \Big(-\sigma_1 h^2 \sqrt{EG} \kappa_1 \kappa_{2g} + \sigma_2 h h_v \sqrt{E} \kappa_1 + \sigma_3 h h_u \sqrt{G} \kappa_{2g} - \sigma_3 h h_v \sqrt{E} \kappa_{1g} \Big) dudv,$$

$$\mathbb{B}_2 = \iint_D \Big(\sigma_1 h h_u \sqrt{G} \kappa_2 + \sigma_2 h^2 \sqrt{EG} \kappa_{1g} \kappa_2 + \sigma_3 h h_u \sqrt{G} \kappa_{2g} - \sigma_3 h h_v \sqrt{E} \kappa_{1g} \Big) dudv,$$

$$\mathbb{B}_3 = \iint_D \Big(\sigma_1 h h_u \sqrt{G} \kappa_2 + \sigma_3 h^2 \sqrt{EG} \kappa_1 \kappa_2 + \sigma_2 h h_v \sqrt{E} \kappa_1 \Big) dudv,$$

$$\mathbb{B}_{12} = \iint_D \Big(\sigma_1 h^2 \sqrt{EG} \kappa_{1g} \kappa_2 - \sigma_1 h h_v \sqrt{E} \kappa_1 - \sigma_2 h^2 \sqrt{EG} \kappa_1 \kappa_{2g} - \sigma_2 h h_u \sqrt{G} \kappa_2 \Big) dudv,$$

$$\begin{aligned} \mathbb{B}_{13} &= \iint_D \Big(\sigma_1 h^2 \sqrt{EG} \kappa_1 \kappa_2 - \sigma_1 h h_u \sqrt{G} \kappa_{2g} - \sigma_3 h^2 \sqrt{EG} \kappa_1 \kappa_{2g} + \sigma_1 h h_v \sqrt{E} \kappa_{1g} \\ &\quad - \sigma_3 h h_u \sqrt{G} \kappa_2 \Big) dudv, \end{aligned}$$

$$\begin{aligned} \mathbb{B}_{23} &= \iint_D \Big(\sigma_2 h^2 \sqrt{EG} \kappa_1 \kappa_2 - \sigma_2 h h_u \sqrt{G} \kappa_{2g} + \sigma_3 h^2 \sqrt{EG} \kappa_{1g} \kappa_2 - \sigma_3 h h_v \sqrt{E} \kappa_1 \\ &\quad + \sigma_2 h h_v \sqrt{E} \kappa_{1g} \Big) dudv, \end{aligned}$$

$$\mathbb{C}_1 = \iint_D \Big(-h\sigma_1 \sqrt{EG} \kappa_1 + h\sigma_3 \sqrt{EG} \kappa_{2g} + \sigma_3 h_u \sqrt{G} \Big) dudv,$$

$$\mathbb{C}_2 = \iint_D \Big(-h\sigma_2 \sqrt{EG} \kappa_2 - h\sigma_3 \sqrt{EG} \kappa_{1g} + \sigma_3 h_v \sqrt{E} \Big) dudv,$$

$$\mathbb{C}_3 = \iint_D \Big(-h_u \sigma_1 \sqrt{G} - h_v \sigma_2 \sqrt{E} - \sigma_3 h \sqrt{EG} \kappa_1 - \sigma_3 h \sqrt{EG} \kappa_2 \Big) dudv,$$

$$\mathcal{F}_X^n = \iint_D \sigma_3 \sqrt{EG} dudv.$$

PROPOSITION 1. Let $\mathbb{M}$ be a regular surface given by the parametric equation $X(u,v)$, and let the parameter curves of $\mathbb{M}$ be lines of curvature on $\mathbb{M}$. Let $\mathbf{r_1} = \frac{X_u}{||X_u||}, \mathbf{r_2} = \frac{X_v}{||X_v||}, \mathbf{n} = \mathbf{r_1} \times \mathbf{r_2}$. All the fixed points (with respect to the frame $\{\mathbf{r_1}, \mathbf{r}_2, \mathbf{n}\}$) whose orbit surfaces have equal projection area in the same direction lie on the same quadric.

