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Coordinated Convex Functions And Inequalities*

Muhammad Aslam Noor†, Khalida Inayat Noor‡, Sabah Iftikhar§
Saima Rashid¶, Muhammad Uzair Awan‖

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Abstract

In this paper, we obtain some new Hermite-Hadamard type integral inequalities for coordinated convex functions. Several integral inequalities can be obtained by taking n to be an even number in our main results. These results can be viewed as a significant refinement of the previous work. The ideas and techniques of this paper may stimulate further research in this dynamic field.

1 Introduction and Preliminaries

Convexity through its numerous applications in different fields of pure and applied sciences has attracted many researchers. These facts have motivated to extend and generalize the classical convex functions in different directions using novel and innovative techniques, see [2, 3, 9, 23]. We now recall the concepts of convex sets and functions.

DEFINITION 1. A set $\mathcal{I} \subset \mathbb{R}$ is said to be a convex set if

$$(1-t)x + ty \in \mathcal{I}, \quad \forall x, y \in \mathcal{I},\ t \in [0,1].$$

DEFINITION 2. A function $f : \mathcal{I} \to \mathbb{R}$ is said to be a convex function if

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y), \quad \forall x, y \in \mathcal{I},\ t \in [0,1].$$

If $t = \frac{1}{2}$, then the convex function f satisfies the inequality

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}, \quad \forall x, y \in \mathcal{I},\ t \in [0,1].$$

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†Department of Mathematics, COMSATS University Islamabad, Park Road, Islamabad, Pakistan
‡Department of Mathematics, COMSATS University Islamabad, Park Road, Islamabad, Pakistan
§Department of Mathematics, COMSATS University Islamabad, Park Road, Islamabad, Pakistan
¶Department of Mathematics, COMSATS University Islamabad, Park Road, Islamabad, Pakistan
‖GC University, Faisalabad, Pakistan

and is called the Jensen convex function.

It is well known that $u \in \mathcal{I}$ is a minimum of the differentiable convex function, if and only if, $u \in \mathcal{I}$ satisfies the inequality

$$\langle f'(u), v-u \rangle \geq 0, \quad \forall v \in \mathcal{I},$$

which is called the variational inequality, introduced and studied by Stampacchia [25] in 1964. For the formulation, applications, numerical solution, stability and other aspects of variational inequalities, see [11, 12, 13, 14, 25] and the references therein.

An other interesting and fascinating aspect of theory of convexity is its close relationship with theory of integral inequalities. Many famous inequalities known in the literature are direct consequences of the applications of classical convex functions. In this regard Hermite-Hadamard's integral inequality is one of the most intensively studied inequality. This inequality was obtained by Hermite and Hadamard independently. It provides us an equivalent condition for convexity. Also it give the upper and lower bounds for the integrals. This famous result of Hermite and Hadamard reads as follows:

Let $f : \mathcal{I} = [a, b] \subset \mathbb{R} \subset \mathbb{R} \to \mathbb{R}$ be an integrable convex function, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}.$$

This integral inequality has remained an area of great interest due to its great utility in different fields of pure and applied sciences particulary in statistics and numerical analysis. In recent years many, new generalizations of Hermite-Hadamard's inequality have been established. Dragomir [7] extended the class of convexity on coordinates. The class of coordinated convex functions is defined as:

Let us consider a bidimensional interval $\Delta = [a, b] \times [c, d] \subset \mathbb{R}^2$ with $a < b$ and $c < d$. A function $f : \Delta \to \mathbb{R}$ is said to be convex function on coordinated Δ, if

$$f(tx+(1-t)z, ty+(1-t)w) \leq tf(x, y)+(1-t)f(z, w), \quad \forall (x, y), (z, w) \in \Delta, t \in [0, 1].$$

A function $f : \Delta \to \mathbb{R}$ is said to be convex on coordinates Δ, if $f_y : [a, b] \to \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}$, $f_x(v) = f(x, v)$ are convex $\forall x \in [a, b]$ and $\forall y \in [c, d]$.

