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Stability Of Wave Equation With A Tip Mass Under Unknown Boundary External Disturbance*

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Abstract

This work studies the stabilization problem of a wave equation with a tip mass, taking into account of Fourier heat conduction, which undergoes unknown bounded disturbance at tip mass. Here the nonlinear feedback control law is used to cancel the effect of external disturbances. We have proved the well-posedness of the close-loop system by the maximal monotone operator theory and the variational principle. Further, we have established the exponential stability of the system by construction of a suitable Lyapunov functional.

1 Introduction

In the past decades, due to extensive applications of the wave equation in engineering and mathematical control theory, the stabilization of wave equation have received great attention. The boundary control due to its easily actual operation in engineering, is widely used as the major control strategy for the dynamic system that is governed by the partial differential equations. In order to stabilize wave equation, various control techniques have been developed for instance, distributed control [15], boundary control [6], sliding mode control (SMC) and active disturbance rejection control (ADRC) [4], [17]. However, a small disturbance can make these controller invalid such as a time delay or an external disturbance. Guo and Jin in [4] used sliding mode control (SMC) and the active disturbance rejection control (ADRC) to deal with a one-dimensional anti-stable wave equation subject to a boundary disturbance. Guo and Zhou extended this method to multi-dimension wave equation, see [5]. Guo and Jin in [7] discussed the boundary output feedback for 1-d wave equation with boundary disturbance. For similar results, we can refer to [2], [3], [6], [10], [11] and [12]. It is observed that the disturbances are uniformly bounded in all papers mentioned above.

In this paper, we consider the stabilization problem of the following wave equation with a tip mass which undergoes the disturbance at the tip mass. The dynamic behavior

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attached with Fourier heat conduction is thus governed by the following system of PDE:

$$\begin{cases} \left.\begin{array}{l} y_{tt} - y_{xx} - \kappa\theta_x = 0 \\ \theta_t - \theta_{xx} - \kappa y_{xt} = 0 \end{array}\right\}, & x \in (0,1),\ t > 0, \\ y_x(1,t) + m y_{tt}(1,t) = u(t) + d(t), & t > 0, \\ y(0,t) = 0,\ \theta(1,t) = 0 = \theta(0,t), & t > 0, \\ y(x,0) = y_0(x),\ y_t(x,0) = y_1(x),\ \theta(x,0) = \theta_0(x), & x \in (0,1), \end{cases} \tag{1}$$

where the symbols y_t denotes the derivative of y with respect to time variable t and y_x the derivative of y with respect to spacial variable x, the coefficient $\kappa > 0$ is a small constant satisfying $\kappa < 1$, $m > 0$ is the tip mass, $u(t)$ is the boundary control force and $d(t)$ is the unknown external disturbance satisfying $|d(t)| \leq M$, $M > 0$.

It is well known that if there is no disturbance, that is, $d(t) \equiv 0$, the system (1) can be stabilized exponentially (cf. [14]) under the feedback control law:

$$u(t) = -\alpha y_{xt}(1,t) - \beta y_t(1,t),$$

where $\alpha, \beta > 0$ are suitable reals.

In the case of $d(t) \neq 0$ identically but uniformly bounded, Ge and his co-authors in [8, 9] established the stabilization by using an adaptive boundary control technique. They proved the closed-loop system is ultimately bounded by means of observation of the effect of the external disturbances taking into account of the system parametric uncertainties and disturbances. In this paper, we will design nonlinear and non-smooth feedback control law for (1) without Fourier heat conduction such that the system is stabilized exponentially. It is well known that the solvability and stability analysis of a system with discontinuous nonlinear term are often more difficult than that of a linear system. In fact, the nonlinear feedback control law will make our work more difficult. In this work, we use the semigroup theory [16] to prove the well-posedness of the closed-loop system and the Lyapunov functional method to establish the exponential stability of the system.

