Dimensional Reduction In A Model Of Current-Induced Magnetization Dynamics*

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Abstract

We consider a mathematical model describing skyrmion dynamics in ferromagnets. The model consists of a modified form of the Landau-Lifshitz-Gilbert equation for the evolution of the magnetization vector in a rigid ferromagnet. We perform classical dimensional reductions by using the so-called energy method. We identify the limit problem both for flat and slender media.

1 Introduction

Magnetic skyrmions are particle-like nanometer-sized spin textures of topological origin found in several magnetic materials, and are characterized by a long lifetime. Skyrmions have been observed both by means of neutron scattering in momentum space and microscopy techniques in real space, and their properties include novel Hall effects, current-driven motion with ultra-low current density and multiferroic behaviour. These properties can be understood from a unified viewpoint, namely the emergent electromagnetism associated with the non-coplanar spin structure of skyrmions. From this description, potential applications of skyrmions as information carriers in magnetic information storage and processing devices are described in [15].

In this paper we are interested in a mathematical model describing magnetic skyrmion dynamics by spin-polarized current in ferromagnets. To write the model equations we consider Ω a bounded and regular open set of $\mathbb{R}^3$. The generic point of $\mathbb{R}^3$ is denoted by x. We assume that a ferromagnetic material occupies the domain Ω. With a prescribed current density $\boldsymbol{J}(t,x)$, the time evolution of the magnetization vector $\boldsymbol{m}(t,x)$ may be described by the Landau-Lifshitz-Gilbert (LLG) equation [14, 20]

$$\partial_t \boldsymbol{m} - \alpha \boldsymbol{m} \times \partial_t \boldsymbol{m} = -\gamma \boldsymbol{m} \times \big(\mathcal{H}_{\text{eff}}(\boldsymbol{m}) + \beta(\boldsymbol{J} \cdot \nabla)\boldsymbol{m} \big) \ \text{ in } (0,T) \times \Omega, \qquad (1)$$

where $T > 0$ is fixed and "$\times$" denotes the cross product in $\mathbb{R}^3$. The term parameterized by a factor α describes Gilbert damping torque. The first term on the right-hand side

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accounts for torque by the effective field $\mathcal{H}_{\text{eff}}(\boldsymbol{m})$ which is given by [14, 20]

$$\mathcal{H}_{\text{eff}}(\boldsymbol{m}) = \Delta \boldsymbol{m} + a \operatorname{curl} \boldsymbol{m} + \nabla \varphi, \tag{2}$$

where curl denotes the rotational operator and φ satisfies in $(0, T) \times \mathbb{R}^3$ the stray field equation

$$\operatorname{div}(\nabla \varphi + \overline{\boldsymbol{m}}) = 0, \tag{3}$$

with

$$\begin{cases} \overline{\boldsymbol{m}} = \boldsymbol{m}, & in \ (0, T) \times \Omega, \\ \overline{\boldsymbol{m}} = 0, & in \ (0, T) \times (\mathbb{R}^3 \backslash \Omega). \end{cases}$$

Let E denote the closure of the space of gradients of smooth functions in the L^2 topology. E is a closed subset of $L^2(\mathbb{R}^3)$. The term $\nabla \varphi$ characterizing the stray field may be more conveniently written by using the orthogonal projector onto E denoted by $\mathbb{P}$. Then

$$\nabla \varphi = -\mathbb{P}(\overline{\boldsymbol{m}}). \tag{4}$$

The term parameterized by the positive constant β expresses current-induced torque on $\boldsymbol{m}$. This torque is most commonly non-adiabatic termed and β characterizes its strength. The parameter $\gamma > 0$ is a gyroscopic ratio, and a denotes Dzyaloshinskii-Moriya exchange coefficient. For more detail see [14, 20]. The initial data satisfied by the magnetization is

$$\boldsymbol{m}(0, x) = \boldsymbol{m}(x), \ \ |\boldsymbol{m}_0| = 1 \ \text{ a.e. in } \Omega. \tag{5}$$

Equation (1) should be solved together with appropriate boundary conditions for the magnetization. We consider homogeneous Neumann boundary condition

$$\partial_\nu \boldsymbol{m} = 0 \quad \text{on } (0, T) \times \partial \Omega, \tag{6}$$

where $\partial_\nu \boldsymbol{m}$ denotes the outward normal derivative of $\boldsymbol{m}$ on the boundary of Ω. Throughout, we make use of the following notations. $\mathbb{L}^2(\Omega) = (L^2(\Omega))^3$ and $\mathbb{H}^1(\Omega) = (H^1(\Omega))^3$ are the usual Hilbert spaces. $(L^\infty(\Omega))^3$ is denoted by $\mathbb{L}^\infty(\Omega)$ equipped with the norm $| \cdot |_\infty$. We set $Q = (0, T) \times \Omega$.

In this paper, we will be interested in the problem (1)-(6) in the case where $\beta = 0$. We note that these simplifications do not limit the proposed analysis (see [22]). We consider the following problem

$$\begin{cases} \partial_t \boldsymbol{m} - \alpha \boldsymbol{m} \times \partial_t \boldsymbol{m} = -\gamma \boldsymbol{m} \times \mathcal{H}_{\text{eff}}(\boldsymbol{m}) \quad \text{in } Q, \\[2mm] \mathcal{H}_{\text{eff}}(\boldsymbol{m}) = \Delta \boldsymbol{m} + a \operatorname{curl} \boldsymbol{m} + \nabla \varphi \quad \text{in } Q, \\[2mm] \operatorname{div}(\nabla \varphi + \overline{\boldsymbol{m}}) = 0 \quad \text{in } (0, T) \times \mathbb{R}^3, \\[2mm] \boldsymbol{m}(0, x) = \boldsymbol{m}(x), \ \ |\boldsymbol{m}_0| = 1 \text{ a.e. in } \Omega, \\[2mm] \partial_\nu \boldsymbol{m} = 0 \quad \text{on } (0, T) \times \partial \Omega. \end{cases} \tag{7}$$