This opened a new venue of research for the researchers working in this field. Dragomir [7] obtained Hermite-Hadamard's integral inequality for the coordinated convex functions.

THEOREM 1. Let $f : \Delta = [a, b] \times [c, d] \subset \mathbb{R}^2 \to \mathbb{R}$ be a co-ordinated convex function on the rectangle Δ. Then

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)\mathrm{d}x + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)\mathrm{d}y\right] \\ &\leq \frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f(x, y)\mathrm{d}y\mathrm{d}x \\ &\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b [f(x, c)+f(x, d)]\mathrm{d}x + \frac{1}{d-c}\int_c^d [f(a, y)+f(b, y)]\mathrm{d}y\right] \\ &\leq \frac{f(a, c)+f(a, d)+f(b, c)+f(b, d)}{4}. \end{aligned}$$

For recent developments and other aspects of integral inequalities for coordinated convex functions, see [1, 4, 5, 6, 8, 10, 15, 16, 17, 18, 19, 20, 21, 22, 24] and the references therein.

In this paper, we prove an auxiliary result, which can be viewed as a significant refinement of composite Simpson's rule. Using this auxiliary result, we derive some new Hermite-Hadamard type integral inequalities via coordinated convex functions. Special cases are also discussed. This is the main motivation of this paper. The ideas and techniques of this paper may be a starting point for further research in this fascinating and dynamic field.

2 Main Results

In this section, we discuss our main results. First of all, we recall the following auxiliary results[13], which plays a crucial part in obtaining our main results.

LEMMA 1. Let $f : \mathcal{I} = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function on $\mathcal{I}$. Then

$$h\sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right) \le \int_a^b f(t)\mathrm{d}t \le \frac{h}{3}\left[f(a) + 4\sum_{k=1}^{\frac{n}{2}} f(x_{2k-1}) + 2\sum_{k=1}^{\frac{n-2}{2}} f(x_{2k}) + f(b)\right]$$

holds, where n is even, $h = \frac{b-a}{n}$, and $x_k = a + kh$, for each $k = 0, 1, ..., n$.

LEMMA 2. Let $f : \mathcal{I} = [a, b] \subset \mathbb{R} \to \mathbb{R}_+$ be a convex function on $\mathcal{I}$. Then

$$\int_a^b f(x)\mathrm{d}x - (b-a)f(t) \le \frac{h}{3}\left[f(a) + 4\sum_{k=1}^{\frac{n}{2}} f(x_{2k-1}) + 2\sum_{k=1}^{\frac{n-2}{2}} f(x_{2k}) + f(b)\right],$$

holds for all $t \in [a, b]$, where n is even, $h = \frac{b-a}{n}$, and $x_k = a+kh$, for each $k = 0, 1, ..., n$.

THEOREM 2. Let $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ be a coordinated convex function. Then

$$\begin{aligned}
& \frac{d-c}{2n}\sum_{k=1}^{n}\int_a^b f\left(x, \frac{y_{k-1}+y_k}{2}\right)\mathrm{d}x + \frac{b-a}{2n}\sum_{k=1}^{n}\int_c^d f\left(\frac{x_{k-1}+x_k}{2}, y\right)\mathrm{d}y \\
\le\ & \int_a^b\int_c^d f(x,y)\mathrm{d}y\mathrm{d}x \\
\le\ & \frac{d-c}{6n}\int_a^b [f(x,c)+f(x,d)]\mathrm{d}x + \frac{b-a}{6n}\int_c^d [f(a,y)+f(b,y)]\mathrm{d}y \\
& + \frac{2(d-c)}{3n}\sum_{k=1}^{\frac{n}{2}}\int_a^b f(x, y_{2k-1})\mathrm{d}x + \frac{2(b-a)}{3n}\sum_{k=1}^{\frac{n}{2}}\int_c^d f(x_{2k-1}, y)\mathrm{d}y \\
& + \frac{d-c}{3n}\sum_{k=1}^{\frac{n-2}{2}}\int_a^b f(x, y_{2k})\mathrm{d}x + \frac{b-a}{3n}\sum_{k=1}^{\frac{n-2}{2}}\int_c^d f(x_{2k}, y)\mathrm{d}y,
\end{aligned}$$

where n is even, $h = \frac{b-a}{n}$, and $x_k = a + kh$, for each $k = 0, 1, ..., n$.