The rest is organized as follows. In Section 2, we design the nonlinear feedback controller which is derived from a Lyapunov functional. In Section 3, we study the well-posedness of the resulting closed-loop system via the theory of nonlinear maximal monotone operators and the variational principle. In Section 4, we finally prove that the closed-loop system is exponentially stable by the Lyapunov functional approach.

2 Boundary Feedback Controller

In this section, we design a boundary feedback controller based on a Lyapunov functional that relates to the total energy of the system. For simplicity, in the sequel, we always use the symbols y_t (or $\dot{y}$) to denote the derivative of y with respect to time variable t and y_x the derivative of y with respect to spacial variable x.

We introduce an auxiliary function $\eta(t)$ as

$$\eta(t) = y_x(1,t) + \alpha y_t(1,t), \qquad \alpha > 0. \tag{2}$$

It follows from the third equation of (1) that

$$m\dot{\eta}(t) = my_{xt}(1,t) + \alpha[-y_x(1,t) + u(t) + d(t)].$$

Then, the system (1) turns into

$$\begin{cases} \left.\begin{array}{c} y_{tt} - y_{xx} - \kappa\theta_x = 0 \\ \theta_t - \theta_{xx} - \kappa y_{xt} = 0 \end{array}\right\}, & x \in (0,1),\ t > 0, \\ m\dot{\eta}(t) = my_{xt}(1,t) + \alpha(-y_x(1,t) + u(t) + d(t)), & t > 0, \\ \eta(t) = y_x(1,t) + \alpha y_t(1,t), & \\ y(0,t) = 0,\ \theta(1,t) = 0 = \theta(0,t), & t > 0, \\ y(x,0) = y_0(x),\ y_t(x,0) = y_1(x),\ \theta(x,0) = \theta_0(x), & x \in (0,1), \end{cases} \tag{3}$$

The Lyapunov functional for (3) is taken as follows:

$$\mathbb{E} = \frac{1}{2}\int_0^1 [y_t^2 + y_x^2 + \theta^2]dx + \frac{m}{2}\eta^2(t).$$

Thus, we have

$$\begin{aligned} \dot{\mathbb{E}}(t) &= y_x(1,t)y_t(1,t) + m\eta(t)\dot{\eta}(t) - \int_0^1 \theta_x^2\, dx \\ &= \frac{1}{2\alpha}\left[y_x^2(1,t) + \alpha^2 y_t^2(1,t) + 2\alpha y_x(1,t)y_t(1,t)\right] - \frac{\alpha}{2}y_t^2(1,t) - \frac{1}{2\alpha}y_x^2(1,t) \\ &\quad + m\eta(t)\dot{\eta}(t) - \int_0^1 \theta_x^2\, dx \\ &= \frac{1}{2\alpha}\eta^2(t) - \frac{\alpha}{2}y_t^2(1,t) - \frac{1}{2\alpha}y_x^2(1,t) \\ &\quad + \eta(t)\Big[my_{xt}(1,t) + \alpha[-y_x(1,t) + u(t) + d(t)]\Big] - \int_0^1 \theta_x^2\, dx. \end{aligned} \tag{4}$$

Now, we design the feedback control as

$$u(t) = -\alpha y_t(1,t) - \beta y_{xt}(1,t) - M\mathrm{sgn}(\eta(t)), \tag{5}$$

where $\alpha, \beta > 0$ satisfying $\alpha\beta = m$, and the symbol sgn is a multi-valued function defined by

$$\mathrm{sgn}(x) = \begin{cases} 1, & x > 0, \\ (-1,1), & x = 0, \\ -1, & x < 0. \end{cases} \tag{6}$$