LEMMA 1. If $(\boldsymbol{m}, \varphi)$ is a weak solution of (7) with $\boldsymbol{m} \in \mathbb{H}^1(Q) \cap L^\infty(0, T, \mathbb{H}^1(\Omega))$ and $\nabla \varphi \in L^\infty(0, T, \mathbb{L}^2(\mathbb{R}^3))$, then we have the following energy estimate

$$\mathcal{E}\big(\boldsymbol{m}(t)\big) + \frac{\alpha}{2\gamma} \int_0^t \int_\Omega |\partial_t \boldsymbol{m}|^2 \, \mathrm{d}x \mathrm{d}s \leq \mathcal{E}(\boldsymbol{m}_0) \left(1 + \frac{2\gamma a^2}{\alpha} t \exp\left(\frac{2\gamma a^2}{\alpha} t\right)\right), \qquad (8)$$

where

$$\mathcal{E}\big(\boldsymbol{m}(t)\big) = \frac{1}{2} \int_\Omega |\nabla \boldsymbol{m}|^2 \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \varphi|^2 \, \mathrm{d}x, \qquad (9)$$

for all $t \in (0, T)$.

PROOF. Taking the cross product of (1) by $\boldsymbol{m}$ and using both the identity $\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) = (\boldsymbol{u} \cdot \boldsymbol{w})\boldsymbol{v} - (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{w}$ and the saturation constraint $|\boldsymbol{m}| = 1$, we obtain

$$\boldsymbol{m} \times \partial_t \boldsymbol{m} + \alpha \partial_t \boldsymbol{m} = \gamma \mathcal{H}_{\mathrm{eff}}(\boldsymbol{m}) - \big(\mathcal{H}_{\mathrm{eff}}(\boldsymbol{m}) \cdot \boldsymbol{m}\big) \boldsymbol{m}.$$

Taking now the scalar product of the above equation with $\partial_t \boldsymbol{m}$, we get

$$\alpha |\partial_t \boldsymbol{m}|^2 = \gamma \mathcal{H}_{\mathrm{eff}}(\boldsymbol{m}) \cdot \partial_t \boldsymbol{m}, \qquad (10)$$

which leads after integration by using the boundary condition (6), and the fact that $\|\mathrm{curl}\, \boldsymbol{m}\|_{\mathbb{L}^2(\Omega)} \leq \sqrt{2} \|\nabla \boldsymbol{m}\|_{\mathbb{L}^2(\Omega)}$ (which we can easily check by a simple calculation) to the estimate

$$\mathcal{E}\big(\boldsymbol{m}(t)\big) + \frac{\alpha}{2\gamma} \int_0^t \int_\Omega |\partial_t \boldsymbol{m}|^2 \, \mathrm{d}x \mathrm{d}s \leq \mathcal{E}(\boldsymbol{m}_0) + \frac{a^2 \gamma}{\alpha} \int_0^t \int_\Omega |\nabla \boldsymbol{m}|^2 \, \mathrm{d}x \mathrm{d}s. \qquad (11)$$

By Gronwall's Lemma in its integral form ([10], Appendix B, page 625), we get (8).

REMARK 1. Estimate (8) leads to a bound for the energy of the local magnetization $\boldsymbol{m}$ on the interval $(0, T)$ for T fixed and finite.

REMARK 2. The operator $\mathbb{P}$ is Lipschitz continuous from $\mathbb{L}^2(\Omega)$ into $\mathbb{L}^2(\mathbb{R}^3)$ since it is bounded and linear.

We can prove the global existence of weak solutions to problem (7) using Faedo-Galerkin combined with Penalty method. We proceed as in [1, 4, 5, 8, 24]. We have the folowing result

THEOREM 1. Let $T > 0$ be fixed and $\boldsymbol{m}_0 \in \mathbb{H}^1(\Omega)$ be such that $|\boldsymbol{m}_0(x)| = 1$ a.e, there exists a global weak solution $\boldsymbol{m}$ of the problem (7) such that $\boldsymbol{m} \in \mathbb{H}^1(Q)$, $|\boldsymbol{m}(t, x)| = 1$ a.e and satisfying the energy estimate (8).

In the context of dimensional reduction, asymptotics of the LLG equation has been studied by several authors, see for example [3, 12, 13, 16, 17, 18]. In [11, 23], the classical dimensional reductions in magnetoelastic interactions 3D-2D and 2D-1D are studied. They adopt a model described by the Landau-Lifshitz-Gilbert equation for the magnetization field coupled with an evolution equation for the displacement. The

limit problem both for flat and slender media by using the so-called energy method was identified. In [21], the passage from 3D to 2D of a phase transition model in ferromagnetism coupling thermodynamic and electromagnetic properties of the ferromagnetic material is considered and the limit problem is also obtained by energy method. We finally mention that an important progress was done to design schemes constructing the weak solutions to the general LLG equation. Several schemes were proposed and their convergence to weak solutions was proved. A significant step forward in the convergence theory of numerical schemes has been done recently, see [2, 6, 7]. This will be helpful to give a strategy for efficient computer implementation which may reflect the true nature of the augmentation of the LLG model considered in this paper.

The next section is devoted to the dimensional reduction 3D-2D and 3D-1D. In the last section concluding remarks and suggestions for future work are given.

2   Dimensional Reductions

2.1   Flat Domains

Let ε be a real parameter taking values in a sequence of positive numbers converging to zero. We consider flat domains represented by the cylinder $\Omega_\varepsilon = B \times (0, \varepsilon)$ of $\mathbb{R}^3$ where $B \subset \mathbb{R}^2$ is a bounded and regular open set representing the cross section of Ω_ε. The generic point of $\mathbb{R}^3$ is denoted by $x = (\hat{x}, x_3)$ with $\hat{x} = (x_1, x_2) \in \mathbb{R}^2$. We assume that a ferromagnetic material occupies the domain Ω_ε. We set $\mathbb{R}^3_+ = \mathbb{R}^2 \times \mathbb{R}^+$ and $Q_\varepsilon = (0, T) \times \Omega_\varepsilon$ for $T > 0$. We shall describe the behavior of solutions when $\varepsilon \to 0$.