PROOF. From the assumption, we have that $f_x : [c, d] \to \mathbb{R}$ defined by $f_x(y) = f(x, y)$ is a convex function on $[c, d], \forall x \in [a, b]$. Applying Lemma 2 to the function f_x, we have

$$\begin{aligned} \frac{d-c}{n}\sum_{k=1}^{n} f_x\left(\frac{y_{k-1}+y_k}{2}\right) &\leq \int_c^d f_x(y)\mathrm{d}y \\ &\leq \frac{d-c}{3n}\left[f_x(c) + 4\sum_{k=1}^{\frac{n}{2}} f_x(y_{2k-1}) + 2\sum_{k=1}^{\frac{n-2}{2}} f_x(y_{2k}) + f_x(d)\right]. \end{aligned}$$

This implies that

$$\begin{aligned} &\frac{d-c}{n}\sum_{k=1}^{n} f\left(x, \frac{y_{k-1}+y_k}{2}\right) \\ \leq\ &\int_c^d f(x, y)\mathrm{d}y \\ \leq\ &\frac{d-c}{3n}\left[f(x, c) + 4\sum_{k=1}^{\frac{n}{2}} f(x, y_{2k-1}) + 2\sum_{k=1}^{\frac{n-2}{2}} f(x, y_{2k}) + f(x, d)\right]. \end{aligned} \tag{2.1}$$

Integrating the above inequality over the interval $[a, b]$, we have

$$\begin{aligned} &\frac{d-c}{n}\sum_{k=1}^{n}\int_a^b f\left(x, \frac{y_{k-1}+y_k}{2}\right)\mathrm{d}x \\ \leq\ &\int_a^b\int_c^d f(x, y)\mathrm{d}y\mathrm{d}x \\ \leq\ &\frac{d-c}{3n}\left[\int_a^b f(x, c)\mathrm{d}x + 4\sum_{k=1}^{\frac{n}{2}}\int_a^b f(x, y_{2k-1})\mathrm{d}x\right. \\ &\left. +2\sum_{k=1}^{\frac{n-2}{2}}\int_a^b f(x, y_{2k})\mathrm{d}x + \int_a^b f(x, d)\mathrm{d}x\right]. \end{aligned} \tag{2.2}$$

Similarly, for the mapping $f_y : [a, b] \to \mathbb{R}$, defined by $f_y(x) = (x, y)$ for all $y \in [c, d]$, we have

$$\begin{aligned} &\frac{b-a}{n}\sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}, y\right) \\ \leq\ &\int_a^b f(x, y)\mathrm{d}x \\ \leq\ &\frac{b-a}{3n}\left[f(a, y) + 4\sum_{k=1}^{\frac{n}{2}} f(x_{2k-1}, y) + 2\sum_{k=1}^{\frac{n-2}{2}} f(x_{2k}, y) + f(b, y)\right]. \end{aligned} \tag{2.3}$$

Integrating the above inequality over the interval $[c,d]$, we have

$$\begin{aligned}\frac{b-a}{n}\sum_{k=1}^{n}\int_c^d f\left(\frac{x_{k-1}+x_k}{2},y\right)\mathrm{d}y &\le \int_a^b\int_c^d f(x,y)\mathrm{d}y\mathrm{d}x\\ &\le \frac{b-a}{3n}\Bigg[\int_c^d f(a,y)\mathrm{d}y+4\sum_{k=1}^{\frac{n}{2}}\int_c^d f(x_{2k-1},y)\mathrm{d}y\\ &\quad +2\sum_{k=1}^{\frac{n-2}{2}}\int_c^d f(x_{2k},y)\mathrm{d}y+\int_c^d f(b,y)\mathrm{d}y\Bigg]. \quad (2.4)\end{aligned}$$