Under the feedback control (5), the expression (4) becomes

$$\begin{aligned} \dot{\mathbb{E}}(t) &= \frac{1}{2\alpha}\eta^2(t) - \frac{\alpha}{2}y_t^2(1,t) - \frac{1}{2\alpha}y_x^2(1,t) \\ &\quad + \eta(t)\left[my_{xt}(1,t) + \alpha(-y_x(1,t) + u(t) + d(t))\right] - \int_0^1 \theta_x^2\, dx \\ &= \frac{1}{2\alpha}\eta^2(t) - \frac{\alpha}{2}y_t^2(1,t) - \frac{1}{2\alpha}y_x^2(1,t) - \alpha\eta^2(t) - \alpha M\eta(t)\mathrm{sgn}(\eta(t)) \end{aligned}$$

$$
\begin{aligned}
& +\alpha\eta(t)d(t) - \int_0^1 \theta_x^2\,dx \\
= \quad & -\left(\alpha - \frac{1}{2\alpha}\right)\eta^2(t) - \frac{\alpha}{2}y_t^2(1,t) - \frac{1}{2\alpha}y_x^2(1,t) - \alpha M|\eta(t)| \\
& +\alpha\eta(t)d(t) - \int_0^1 \theta_x^2\,dx.
\end{aligned}
$$

In view of (2), (5) and (6), we have

$$\beta\dot{\eta}(t) = -\eta(t) - M\mathrm{sgn}(\eta(t)) + d(t). \tag{7}$$

Under the inclusion of the differential equation (7), the closed-loop system is equivalent to the following

$$
\begin{cases}
\left.\begin{array}{l} y_{tt} - y_{xx} - \kappa\theta_x = 0 \\ \theta_t - \theta_{xx} - \kappa y_{xt} = 0 \end{array}\right\}, & x \in (0,1),\, t > 0, \\
\beta\dot{\eta}(t) = -\eta(t) - M\mathrm{sgn}(\eta(t)) + d(t) & t > 0, \\
\eta(t) = y_x(1,t) + \alpha y_t(1,t), & \\
y(0,t) = 0,\, \theta(1,t) = 0 = \theta(0,t), & t > 0, \\
y(x,0) = y_0(x),\, y_t(x,0) = y_1(x),\ \theta(x,0) = \theta_0(x), & x \in (0,1).
\end{cases} \tag{8}
$$

3 Well-Posedness of the Closed-Loop System

In this section, we will consider the existence and uniqueness of the solution of the closed-loop system (8). For this purpose, we introduce the state space as

$$\mathcal{H} = H_*^1(0,1) \times L^2(0,1) \times L^2(0,1) \times \mathbb{R},$$

where $H_*^1(0,1) = \{y \in H^1(0,1) : y(0) = 0\}$ and $L^2(0,1)$, $H^1(0,1)$, $H_0^1(0,1)$ are defined as usual. In $\mathcal{H}$, the inner product is defined as

$$\langle Y_1, Y_2\rangle_{\mathcal{H}} = \int_0^1 [f_{1x}f_{2x} + g_1g_2 + \tau_1\tau_2]\,dx + K\eta_1\eta_2,$$

for any $Y_i = (f_i, g_i, \tau_i, \eta_i)^T \in \mathcal{H}$, $i = 1, 2$, where $K > \frac{\beta^2}{2m}$ is a constant. Clearly, $(\mathcal{H}, ||.||_{\mathcal{H}})$ is a Hilbert space.

Now, we define an operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ as

$$\mathcal{A}Y = \begin{pmatrix} g \\ f_{xx} + \kappa\tau_x \\ \tau_{xx} + \kappa g_x \\ -\frac{1}{\beta}(\eta + M\mathrm{sgn}(\eta)) \end{pmatrix}, \qquad \forall\, Y = (f, g, \tau, \eta)^T \in \mathcal{D}(\mathcal{A}). \tag{9}$$

Thus the closed-loop system (8) can be written as

$$
\begin{cases}
\dot{Y} - \mathcal{A}Y = s(t), & t > 0, \\
Y(0) = (y_0, y_1, \theta_0, \eta_0)^T,
\end{cases} \tag{10}
$$

where $s(t) = (0,0,0,\frac{d(t)}{\beta})$. So, instead of dealing with (8), we will consider (10) in the Hilbert space $\mathcal{H}$, with the domain $\mathcal{D}(\mathcal{A})$ of the operator $\mathcal{A}$ given by

$$\begin{aligned}\mathcal{D}(\mathcal{A}) \quad = \quad & \Big\{(f,g,\tau,\eta)^T \in H^2(0,1)\cap H^1_*(0,1)\times H^1_0(0,1)\times L^2(0,1)\times\mathbb{R}: \\ & \eta = f_x(1)+\alpha g(1), \tau(0)=0=\tau(1)\Big\}.\end{aligned}$$

We are now ready to state our existence result as follows.