Let $(\boldsymbol{m}, \varphi)$ be the solution of the problem posed in Ω_ε. We introduce the change of variables

$$(x_1, x_2, x_3) = (x, y, \varepsilon z) \;\; \text{with} \;\; X = (X^{'}, z) \in \Omega = B \times (0, 1), \;\; X^{'} = (x, y). \qquad (12)$$

Let $(\boldsymbol{m}^\varepsilon, \varphi^\varepsilon)$ be the fields associated with $(\boldsymbol{m}, \varphi)$. Then $\boldsymbol{m}^\varepsilon$ satisfies in Q_ε the following equations

$$
\begin{aligned}
\partial_t \boldsymbol{m}^\varepsilon - \alpha \boldsymbol{m}^\varepsilon \times \partial_t \boldsymbol{m}^\varepsilon \;\; = \;\; & -\gamma red \boldsymbol{m}^\varepsilon \times \left(\widehat{\Delta} \boldsymbol{m}^\varepsilon + \frac{1}{\varepsilon^2} \partial_z \boldsymbol{m}^\varepsilon + a \, \widehat{\mathrm{curl}} \boldsymbol{m}^\varepsilon \mathbf{e_3} \right. \\
& \left. + a \, L(\boldsymbol{m}^\varepsilon) + \widehat{\mathrm{grad}} \varphi^\varepsilon + \frac{1}{\varepsilon} \partial_z \varphi^\varepsilon \mathbf{e_3} \right),
\end{aligned}
\qquad (13)
$$

where $(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$ represents the canonical basis of $\mathbb{R}^3$, $\widehat{\mathrm{curl}} \boldsymbol{m}^\varepsilon = \partial_x m_2^\varepsilon - \partial_y m_1^\varepsilon$.

The operators $\widehat{\mathrm{div}}, \widehat{\mathrm{grad}}, \widehat{\Delta}$ represent divergence, gradient and Laplacian operators, respectively, with respect to the variable $\hat{x}$ and $\hat{\boldsymbol{m}} = (m_1, m_2, 0)$ with (m_1, m_2) are the first two components of $\boldsymbol{m}$. The vector $\widehat{\mathrm{grad}} \varphi$ may also be considered as a 2D vector or a 3D vector where the third component is 0.

The operator L is defined by

$$L(\boldsymbol{m}^\varepsilon) = \left(\partial_y m_3^\varepsilon - \frac{1}{\varepsilon} \partial_z m_2^\varepsilon, \; \frac{1}{\varepsilon} \partial_z m_1^\varepsilon - \partial_x m_3^\varepsilon, \; 0 \right). \qquad (14)$$

The magnetic potentiel φ^ε satisfies the equations

$$\begin{cases} \widehat{\operatorname{div}} \left(\widehat{\operatorname{grad}} \, \varphi^\varepsilon + \chi(\Omega)\hat{\boldsymbol{m}}^\varepsilon \right) + \partial_z \left(\frac{1}{\varepsilon^2} \partial_z \varphi^\varepsilon + \frac{1}{\varepsilon}\chi(\Omega)\boldsymbol{m}^\varepsilon \cdot \mathbf{e_3} \right) = 0 \ \text{ in } \ (0,T) \times \mathbb{R}^3_+, \\ \left((\widehat{\operatorname{grad}} \, \varphi^\varepsilon, \frac{1}{\varepsilon}\partial_z \varphi^\varepsilon) + \chi(\Omega)\boldsymbol{m}^\varepsilon \right) \cdot \boldsymbol{n} = 0 \ \text{ on } \ (0,T) \times \partial\Omega_\varepsilon, \end{cases}$$

$$(15)$$

where $\boldsymbol{n}$ is the outer unit normal at the boundary $\partial\Omega_\varepsilon$.

The existence of solutions $(\boldsymbol{m}^\varepsilon, \varphi^\varepsilon)$ of the new system is guaranteed by Theorem 1. The energy estimate satisfied by $(\boldsymbol{m}^\varepsilon, \varphi^\varepsilon)$ is

$$\mathcal{E}^\varepsilon\big(\boldsymbol{m}^\varepsilon(t)\big) + \frac{\alpha}{\gamma} \int_0^t \int_\Omega |\partial_t \boldsymbol{m}^\varepsilon|^2 \, \mathrm{d}X \mathrm{d}s \leq \mathcal{E}^\varepsilon(\boldsymbol{m}_0^\varepsilon)\left(1 + \frac{2\gamma a^2}{\alpha} t \exp\left(\frac{2\gamma a^2}{\alpha} t\right)\right), \quad (16)$$

where

$$\begin{aligned} \mathcal{E}^\varepsilon\big(\boldsymbol{m}^\varepsilon(t)\big) \ &= \ \frac{1}{2}\Bigg(\int_\Omega |\widehat{\operatorname{grad}} \, \boldsymbol{m}^\varepsilon|^2 \, \mathrm{d}X + \frac{1}{\varepsilon^2} \int_\Omega |\partial_z \boldsymbol{m}^\varepsilon|^2 \, \mathrm{d}X + \int_{\mathbb{R}^3_+} |\widehat{\operatorname{grad}} \, \varphi^\varepsilon|^2 \, \mathrm{d}X \\ &\quad + \frac{1}{\varepsilon^2} \int_\Omega |\partial_z \varphi^\varepsilon|^2 \, \mathrm{d}X \Bigg), \end{aligned}$$

and $\mathcal{E}^\varepsilon(\boldsymbol{m}_0^\varepsilon)$ is given by

$$\begin{aligned} \mathcal{E}^\varepsilon(\boldsymbol{m}_0^\varepsilon) \ &= \ \frac{1}{2}\Bigg(\int_\Omega |\widehat{\operatorname{grad}} \, \boldsymbol{m}_0^\varepsilon|^2 \, \mathrm{d}X + \frac{1}{\varepsilon^2} \int_\Omega |\partial_z \boldsymbol{m}_0^\varepsilon|^2 \, \mathrm{d}X + \int_{\mathbb{R}^3_+} |\widehat{\operatorname{grad}} \, \varphi_0^\varepsilon|^2 \, \mathrm{d}X \\ &\quad + \frac{1}{\varepsilon^2} \int_\Omega |\partial_z \varphi_0^\varepsilon|^2 \, \mathrm{d}X \Bigg). \end{aligned}$$

To get uniform bounds for the solutions we discuss the admissibility criterion for the initial data.