Adding (2.2) and (2.4), we obtain

$$\begin{aligned}&\frac{d-c}{2n}\sum_{k=1}^{n}\int_a^b f\left(x,\frac{y_{k-1}+y_k}{2}\right)\mathrm{d}x+\frac{b-a}{2n}\sum_{k=1}^{n}\int_c^d f\left(\frac{x_{k-1}+x_k}{2},y\right)\mathrm{d}y\\ \le\ &\int_a^b\int_c^d f(x,y)\mathrm{d}y\mathrm{d}x\\ \le\ &\frac{d-c}{6n}\int_a^b[f(x,c)+f(x,d)]\mathrm{d}x+\frac{b-a}{6n}\int_c^d[f(a,y)+f(b,y)]\mathrm{d}y\\ &+\frac{2(d-c)}{3n}\sum_{k=1}^{\frac{n}{2}}\int_a^b f(x,y_{2k-1})\mathrm{d}x+\frac{2(b-a)}{3n}\sum_{k=1}^{\frac{n}{2}}\int_c^d f(x_{2k-1},y)\mathrm{d}y\\ &+\frac{d-c}{3n}\sum_{k=1}^{\frac{n-2}{2}}\int_a^b f(x,y_{2k})\mathrm{d}x+\frac{b-a}{3n}\sum_{k=1}^{\frac{n-2}{2}}\int_c^d f(x_{2k},y)\mathrm{d}y.\end{aligned}$$

This completes the proof.

COROLLARY 1. Let $f:\Delta=[0,1]\times[0,1]\to\mathbb{R}$ be a coordinated convex function. Then

$$\begin{aligned}&\frac{1}{4}\int_0^1\left[f\left(x,\frac{1}{4}\right)+f\left(x,\frac{3}{4}\right)\right]\mathrm{d}x+\frac{1}{4}\int_0^1\left[f\left(\frac{1}{4},y\right)+f\left(\frac{3}{4},y\right)\right]\mathrm{d}x\\ \le\ &\int_0^1\int_0^1 f(x,y)\mathrm{d}y\mathrm{d}x\\ \le\ &\frac{1}{12}\int_0^1\Big[f(x,0)+f(x,1)\Big]\mathrm{d}x+\frac{1}{12}\int_0^1\Big[f(0,y)+f(1,y)\Big]\mathrm{d}y\\ &+\frac{1}{3}\int_0^1 f\left(x,\frac{1}{2}\right)\mathrm{d}x+\frac{1}{3}\int_0^1 f\left(\frac{1}{2},y\right)\mathrm{d}y.\end{aligned}$$

PROOF. The desired inequalities are obtained by taking $n=2$ in Theorem 2 and observing that $x_0=y_0=0$, $x_1=y_1=\frac{1}{2}$ and $x_2=y_2=1$.

THEOREM 3. Under the assumptions of Theorem 2, we obtain

$$\begin{aligned}
&\frac{3n}{2(b-a)}\int_a^b[f(x,c)+f(x,d)]\mathrm{d}x+\frac{3n}{2(d-c)}\int_c^d[f(a,y)+f(b,y)]\mathrm{d}y\\
\le\ & f(a,c)+f(a,d)+f(b,c)+f(b,d)\\
&+2\sum_{k=1}^{\frac{n}{2}}[f(x_{2k-1},c)+f(x_{2k-1},d)+f(a,y_{2k-1})+f(b,y_{2k-1})]\\
&+\sum_{k=1}^{\frac{n-2}{2}}[f(x_{2k},c)+f(x_{2k},d)+f(a,y_{2k})+f(b,y_{2k})\Big],
\end{aligned}$$

where n is even, $h=\frac{b-a}{n}$ and $x_k=a+kh$, for each $k=0,1,...,n$.