THEOREM 1. Let $\mathcal{A}$ be defined as in (9). Then for any initial value $Y_0 \in \mathcal{D}(\mathcal{A})$, there is a unique solution to the system (8).

PROOF. Firstly, we prove $-\mathcal{A}$ is monotone. For any $Y_i = (f_i, g_i, \tau_i, \eta_i) \in \mathcal{D}(\mathcal{A})$, $i = 1, 2$, we have

$$\begin{aligned}
& \langle \mathcal{A}Y_1 - \mathcal{A}Y_2, Y_1 - Y_2\rangle_{\mathcal{H}} \\
= \quad & \int_0^1 (g_{1x}-g_{2x})(f_{1x}-f_{2x})dx + \int_0^1 \left[(f_{1xx}-f_{2xx})+\kappa(\tau_{1x}-\tau_{2x})\right](g_1-g_2)dx \\
& + \int_0^1 \left[(\tau_{1xx}-\tau_{2xx})+\kappa(g_{1x}-g_{2x})\right](\tau_1-\tau_2)dx + K(\dot{\eta_1}-\dot{\eta_2})(\eta_1-\eta_2) \\
= \quad & \int_0^1 (g_{1x}-g_{2x})(f_{1x}-f_{2x})dx + \int_0^1 (f_{1xx}-f_{2xx})(g_1-g_2)dx \\
& +\kappa\int_0^1 (\tau_{1x}-\tau_{2x})(g_1-g_2)dx + \int_0^1 (\tau_{1xx}-\tau_{2xx})(\tau_1-\tau_2)dx \\
& +\kappa\int_0^1 (g_{1x}-g_{2x})(\tau_1-\tau_2)dx - \frac{K}{\beta}(\eta_1-\eta_2)^2 \\
& -\frac{KM}{\beta}(\eta_2-\eta_1)(\mathrm{sgn}(\eta_2)-\mathrm{sgn}(\eta_1)) \\
= \quad & (g_1(1)-g_2(1))(f_{1x}(1)-f_{2x}(1)) - \int_0^1 (\tau_{1x}-\tau_{2x})^2 dx \\
& -\frac{K}{\beta}(\eta_1-\eta_2)^2 - \frac{KM}{\beta}(\eta_2-\eta_1)(\mathrm{sgn}(\eta_2)-\mathrm{sgn}(\eta_1)) \\
= \quad & -\alpha(g_1(1)-g_2(1))^2 + (\eta_1-\eta_2)(g_1(1)-g_2(1)) - \int_0^1 (\tau_{1x}-\tau_{2x})^2 dx \\
& -\frac{K}{\beta}(\eta_1-\eta_2)^2 - \frac{KM}{\beta}(\eta_2-\eta_1)(\mathrm{sgn}(\eta_2)-\mathrm{sgn}(\eta_1)) \\
= \quad & -\alpha\left[(g_1(1)-g_2(1)) - \frac{1}{2\alpha}(\eta_1-\eta_2)\right]^2 - \left[\frac{K}{\beta}-\frac{1}{4\alpha}\right](\eta_1-\eta_2)^2 \\
& - \int_0^1 (\tau_{1x}-\tau_{2x})^2 dx - \frac{KM}{\beta}(\eta_2-\eta_1)(\mathrm{sgn}(\eta_2)-\mathrm{sgn}(\eta_1)).
\end{aligned}$$

Since $\alpha\beta = m$ and $K > \frac{\beta^2}{2m}$, we have $\frac{K}{\beta} > \frac{\beta}{2m} > \frac{\beta}{4m} = \frac{1}{4\alpha}$. Therefore,

$$\langle \mathcal{A}Y_1 - \mathcal{A}Y_2, Y_1 - Y_2\rangle_{\mathcal{H}} \le 0.$$

Thus $-\mathcal{A}$ is monotone.