DEFINITION 1. An initial datum $(\boldsymbol{m}_0^\varepsilon, \varphi_0^\varepsilon)$ is said to be admissible if we have

$$\mathcal{E}^\varepsilon(\boldsymbol{m}_0^\varepsilon) < +\infty.$$

The admissibility criterion means

$$\int_\Omega |\widehat{\operatorname{grad}} \, \boldsymbol{m}_0^\varepsilon|^2 \, \mathrm{d}X + \frac{1}{\varepsilon^2} \int_\Omega |\partial_z \boldsymbol{m}_0^\varepsilon|^2 \, \mathrm{d}X + \int_{\mathbb{R}^3_+} |\widehat{\operatorname{grad}} \, \varphi_0^\varepsilon|^2 \, \mathrm{d}X + \frac{1}{\varepsilon^2} \int_\Omega |\partial_z \varphi_0^\varepsilon|^2 \, \mathrm{d}X < +\infty.$$

Thus, since $|\boldsymbol{m}_0^\varepsilon|^2 = 1$ a.e., to satisfy the criterion, we assume that there exists $C > 0$ independent of ε such that

$$\int_\Omega |\widehat{\operatorname{grad}} \, \boldsymbol{m}_0^\varepsilon|^2 \, \mathrm{d}X < C, \quad \int_\Omega |\partial_z \boldsymbol{m}_0^\varepsilon|^2 \, \mathrm{d}X < C\varepsilon^2, \quad |\boldsymbol{m}_0^\varepsilon(x,y)|^2 = 1 \ \ a.e, \quad (17)$$

$$\int_{\mathbb{R}^3_+} |\widehat{\operatorname{grad}} \, \varphi_0^\varepsilon|^2 \, \mathrm{d}X < C, \quad \int_\Omega |\partial_z \varphi_0^\varepsilon|^2 \, \mathrm{d}X < C\varepsilon^2. \quad (18)$$

Conditions (17) and (18) mean that the couple $(\boldsymbol{m}_0^\varepsilon, \varphi_0^\varepsilon)$ is essentially independent of the variable z and its strong limit $(\boldsymbol{m}_0, \varphi_0)$ is independent of z.

Let $(\boldsymbol{m}^\varepsilon, \varphi^\varepsilon)$ be a solution of the problem associated with an admissible initial datum $(\boldsymbol{m}_0^\varepsilon, \varphi_0^\varepsilon)$. We have

$$\boldsymbol{m}_0^\varepsilon \rightharpoonup \boldsymbol{m}_0 \ \text{ weakly in } \ \mathbb{H}^1(\Omega),$$

$$\widehat{\text{grad}} \, \varphi_0^\varepsilon \rightharpoonup \widehat{\text{grad}} \, \varphi_0 \ \text{ weakly in } \ \mathbb{L}^2(\mathbb{R}_+^3).$$

Moreover $\boldsymbol{m}_0(X) = \boldsymbol{m}_0(X^{'})$ is independent of z. For subsequences, the solutions verify the convergences

$$\boldsymbol{m}^\varepsilon \rightharpoonup \boldsymbol{m} \ \text{ weakly-}\star \text{ in } \ L^\infty\big(0, T, \mathbb{H}^1(\Omega)\big),$$

$$\partial_t \boldsymbol{m}^\varepsilon \rightharpoonup \partial_t \boldsymbol{m} \ \text{ weakly in } \ L^2\big(0, T, \mathbb{L}^2(\Omega)\big),$$

$$\partial_z \boldsymbol{m}^\varepsilon \to 0 \ \text{ strongly in } \ L^\infty\big(0, T, \mathbb{L}^2(\Omega)\big),$$

$$\text{curl} \, \boldsymbol{m}^\varepsilon \rightharpoonup \text{curl} \, \boldsymbol{m} \ \text{ weakly in } \ L^\infty\big(0, T, \mathbb{L}^2(\Omega)\big),$$

$$\boldsymbol{m}^\varepsilon \to \boldsymbol{m} \ \text{ strongly in } \ L^2\big(0, T, \mathbb{L}^2(\Omega)\big), \tag{19}$$

$$\widehat{\text{grad}} \, \varphi^\varepsilon \rightharpoonup \widehat{\text{grad}} \, \varphi \ \text{ weakly in } \ L^\infty\big(0, T, \mathbb{L}^2(\mathbb{R}_+^3)\big),$$

$$\partial_z \varphi^\varepsilon \to 0 \ \text{ strongly in } \ L^\infty\big(0, T, L^2(\mathbb{R}_+^3)\big).$$

The strong convergence of (19) is a consequence of the classical use of Aubin's compactness Lemma (See [19]).

We multiply the first equation of (15) by a test functions $\boldsymbol{k} \in \mathcal{D}((0, T) \times \mathbb{R}_+^3)$, we get

$$\int_{(0,T) \times \mathbb{R}_+^3} \widehat{\text{div}} \left(\widehat{\text{grad}} \varphi^\varepsilon + \chi(\Omega)\hat{\boldsymbol{m}}^\varepsilon \right) \cdot \boldsymbol{k} \, \mathrm{d}X \mathrm{d}t$$
$$+ \int_{(0,T) \times \mathbb{R}_+^3} \partial_z \left(\frac{1}{\varepsilon^2} \partial_z \varphi^\varepsilon + \frac{1}{\varepsilon} \chi(\Omega) \boldsymbol{m}^\varepsilon \cdot \mathbf{e_3} \right) \cdot \boldsymbol{k} \, \mathrm{d}X \mathrm{d}t = 0. \tag{20}$$