4 Holditch-type Theorem for Projection Areas

Let P and S be two fixed points and Q be another point on the line segment PS. Then we may write

$$q_i = \lambda p_i + \mu s_i\,, \quad \lambda + \mu = 1, \quad 1 \leq i \leq 3.$$

Using (19), we get

$$\mathcal{F}_Q^n = \lambda^2 \mathcal{F}_P^n + 2\lambda\mu \mathcal{F}_{PS}^n + \mu^2 \mathcal{F}_S^n, \tag{20}$$

where

$$\mathcal{F}_{PS}^n = \mathcal{F}_X^n + \frac{1}{2}\sum_{i=1}^{3} \mathbb{C}_i (p_i + s_i) + \sum_{i=1}^{3} \mathbb{B}_i p_i s_i + \frac{1}{2}\sum_{1=i<j}^{3} \mathbb{B}_{ij}(p_i s_j + p_j s_i)$$

is called the mixture projection area, and it satisfies $\mathcal{F}_{PS}^n = \mathcal{F}_{SP}^n$ and $\mathcal{F}_{PP}^n = \mathcal{F}_P^n$. Since

$$\mathcal{F}_P^n - 2\mathcal{F}_{PS}^n + \mathcal{F}_S^n = \sum_{i=1}^{3} \mathbb{B}_i (p_i - s_i)^2 + \sum_{1=i<j}^{3} \mathbb{B}_{ij}(p_i - s_i)(p_i - s_j),$$

we can rewrite (20) as follows:

$$\mathcal{F}_Q^n = \lambda \mathcal{F}_P^n + \mu \mathcal{F}_S^n - \lambda\mu \left\{ \sum_{i=1}^{3} \mathbb{B}_i (p_i - s_i)^2 + \sum_{1=i<j}^{3} \mathbb{B}_{ij}(p_i - s_i)(p_j - s_j) \right\}. \tag{21}$$

If we define the oriented distance $\mathcal{D}(P,S)$ between the points P, S by

$$\mathcal{D}^2(P,S) = \varepsilon \left\{ \sum_{i=1}^{3} \mathbb{B}_i (p_i - s_i)^2 + \sum_{1=i<j}^{3} \mathbb{B}_{ij}(p_i - s_i)(p_j - s_j) \right\}, \quad \varepsilon = \mp 1, \tag{22}$$

from (21) we have

$$\mathcal{F}_Q^n = \lambda \mathcal{F}_P^n + \mu \mathcal{F}_S^n - \varepsilon\lambda\mu \mathcal{D}^2(P,S).$$

Since $\mathcal{D}$ satisfies

$$\mathcal{D}(P,Q) + \mathcal{D}(Q,S) = \mathcal{D}(P,S),$$

denoting

$$\lambda = \frac{\mathcal{D}(Q,S)}{\mathcal{D}(P,S)}, \quad \mu = \frac{\mathcal{D}(P,Q)}{\mathcal{D}(P,S)}$$

yield

$$\mathcal{F}_Q^n = \frac{1}{\mathcal{D}(P,S)} \left\{ \mathcal{D}(Q,S)\mathcal{F}_P^n + \mathcal{D}(P,Q)\mathcal{F}_S^n \right\} - \varepsilon \mathcal{D}(P,Q)\mathcal{D}(Q,S). \tag{23}$$

Now, we consider that the points P and S trace the same orbit surface. In this case, for the projection area in the direction of $\mathbf{e}$ we have $\mathcal{F}_P^n = \mathcal{F}_S^n$. Then, from (23) we obtain

$$\mathcal{F}_P^n - \mathcal{F}_Q^n = \varepsilon \mathcal{D}(P,Q)\mathcal{D}(Q,S)$$

which is the analog result given by H. R. Müller [9]. It should be noted that the distance function defined here depends on not only the geodesic curvatures of the parameter curves and principal curvatures of the surface but also on the homothetic scale.

PROPOSITION 2. Let $\mathbb{M}$ be a regular surface given by the parametric equation $X(u,v)$ whose parameter curves are lines of curvature. Let $\mathbf{r_1} = \frac{X_u}{||X_u||}, \mathbf{r_2} = \frac{X_v}{||X_v||}, \mathbf{n} = \mathbf{r_1} \times \mathbf{r_2}$, and $\overline{PS}$ be a line segment with constant length. If P and S trace the same orbit surface during the homothetic motion of $\{\mathbf{r_1}, \mathbf{r_2}, \mathbf{n}\}$, then, the point Q on this line segment traces another surface. The difference between the projection areas of these surfaces (in the direction of a unit vector $\mathbf{e}$) depends on the distances (defined by (22)) of Q from the endpoints.

5 Applications

Let us consider the cylinder $\mathcal{C}$ given by the parametric equation $X(u,v) = (\cos u, \sin u, v)$ with $D : 0 \le u \le \pi,\ -1 \le v \le 1$. We have $\mathbf{r}_1 = (-\sin u, \cos u, 0)$, $\mathbf{r}_2 = (0,0,1)$, $\mathbf{n} = (\cos u, \sin u, 0)$, $\kappa_{1g} = \kappa_{2g} = 0$, $\kappa_1 = -1, \kappa_2 = 0$. If we take $\mathbf{e} = (0,1,0)$, then we obtain $\sigma_1 = \cos u$, $\sigma_2 = 0$, $\sigma_3 = \sin u$.