Next, we show that $-\mathcal{A}$ is maximal. According to the definition of the maximal operator, we only need to show that $\mathcal{R}(I-\mathcal{A})=\mathcal{H}$, i.e., for any $F=(u,v,\phi,w)\in\mathcal{H}$, there exist $Y=(f,g,\tau,\eta)\in\mathcal{D}(\mathcal{A})$ such that $F=\mathcal{A}Y$, i.e.,

$$\begin{cases} f-g=u, \\ g-f_{xx}-\kappa\,\tau_x=v, \\ \tau-\tau_{xx}-\kappa\,g_x=\phi, \\ \eta+\frac{1}{\beta}\left(\eta+M\operatorname{sgn}(\eta)\right)=w, \end{cases} \tag{11}$$

with the boundary conditions

$$\begin{cases} f(0)=0, \\ f_x(1)=\eta-\alpha\,g(1), \\ \tau(0)=0=\tau(1). \end{cases} \tag{12}$$

Solving fourth equation of (11), we have

$$\eta=\begin{cases} \frac{\beta\,w-M}{\beta+1}, & \text{when} \quad w>\frac{M}{\beta}, \\ 0, & \text{when} \quad w\in\left[-\frac{M}{\beta},\frac{M}{\beta}\right], \\ \frac{\beta\,w+M}{\beta+1}, & \text{when} \quad w<-\frac{M}{\beta}. \end{cases}$$

Now, from the first equation of (11), we get $g=f-u$. Hence, in view of the second and third equations of (11) with the boundary conditions in (12), we get

$$\begin{cases} \left.\begin{array}{l} f(x)-f_{xx}(x)-\kappa\tau_x(x)=v(x)+u(x) \\ \tau(x)-\tau_{xx}(x)-\kappa f_x(x)=\phi(x)-\kappa u_x(x) \end{array}\right\}, \quad x\in(0,1),\ t>0, \\ f(0)=0, \\ f_x(1)=\eta-\alpha g(1), \\ \eta(0)=0=\tau(1). \end{cases} \tag{13}$$

To solve (13), we consider the bilinear form

$$\mathcal{M}:\Big(H^1_*(0,1)\times H^1_0(0,1)\Big)\times\Big(H^1_*(0,1)\times H^1_0(0,1)\Big)\to R$$

given by

$$\begin{aligned} \mathcal{M}((f,\tau),(h,\sigma)) \;=\;& \int_0^1 f(x)h(x)\,dx+\int_0^1 f_x(x)h_x(x)\,dx+\kappa\int_0^1\tau(x)h_x(x)\,dx \\ &+\alpha f(1)h(1)+\int_0^1\tau(x)\sigma(x)\,dx+\int_0^1\tau_x(x)\sigma_x(x)\,dx \\ &+\kappa\int_0^1 f(x)\sigma_x(x)\,dx \end{aligned}$$

and the linear form $\mathcal{J}:H^1_*(0,1)\times H^1_0(0,1)\to R$ given by

$$\mathcal{J}(h,\sigma)=\int_0^1(u(x)+v(x))h(x)\,dx+(\alpha u(1)+\eta)h(1)+\int_0^1(\phi(x)-\kappa u_x(x))\sigma(x)\,dx.$$