Arguing that in the limit, φ is independent of the variable z we choose test functions $\boldsymbol{k}$ which are independent of z hence (20) becomes

$$-\int_{(0,T) \times \mathbb{R}_+^3} \widehat{\text{grad}} \, \varphi^\varepsilon \cdot \widehat{\text{grad}} \, \boldsymbol{k} \, \mathrm{d}X \mathrm{d}t - \int_{(0,T) \times \mathbb{R}_+^3} \chi(\Omega)\hat{\boldsymbol{m}}^\varepsilon \cdot \widehat{\text{grad}} \, \boldsymbol{k} \, \mathrm{d}X \mathrm{d}t = 0. \tag{21}$$

Let us pass to the limit into (21), we get

$$-\int_{(0,T) \times \mathbb{R}_+^3} \widehat{\text{grad}} \, \varphi \cdot \widehat{\text{grad}} \, \boldsymbol{k} \, \mathrm{d}X^{'} \mathrm{d}t - \int_{(0,T) \times \mathbb{R}_+^3} \chi(\Omega)\hat{\boldsymbol{m}} \cdot \widehat{\text{grad}} \, \boldsymbol{k} \, \mathrm{d}X^{'} \mathrm{d}t = 0.$$

Thus

$$\widehat{\Delta}\varphi + \chi(\Omega)\widehat{\text{div}} \, \hat{\boldsymbol{m}} = 0 \ \text{ in } \ (0, T) \times \mathbb{R}_+^3.$$

Let $\boldsymbol{g}^{\varepsilon}(t, \hat{x}, z)$ be a regular test functions depending on ε. Multiplying equation (13) by $\boldsymbol{g}^{\varepsilon}$ and integrating by parts, we get the weak formulation

$$\frac{1}{\gamma} \int_Q \partial_t \boldsymbol{m}^{\varepsilon} \cdot \boldsymbol{g}^{\varepsilon} \, \mathrm{d}X \mathrm{d}t - \frac{\alpha}{\gamma} \int_Q \boldsymbol{m}^{\varepsilon} \times \partial_t \boldsymbol{m}^{\varepsilon} \cdot \boldsymbol{g}^{\varepsilon} \mathrm{d}X \mathrm{d}t$$

$$= \int_Q \boldsymbol{m}^{\varepsilon} \times \widehat{\mathrm{grad}} \boldsymbol{m}^{\varepsilon} \cdot \widehat{\mathrm{grad}} \, \boldsymbol{g}^{\varepsilon} \mathrm{d}X \mathrm{d}t + \frac{1}{\varepsilon^2} \int_Q \boldsymbol{m}^{\varepsilon} \times \partial_z \boldsymbol{m}^{\varepsilon} \cdot \partial_z \boldsymbol{g}^{\varepsilon} \, \mathrm{d}X \mathrm{d}t$$

$$- a \int_Q \boldsymbol{m}^{\varepsilon} \times L(\boldsymbol{m}^{\varepsilon}) \cdot \boldsymbol{g}^{\varepsilon} \, \mathrm{d}X \mathrm{d}t - \int_Q \boldsymbol{m}^{\varepsilon} \times \widehat{\mathrm{grad}} \varphi^{\varepsilon} \cdot \boldsymbol{g}^{\varepsilon} \, \mathrm{d}X \mathrm{d}t$$

$$- \frac{1}{\varepsilon} \int_Q \boldsymbol{m}^{\varepsilon} \times \partial_z \varphi^{\varepsilon} \mathbf{e_3} \cdot \boldsymbol{g}^{\varepsilon} \, \mathrm{d}X \mathrm{d}t - a \int_Q \boldsymbol{m}^{\varepsilon} \times \mathrm{curl} \, \boldsymbol{m}^{\varepsilon} \mathbf{e_3} \cdot \boldsymbol{g}^{\varepsilon} \, \mathrm{d}X \mathrm{d}t.$$

In order to pass to the limit as $\varepsilon \to 0$, we use the fact that in the limit, $\boldsymbol{m}$ is independent of the variable z. We use test functions of the type

$$\boldsymbol{g}^{\varepsilon}(t, X^{'}, z) = \boldsymbol{g}_0(t, X^{'}) + \varepsilon \boldsymbol{g}(t, X^{'}, \varepsilon z).$$

Note that

$$\partial_z \boldsymbol{g}^{\varepsilon} = \varepsilon^2 (\partial_z \boldsymbol{g})(\varepsilon z) \quad \text{and} \quad \partial_{X'} \boldsymbol{g}^{\varepsilon} = \partial_{X'} \boldsymbol{g}_0 + \varepsilon \partial_{X'} \boldsymbol{g}(\varepsilon z).$$

The weak convergence of $\partial_t \boldsymbol{m}^{\varepsilon}$ to $\partial_t \boldsymbol{m}$ in $L^2(0, T, \mathbb{L}^2(\Omega))$ and the strong convergence of $\boldsymbol{m}^{\varepsilon}$ to $\boldsymbol{m}$ in $L^2(0, T, \mathbb{L}^2(\Omega))$ allow to get

$$\int_Q \partial_t \boldsymbol{m}^{\varepsilon} \cdot \boldsymbol{g}^{\varepsilon} \, \mathrm{d}X \mathrm{d}t \longrightarrow \int_{Q'} \partial_t \boldsymbol{m} \cdot \boldsymbol{g}_0 \, \mathrm{d}X^{'} \mathrm{d}t,$$

$$\int_Q \boldsymbol{m}^{\varepsilon} \times \partial_t \boldsymbol{m}^{\varepsilon} \cdot \boldsymbol{g}^{\varepsilon} \, \mathrm{d}X \mathrm{d}t \longrightarrow \int_{Q'} \boldsymbol{m} \times \partial_t \boldsymbol{m} \cdot \boldsymbol{g}_0 \, \mathrm{d}X^{'} \mathrm{d}t.$$