If we choose $h(u,v) = 1$, then we obtain the oriented projection area formula as $\mathcal{F}_\psi^n = 4 + 4\alpha_3$. This means all the fixed points with $\alpha_3 = -1$ have vanishing oriented projection areas. Also, all the fixed points on the tangent plane of the cylinder have the same projection area as the given cylinder's projection area. Figure 1 shows some orbit surfaces with vanishing oriented projection areas from different view angles. Figure 2 shows some orbit surfaces with the projection area 4 from different view angles.

If we choose $h(u,v) = \frac{u+v}{3}$, we obtain the oriented projection area formula as

$$\mathcal{F}_\psi^n = 4 - \frac{4}{9}\alpha_1\alpha_2 + \frac{2\pi}{9}\alpha_2\alpha_3 + \frac{4}{3}\alpha_2 + \frac{2\pi}{3}\alpha_3.$$

In this case, we obtain the oriented projection area of the orbit surface of $\varphi(u,v) = \mathbf{r}_1 - 2\mathbf{r}_2 + 3\mathbf{n}$ as $\frac{20}{9} + \frac{6\pi}{9}$. Figure 3 shows its orbit surface from different view angles.

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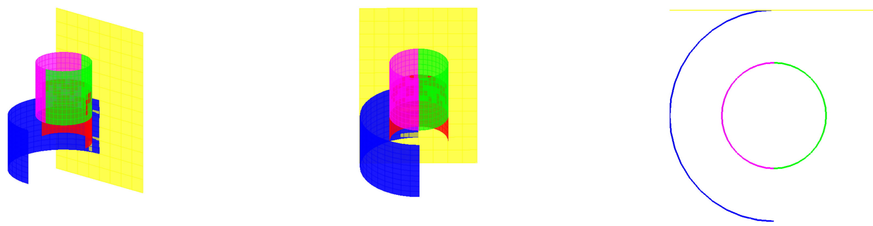
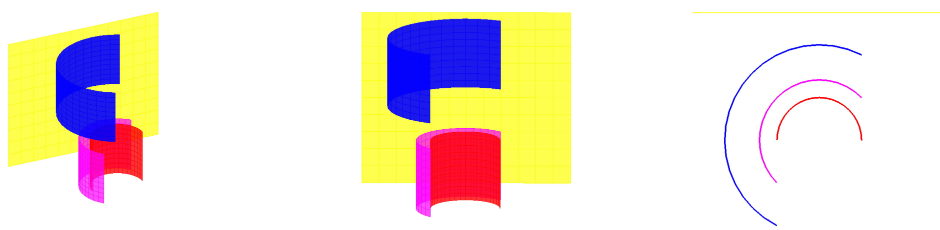


(a) (b) (c) seen from top

Figure 1: The given cylinder $\mathcal{C}$ (red) and the projection plane (yellow) together with the orbit surfaces of the points: $\varphi(u,v) = \mathbf{r}_1 + \mathbf{r}_2 - \mathbf{n}$ (magenta), $\varphi(u,v) = 2\mathbf{r}_1 - \mathbf{r}_2 - \mathbf{n}$ (blue), $\varphi(u,v) = -\mathbf{r}_1 + \mathbf{r}_2 - \mathbf{n}$ (green).



(a) (b) (c) seen from top

Figure 2: The given cylinder $\mathcal{C}$ (red) and the projection plane (yellow) together with the orbit surfaces of the points: $\varphi(u,v) = \mathbf{r}_1$ (magenta), $\varphi(u,v) = 2\mathbf{r}_1 + 3\mathbf{r}_2$ (blue).

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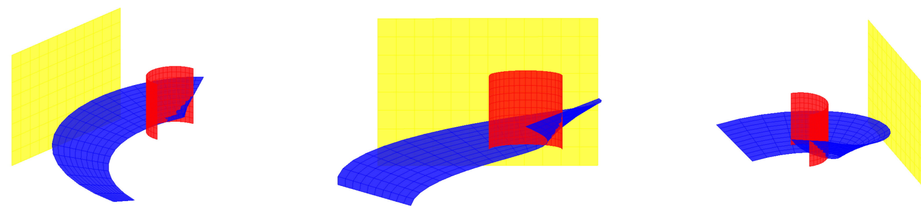

(a) (b) (c)

Figure 3: The given cylinder $\mathcal{C}$ (red) and the projection plane (yellow) together with the orbit surface of the point $\varphi(u,v) = \mathbf{r}_1 - 2\mathbf{r}_2 + 3\mathbf{n}$ (blue).

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