PROOF. Now using the second part of (2.2) and (2.4), we have

$$\int_a^b f(x,c)\mathrm{d}x\le\frac{b-a}{3n}\Bigg[f(a,c)+4\sum_{k=1}^{\frac{n}{2}}f(x_{2k-1},c)+2\sum_{k=1}^{\frac{n-2}{2}}f(x_{2k},c)+f(b,c)\Bigg],$$

$$\int_a^b f(x,d)\mathrm{d}x\le\frac{b-a}{3n}\Bigg[f(a,d)+4\sum_{k=1}^{\frac{n}{2}}f(x_{2k-1},d)+2\sum_{k=1}^{\frac{n-2}{2}}f(x_{2k},d)+f(b,d)\Bigg],$$

$$\int_c^d f(a,y)\mathrm{d}y\le\frac{d-c}{3n}\Bigg[f(a,c)+4\sum_{k=1}^{\frac{n}{2}}f(a,y_{2k-1})+2\sum_{k=1}^{\frac{n-2}{2}}f(a,y_{2k})+f(a,d)\Bigg],$$

and

$$\int_c^d f(b,y)\mathrm{d}y\le\frac{d-c}{3n}\Bigg[f(b,c)+4\sum_{k=1}^{\frac{n}{2}}f(b,y_{2k-1})+2\sum_{k=1}^{\frac{n-2}{2}}f(b,y_{2k})+f(b,d).$$

By adding the above inequalities, we obtain the required result.

THEOREM 4. Let $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ be a coordinated convex function. Then

$$\begin{aligned}
\int_a^b\int_c^d f(x,y)\mathrm{d}y\mathrm{d}x\ \le\ & \frac{b-a}{12n}\Bigg[(2+3n)\int_c^d f(a,y)\mathrm{d}y+(2+3n)\int_c^d f(b,y)\mathrm{d}y\\
&+8\sum_{k=1}^{\frac{n}{2}}\int_c^d f(x_{2k-1},y)\mathrm{d}y+4\sum_{k=1}^{\frac{n-2}{2}}\int_c^d f(x_{2k},y)\Bigg]\mathrm{d}y\\
&+\frac{d-c}{12n}\Bigg[(2+3n)\int_a^b f(x,c)\mathrm{d}x+(2+3n)\int_a^b f(x,d)\mathrm{d}x\\
&+8\sum_{k=1}^{\frac{n}{2}}\int_a^b f(x,y_{2k-1})\mathrm{d}x+4\sum_{k=1}^{\frac{n-2}{2}}\int_a^b f(x,y_{2k})\mathrm{d}x\Bigg],
\end{aligned}$$

where n is even, $h=\frac{b-a}{n}$ and $x_k=a+kh$, for each $k=0,1,...,n$.

PROOF. Applying the inequality of Lemma 2 to the function $f_y:[a,b]\to\mathbb{R}$ defined as $f_y(x)=f(x,y)$ at $t=b$,

$$\int_a^b f_y(x)\mathrm{d}x-(b-a)f_y(b)\le\frac{h}{3}\left[f_y(a)+4\sum_{k=1}^{\frac{n}{2}}f_y(x_{2k-1})+2\sum_{k=1}^{\frac{n-2}{2}}f_y(x_{2k})+f_y(b)\right].$$

This implies

$$\begin{aligned}&\int_a^b f(x,y)\mathrm{d}x-(b-a)f(b,y)\\ \le\quad&\frac{b-a}{3n}\left[f(a,y)+4\sum_{k=1}^{\frac{n}{2}}f(x_{2k-1},y)+2\sum_{k=1}^{\frac{n-2}{2}}f(x_{2k},y)+f(b,y)\right].\end{aligned}$$

Integrating the above inequality over $[c,d]$, we have

$$\begin{aligned}\int_a^b\int_c^d f(x,y)\mathrm{d}y\mathrm{d}x\quad\le\quad&\frac{b-a}{3n}\left[\int_c^d f(a,y)\mathrm{d}y+(1+3n)\int_c^d f(b,y)\mathrm{d}y\right.\\&\left.+4\sum_{k=1}^{\frac{n}{2}}\int_c^d f(x_{2k-1},y)\mathrm{d}y+2\sum_{k=1}^{\frac{n-2}{2}}\int_c^d f(x_{2k},y)\mathrm{d}y\right].\end{aligned}$$