Next, we have

$$\int_Q \boldsymbol{m}^{\varepsilon} \times \widehat{\mathrm{grad}} \, \boldsymbol{m}^{\varepsilon} \cdot \widehat{\mathrm{grad}} \, \boldsymbol{g}^{\varepsilon} \, \mathrm{d}X \mathrm{d}t \longrightarrow \int_{Q'} \boldsymbol{m} \times \widehat{\mathrm{grad}} \, \boldsymbol{m} \cdot \widehat{\mathrm{grad}} \, \boldsymbol{g}_0 \, \mathrm{d}X^{'} \mathrm{d}t,$$

and

$$\frac{1}{\varepsilon^2} \int_Q \boldsymbol{m}^{\varepsilon} \times \partial_z \boldsymbol{m}^{\varepsilon} \cdot \partial_z \boldsymbol{g}^{\varepsilon} \, \mathrm{d}X \mathrm{d}t \longrightarrow 0,$$

$$\int_Q \boldsymbol{m}^{\varepsilon} \times \widehat{\mathrm{grad}} \, \varphi^{\varepsilon} \cdot \boldsymbol{g}^{\varepsilon} \, \mathrm{d}X \mathrm{d}t \longrightarrow \int_{Q'} \boldsymbol{m} \times \widehat{\mathrm{grad}} \, \varphi \cdot \boldsymbol{g}_0 \, \mathrm{d}X^{'} \mathrm{d}t,$$

$$\int_Q \boldsymbol{m}^{\varepsilon} \times \widehat{\mathrm{curl}} \, \boldsymbol{m}^{\varepsilon} \mathbf{e_3} \cdot \boldsymbol{g}^{\varepsilon} \, \mathrm{d}X \mathrm{d}t \longrightarrow \int_{Q'} \boldsymbol{m} \times \widehat{\mathrm{curl}} \, \boldsymbol{m} \, \mathbf{e_3} \cdot \boldsymbol{g}_0 \, \mathrm{d}X^{'} \mathrm{d}t,$$

where $Q^{'} = (0, T) \times B$. Recall that

$$L(\boldsymbol{m}^{\varepsilon}) = \left(\partial_y m_3^{\varepsilon} - \frac{1}{\varepsilon} \partial_z m_2^{\varepsilon}, \ \frac{1}{\varepsilon} \partial_z m_1^{\varepsilon} - \partial_x m_3^{\varepsilon}, \ 0\right).$$

To pass to the limit in term involving $L(\boldsymbol{m}^{\varepsilon})$, we have the following convergence result.

LEMMA 2. Define $\boldsymbol{\Psi}_\varepsilon = \frac{1}{\varepsilon}\partial_z \boldsymbol{m}^\varepsilon$. Then

$$\boldsymbol{\Psi}_\varepsilon \rightharpoonup \boldsymbol{\Psi} \quad \text{weakly-}\star\ \ L^\infty\big(0, T, \mathbb{L}^2(\Omega)\big),$$

where $\boldsymbol{\Psi}$ is given by

$$\boldsymbol{\Psi} = \boldsymbol{F} \times \boldsymbol{m} + \alpha_0 \boldsymbol{m}, \tag{22}$$

and $\boldsymbol{F}$ is a function of the variable X' and $\alpha_0 \in \mathbb{R}$.

PROOF. We multiply equation (13) by ε and choose $\boldsymbol{g}^\varepsilon = \boldsymbol{g} \in \mathcal{D}(Q)$ independent of ε. We get

$$\varepsilon\left(\frac{1}{\gamma}\int_Q \partial_t \boldsymbol{m}^\varepsilon \cdot \boldsymbol{g}\, \mathrm{d}X\mathrm{d}t - \frac{\alpha}{\gamma}\int_Q \boldsymbol{m}^\varepsilon \times \partial_t \boldsymbol{m}^\varepsilon \cdot \boldsymbol{g}\, \mathrm{d}X\mathrm{d}t\right)$$

$$= \varepsilon\bigg(\int_Q \boldsymbol{m}^\varepsilon \times \widehat{\mathrm{grad}}\,\boldsymbol{m}^\varepsilon \cdot \widehat{\mathrm{grad}}\,\boldsymbol{g}\, \mathrm{d}X\mathrm{d}t - a\int_Q \boldsymbol{m}^\varepsilon \times L(\boldsymbol{m}^\varepsilon) \cdot \boldsymbol{g}\, \mathrm{d}X\mathrm{d}t$$

$$- \int_Q \boldsymbol{m}^\varepsilon \times \widehat{\mathrm{grad}}\,\varphi^\varepsilon \cdot \boldsymbol{g}\, \mathrm{d}X\mathrm{d}t - \frac{1}{\varepsilon}\int_Q \boldsymbol{m}^\varepsilon \times \partial_z \varphi^\varepsilon \mathbf{e_3} \cdot \boldsymbol{g}\, \mathrm{d}X\mathrm{d}t$$

$$- a\int_Q \boldsymbol{m}^\varepsilon \times \mathrm{curl}\,\boldsymbol{m}^\varepsilon \mathbf{e_3} \cdot \boldsymbol{g}\, \mathrm{d}X\mathrm{d}t\bigg) + \frac{1}{\varepsilon}\int_Q \boldsymbol{m}^\varepsilon \times \partial_z \boldsymbol{m}^\varepsilon \cdot \partial_z \boldsymbol{g}\, \mathrm{d}X\mathrm{d}t.$$

Hence, passing to the limit in the above equation, by using previous convergences, we deduce that the weak-$\star$ limit $\boldsymbol{\Psi}$ of the sequence $\boldsymbol{\Psi}_\varepsilon$ satisfies $\partial_z\big(\boldsymbol{m} \times \boldsymbol{\Psi}\big) = 0$. Recall that if $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{X}$ are three-dimensional vectors such that $\boldsymbol{u} \neq 0$, then

$$\boldsymbol{u} \times \boldsymbol{X} = \boldsymbol{v} \quad \text{means} \quad \exists \lambda \in \mathbb{R}, \quad \boldsymbol{X} = -\frac{\boldsymbol{u} \times \boldsymbol{v}}{|\boldsymbol{u}|} + \lambda\boldsymbol{u}.$$

This allows to get (22). By lemma 2, we have

$$\int_Q \boldsymbol{m}^\varepsilon \times L(\boldsymbol{m}^\varepsilon) \cdot \boldsymbol{g}^\varepsilon\, \mathrm{d}X\mathrm{d}t \longrightarrow \int_{Q'} \boldsymbol{m} \times L(\boldsymbol{m}) \cdot \boldsymbol{g}_0\, \mathrm{d}X'\mathrm{d}t,$$

where $L(\boldsymbol{m})$ is given by

$$L(\boldsymbol{m}) = \big(\partial_y m_3 - \Psi_2\,,\ \ \Psi_1 - \partial_x m_3\,,\ \ 0\big),$$

and $\Psi_i, i = 1, 2, 3$ are the components of the vector $\boldsymbol{\Psi}$. To pass to the limit in the term $\frac{1}{\varepsilon}\int_Q \boldsymbol{m}^\varepsilon \times \partial_z \varphi^\varepsilon \mathbf{e_3} \cdot \boldsymbol{g}^\varepsilon\, \mathrm{d}X\mathrm{d}t$, we need the following Lemma.

LEMMA 3. Define $\Theta_\varepsilon = \frac{1}{\varepsilon}\partial_z \varphi^\varepsilon$, then

$$\Theta_\varepsilon \rightharpoonup \Theta \quad \text{weakly-}\star \text{ in } L^\infty\big(0, T, \mathbb{L}^2(\mathbb{R}^3_+)\big),$$

where Θ is given by

$$\Theta = -\chi(\Omega)\boldsymbol{m} \cdot \mathbf{e_3} + G \quad \text{in } (0, T) \times \mathbb{R}^3_+.$$

PROOF. We multiply equation (20) by ε, and we choose a test function $\boldsymbol{k} \in \mathcal{D}((0,T) \times \mathbb{R}_+^3)$ independent of ε. Passing to the limit in the resulting equation, we get $\partial_z(\Theta + \chi(\Omega)\boldsymbol{m} \cdot \mathbf{e_3}) = 0$. Therfore $\Theta = -\chi(\Omega)\boldsymbol{m} \cdot \mathbf{e_3} + G$ in $(0,T) \times \mathbb{R}_+^3$ where G is a function depending only on $X^{'}$.

REMARK 3. In the sequel and without loss of generality we will assume that $G \equiv 0$.

By Lemma 3, we obtain

$$\frac{1}{\varepsilon} \int_Q \boldsymbol{m}^\varepsilon \times \partial_z \varphi^\varepsilon \mathbf{e_3} \cdot \boldsymbol{g}^\varepsilon \, \mathrm{d}X \mathrm{d}t \longrightarrow -\int_{Q^{'}} \boldsymbol{m} \times (\boldsymbol{m} \cdot \mathbf{e_3})\mathbf{e_3} \cdot \boldsymbol{g}_0 \, \mathrm{d}X^{'} \mathrm{d}t.$$

Finally we obtain

$$\partial_t \boldsymbol{m} - \alpha \boldsymbol{m} \times \partial_t \boldsymbol{m} \quad = \quad -\gamma \boldsymbol{m} \times \left(\widehat{\Delta}\boldsymbol{m} + a \, \widehat{\mathrm{curl}}\boldsymbol{m} \, \mathbf{e_3} + a \, L(\boldsymbol{m}) \right.$$
$$\left. + \widehat{\mathrm{grad}}\varphi - (\boldsymbol{m} \cdot \mathbf{e_3})\mathbf{e_3} \right),$$

in $(0,T) \times B$. Gathering all convergence results, we get

THEOREM 2. Let $(\boldsymbol{m}^\varepsilon, \varphi^\varepsilon)$ be a global weak solution of problem (13)-(15) associated with the admissible initial datum $(\boldsymbol{m}_0^\varepsilon, \varphi_0^\varepsilon)$. Let $(\boldsymbol{m}, \varphi)$ be the weak-$\star$ limit of a subsequence of $(\boldsymbol{m}^\varepsilon, \varphi^\varepsilon)$. Then $(\boldsymbol{m}, \varphi)$ is independent of the variable z and satisfies in $(0,T) \times B$, $|\boldsymbol{m}(t, X^{'})| = 1$ and the equation

$$\begin{cases} \partial_t \boldsymbol{m} - \alpha \boldsymbol{m} \times \partial_t \boldsymbol{m} = -\gamma \boldsymbol{m} \times \mathcal{H}^{fl}(\boldsymbol{m}), \\ \mathcal{H}^{fl}(\boldsymbol{m}) = \widehat{\Delta}\boldsymbol{m} + a \, \widehat{\mathrm{curl}} \, \boldsymbol{m} \, \mathbf{e_3} + a \, L(\boldsymbol{m}) + \widehat{\mathrm{grad}}\varphi - (\boldsymbol{m} \cdot \mathbf{e_3})\mathbf{e_3}, \end{cases} \tag{23}$$

and in $(0,T) \times \mathbb{R}_+^3$ the equation

$$\widehat{\Delta}\varphi + \chi(\Omega)\widehat{\mathrm{div}}\boldsymbol{m} = 0, \tag{24}$$

where

$$\begin{cases} L(\boldsymbol{m}) = \left(\partial_y m_3 - \Psi_2, \; \Psi_1 - \partial_x m_3, \; 0 \right), \\ \boldsymbol{\Psi} = \boldsymbol{F} \times \boldsymbol{m} + \alpha_0 \boldsymbol{m} \end{cases} \tag{25}$$

and $\boldsymbol{F}$ is a function of the variable $X^{'}$ and $\alpha_0 \in \mathbb{R}$. The associated initial and boundary conditions are given by

$$\boldsymbol{m}(0, X^{'}) = \boldsymbol{m}_0, \quad |\boldsymbol{m}_0| = 1 \quad \text{in} \; B, \tag{26}$$

$$\partial_\nu \boldsymbol{m} = 0 \quad \text{on} \; \partial B. \tag{27}$$

REMARK 4. Let $T > 0$ and assume that $\boldsymbol{F} \in L^2(0, T, \mathbb{L}^\infty(B))$. Then any regular solution of (23)-(27) satisfies the following energy estimate