Applying again Lemma 2 to the mapping f_y at $t=a$ and integrating over $[c,d]$, we have

$$\begin{aligned}\int_a^b\int_c^d f(x,y)\mathrm{d}y\mathrm{d}x\quad\le\quad&\frac{b-a}{3n}\left[(1+3n)\int_c^d f(a,y)\mathrm{d}y+\int_c^d f(b,y)\mathrm{d}y\right.\\&\left.+4\sum_{k=1}^{\frac{n}{2}}\int_c^d f(x_{2k-1},y)\mathrm{d}y+2\sum_{k=1}^{\frac{n-2}{2}}\int_c^d f(x_{2k},y)\mathrm{d}y\right].\end{aligned}$$

Adding the above inequalities, we have

$$\begin{aligned}\int_a^b\int_c^d f(x,y)\mathrm{d}y\mathrm{d}x\quad\le\quad&\frac{b-a}{6n}\left[(2+3n)\int_c^d f(a,y)\mathrm{d}y+(2+3n)\int_c^d f(b,y)\mathrm{d}y\right.\\&\left.+8\sum_{k=1}^{\frac{n}{2}}\int_c^d f(x_{2k-1},y)\mathrm{d}y+4\sum_{k=1}^{\frac{n-2}{2}}\int_c^d f(x_{2k},y)\mathrm{d}y\right].\quad(2.5)\end{aligned}$$

Similarly, for the mapping $f_x:[c,d]\to\mathbb{R}$, defined as $f_x(y)=f(x,y)$ at $t=c$ and $t=d$ and then integrating over $[a,b]$, we have

$$\begin{aligned}\int_a^b\int_c^d f(x,y)\mathrm{d}y\mathrm{d}x\quad\le\quad&\frac{d-c}{6n}\left[(2+3n)\int_a^b f(x,c)\mathrm{d}x+(2+3n)\int_a^b f(x,d)\mathrm{d}x\right.\\&\left.+8\sum_{k=1}^{\frac{n}{2}}\int_a^b f(x,y_{2k-1})\mathrm{d}x+4\sum_{k=1}^{\frac{n-2}{2}}\int_a^b f(x,y_{2k})\mathrm{d}x\right].\quad(2.6)\end{aligned}$$

By adding (2.6) and (2.5), we obtain the required result.

For $n = 2$ in Theorem 2, we have

COROLLARY 2. Let $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ be a coordinated convex function. Then

$$\int_a^b \int_c^d f(x,y)\mathrm{d}y\mathrm{d}x \leq \frac{b-a}{3}\left[\int_c^d f(a,y)\mathrm{d}y + \int_c^d f(b,y)\mathrm{d}y + \int_c^d f\left(\frac{a+b}{2}, y\right)\mathrm{d}y\right] + \frac{d-c}{3}\left[\int_a^b f(x,c)\mathrm{d}x + \int_a^b f(x,d)\mathrm{d}x + \int_a^b f\left(x, \frac{c+d}{2}\mathrm{d}x\right)\right].$$

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References

[1] A. O. Akdemir and M. E. Ozdemir, Some Hadamard-type inequalities for coordinated P-convex functions and Godunova-Levin functions, American Inst. Physics(AIP) Conference Proced.1309, Kasim-2010, 7–15.

[2] M. Alomari and M. Darus, The Hadamard's inequality for s-convex function of 2-variables on the co-ordinates, Int. J. Math. Anal., 2(2008), 629–638.

[3] M. Alomari and M. Darus, On the Hadamard's inequality for log-convex function on the co-ordinates, J. Inequal. Appl., (2009), Art. ID 283147, 13.

[4] M. K. Bakulla, An improvement of the Hermite-Hadamard iequality for functions convex on the coordinates, Austr. J. Math. Anal. Appl., 11(2014), Art. 3, 17.