$$\mathcal{E}(t) + \frac{\alpha}{2\gamma} \int_0^t \int_B |\partial_t \boldsymbol{m}|^2 \, \mathrm{d}X' \mathrm{d}s \leq \exp\left(\frac{16a^2\gamma}{\alpha} T\right) \left[\mathcal{E}(0) + \int_0^t \eta(s) \, \mathrm{d}s\right], \qquad (28)$$

for all $t \in (0, T)$ where

$$\mathcal{E}(t) = \frac{1}{2} \int_B |\widehat{\mathrm{grad}}\boldsymbol{m}|^2 \, \mathrm{d}X' + \frac{1}{2} \int_{\mathbb{R}^2} |\widehat{\mathrm{grad}}\varphi|^2 \, \mathrm{d}X',$$

and η is given by

$$\eta(t) = \frac{2\gamma}{\alpha} \left[4a^2(|\boldsymbol{F}|_\infty^2 + \alpha_0^2) + 1\right] \mathrm{meas}(B).$$

The energy estimate (28) allows to prove global existence of weak solutions for the reduced model (23).

2.2   Slender Domains

Note that we can proceed as above to get result for the 3D-1D dimensional reduction. In fact, we have the following theorem which we state without proof.

THEOREM 3.   Let $(\boldsymbol{m}^\varepsilon, \varphi^\varepsilon)$ be a weak solution of the problem associated with the admissible initial datum $(\boldsymbol{m}_0^\varepsilon, \varphi_0^\varepsilon)$. Then, one has $(\boldsymbol{m}^\varepsilon, \varphi^\varepsilon) \rightharpoonup (\boldsymbol{m}, \varphi)$ weakly-$\star$ in $L^\infty(0, T, \mathbb{H}^1(\Omega))$ and $\boldsymbol{m}^\varepsilon \to \boldsymbol{m}$ strongly in $L^2(0, T, \mathbb{L}^2(\Omega))$. The couple $(\boldsymbol{m}, \varphi)$ is independent of the variable X' and satisfies in $(0, T) \times (0, 1)$, $|\boldsymbol{m}(t, z)|^2 = 1$ and the following one-dimensional coupled system

$$\begin{cases} \partial_t \boldsymbol{m} - \alpha \boldsymbol{m} \times \partial_t \boldsymbol{m} = -\gamma \boldsymbol{m} \times \mathcal{H}_{\mathrm{eff}}^{sl}(\boldsymbol{m}), \\ \partial_{zz}\varphi + \chi(\Omega)\partial_z \boldsymbol{m} = 0, \end{cases}$$

where

$$\begin{cases} \mathcal{H}_{\mathrm{eff}}^{sl}(\boldsymbol{m}) = \partial_{zz}\boldsymbol{m} + a\,\widehat{\mathrm{curl}}\boldsymbol{m} + a\,L(\boldsymbol{m}) + \partial_z\varphi\,\mathbf{e_3}, \\ L(\boldsymbol{m}) = \left(d_3',\ -d_3,\ d_1 - d_1'\right). \end{cases}$$

The operator $\widehat{\mathrm{curl}}\boldsymbol{m}$ is defined by

$$\widehat{\mathrm{curl}}\boldsymbol{m} = \begin{pmatrix} -\partial_z m_2 \\ \partial_z m_1 \\ 0 \end{pmatrix},$$

and $d_i, d_i', i = 1, 2, 3$ are the components of the vectors

$$\boldsymbol{d} = \boldsymbol{m} \times \boldsymbol{F}(z) + c_1 \boldsymbol{m}, \quad \boldsymbol{d}' = \boldsymbol{m} \times \boldsymbol{g}(z) + c_2 \boldsymbol{m}, \quad (c_1, c_2) \in \mathbb{R}^2,$$

where $\boldsymbol{F}$ and $\boldsymbol{g}$ are arbitrary functions depending only on z. The associated initial and boundary conditions are given by

$$\boldsymbol{m}(0, z) = \boldsymbol{m}_0, \quad |\boldsymbol{m}_0| = 1 \ \text{ in } \ (0, 1),$$

$$\partial_z \boldsymbol{m}(t, j) = 0 \ \text{ for } \ j = 0, 1.$$

REMARK 5. Note that one can derive an energy estimate for the above limit problem allowing to prove global existence of weak solutions.

3 Concluding Remarks

In this paper, we have considered a mathematical model describing current-induced skyrmion dynamics in ferromagnets. We derived thin layer models both for flat and slender domains. The dimensional reductions are performed by using a scaling technique combined with the method of oscillating test functions. The obtained limit problems may be used in numerical simulations and thus avoiding high cost of direct numerical simulations.

A direction for future research concerns magnetic domain walls (DWs) which are boundaries in magnetic materials that divide regions with distinct magnetization directions. The manipulation and control of DWs in ferromagnetic nanowires (essentially one dimensional models) has recently become a subject of intense experimental and theoretical research, see, for example, Carbou and Labbé [9]. The rapidly growing interest in the physics of the DW motion can be mainly explained by a promising possibility of using DWs as the basis for next-generation memory and logic devices. However, in order to realize such devices in practice it is essential to be able to position individual DWs precisely along magnetic nanowires. It would be interesting to address within the context of the present paper the stability of the propagation of such processing DWs with respect to perturbations of the initial magnetization profile, some anisotropy properties of the nanowire, and applied magnetic field.

Finally, a valuable direction for future research is the effect of very small domain irregularities on the limiting problems. More precisely, the roughness may be defined by means of a periodical function h^ε with period for example of order ε^2 ($\varepsilon > 0$). So that the three-dimensional domain may be represented by $\Omega^\varepsilon = \{(x, z) \in \mathbb{R}^2 \times \mathbb{R} : x \in \omega, 0 < z < h^\varepsilon(x)\}$ where ω is a domain of $\mathbb{R}^2$. Various limit models may be obtained depending on the ratio between the size of rugosities and the mean height of the domain.

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