[5] M. K. Bakula and J. Pecaric, On the Jensen's inequality for convex functions on the co-ordinates in a rectangle from the plane, Taiwanese J. Math., 5(2006), 127–1292.

[6] F. Chen, A note on the Hermite-hadamard inequality for convex functions on the co-ordinates, J. Math. Inequal., 8(4)(2014), 915-923.

[7] S. S. Dragomir, On the Hadamard's inequality for convex function on the co-ordinates in a rectangle from the plane, Taiwansese J. Math., 5(2001), 775–788.

[8] D. Y. Hwang, K. L. Tseng and G. S. Yang, Some Hadamard's inequalities for co-ordinatet]d convex functions in a rectangle from the plane, Taiwanese J. Math., 11(2007), 63–73.

[9] M. A. Latif and M. Alomari, On Hadamard's type inequalities for h-convex functions on the co-ordinates, Int. J. Math. Anal., 3(2009), 1645–1656.

[10] M. A. Latif and S. S. Dragomir, On some new inequalities for differentiable co-ordinated convex functions, J. Inequal. Appl., 2012, 28(2012).

[11] M. A. Noor, New approximation schemes for general variational inequalities, J. Math. Anal. Appl., 251(2000), 217–230.

[12] M. A. Noor, Some developments in general variational inequalites, Appl. Math. Comput., 152(2004), 199–277.

[13] M. A. Noor, Advanced Convex Analysis and Optimization, Lecture Notes, Mathematics Department, COMSATS University Islamabad, Islamabad, Pakistan, (2008–2018).

[14] M. A. Noor, K. I. Noor and T. M. Rassias, Some aspects of variational inequalities, J. Comput. Appl. Math., 47(1993), 285–312.

[15] M. A. Noor, K. I. Noor and M. U. Awan, Integral inequalities for coordinated harmonic harmonically convex functions, Complex Var. Ellipt. Eq., 60(20015),776–786.

[16] M. A. Noor, K. I. Noor, S. Iftikhar and C. Ionescu, Hermite-Hadamard inequalities for coordinated harmonic convex functions, U.P.B. Sci. Bull., Series A, 7(2017), 25–34.

[17] E. R. Nwaeze, Generalized Hermite-Hadamard's inequality for functions convex on the coordinates, Appl. Math. E-Notes, 18(2018), 275–283.

[18] M. E. Ozdemir, A. O. Akdemir and H. Kavurmaci, On the Simson's inequalitiy for co-ordinated convex functions, Turkish J. Anal. Number Theory, 2(2014),165–169.

[19] M. E. Ozdemir, A. O. Akdemir and C. Yildiz, On co-ordinated quasi-convex functions, Czechoslovak Math. J., 62(2012), 889-900.

[20] M. E. Ozdemir, M. A. Latif and A. O. Akdmir,On some Hadamard-type inequalities for product of two s-convex functions on the co-ordinates, J. Inequal. Appl., 2012(2012),Article ID: 21.

[21] M. E. Ozdemir, E. Set and M. Z. Sarikaya, Some new Hadamard-type inequalites for co-ordinated m-convex and (α, m)-convex functions, Hacettepe J. Math. Stat., 40(2011), 219–229.

[22] M. E. Ozdemir, H. Kavurmaci, A. O. Akdemir and M. Avic, Inequalities for convex and s-convex functions on $\Delta = [a, b] \times [c, d]$, J. Inequal. Appl., 2012, 20(2012).

[23] M. E. Ozdemir, X. Yildiz and A. O. Akdemir, On some new Hadamard-Type-inequalities for co-ordinated quasi-convex functions, Hacettepe. J. Math. Stat., 41(2012), 697–707.

[24] M. Z. Sarikaya, E. Set, M. E. Ozdemir and S. S. Dragomir, New some Hadamard's type inequalities for co-ordinated convex functions, Tamsui Oxford J.Inform. Math. Sci., 28(2012), 137–152.

[25] G. Stampacchia, Formes bilineaires coercivities sur les ensebles convexes, Comptes Rendus del Academie des sciences, Paris, 258(1964), 4413–4416.