Multiplying both side of the first equation of (13) by any function $z(x) \in H_*^1(0,1)$, the second equation by $\xi(x) \in H_0^1(0,1)$ and then integrating over $[0,1]$, we get a variational equation

$$\begin{aligned} & \int_0^1 [f(x)z(x) + f_x(x)z_x(x) + \kappa\,\tau(x)z_x(x)]\,dx + \alpha f(1)z(1) \\ & \qquad\qquad + \int_0^1 [\tau(x)\xi(x) + \tau_x(x)\xi_x(x) + \kappa\, f(x)\xi_x(x)]\,dx \\ = & \int_0^1 (u(x)+v(x))z(x)\,dx + (\alpha u(1)+\eta)z(1) + \int_0^1 (\phi(x) - \kappa u_x(x))\xi(x)\,dx. \end{aligned}$$

i.e., $\mathcal{M}((f,\tau),(z,\xi)) = \mathcal{J}(z,\xi)$. Also, $\mathcal{M}$ and $\mathcal{J}$ satisfy the following conditions:

I. $\mathcal{M}$ is bounded on $H_*^1(0,1) \times H_0^1(0,1)$, since

$$\begin{aligned} & |\mathcal{M}((f,\tau),(h,\sigma))| \\ \le & \left|\int_0^1 [f(x)h(x) + f_x(x)h_x(x)]\,dx + \alpha f(1)h(1)\right| + \left|\int_0^1 [\tau(x)\sigma(x) + \tau_x(x)\sigma_x(x)]\,dx\right| \\ & + \left|\kappa \int_0^1 [\tau(x)h_x(x) + f(x)\sigma_x(x)]\ dx\right| \\ \le & \left(\int_0^1 \left[|f(x)|^2 + |f_x(x)|^2 + \alpha|f(1)|^2\right] dx\right)^{\frac{1}{2}} \left(\int_0^1 \left[|h(x)|^2 + |h_x(x)|^2 + \alpha|h(1)|^2\right] dx\right)^{\frac{1}{2}} \\ & + \left(\int_0^1 \left[|\tau(x)|^2 + |\tau_x(x)|^2\right] dx\right)^{\frac{1}{2}} \left(\int_0^1 \left[|\sigma(x)|^2 + |\sigma_x(x)|^2\right] dx\right)^{\frac{1}{2}} \\ & +\kappa \left(\int_0^1 |\tau(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_0^1 |h_x(x)|^2 dx\right)^{\frac{1}{2}} + \kappa \left(\int_0^1 |f(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_0^1 |\sigma_x(x)|^2 dx\right)^{\frac{1}{2}} \\ \le & (\alpha+1)||f||^2_{H_*^1(0,1)}||h||^2_{H_*^1(0,1)} + ||\tau||^2_{H_0^1(0,1)}||\sigma||^2_{H_0^1(0,1)} \\ & +\kappa\left(||h_x||^2_{H_*^1(0,1)}||\tau||^2_{H_0^1(0,1)} + ||f||^2_{H_*^1(0,1)}||\sigma_x||^2_{H_0^1(0,1)}\right). \end{aligned}$$

II. $\mathcal{M}$ is coercive, because

$$\begin{aligned} |\mathcal{M}((f,\tau),(h,\sigma))| &= \int_0^1 \left[|f(x)|^2 + |f_x(x)|^2 + |\tau(x)|^2 + |\tau_x(x)|^2\right] dx + \alpha|f(1)|^2 \\ &\ge \left(||f||^2_{H_*^1(0,1)} + ||\tau||^2_{H_0^1(0,1)}\right). \end{aligned}$$

III. $\mathcal{J}$ is bounded, i.e.,

$$\begin{aligned} & |\mathcal{J}(h,\sigma)| \\ = & \left|\int_0^1 (u(x)+v(x))h(x)\,dx + (\alpha u(1)+\eta)h(1) + \int_0^1 (\phi(x) - \kappa u_x(x))\sigma(x)\,dx\right| \\ \le & \left|\int_0^1 (u(x)+v(x))h(x)\,dx + (\alpha u(1)+\eta)h(1)\right| + \left|\int_0^1 (\phi(x) - \kappa u_x(x))\sigma(x)\,dx\right| \end{aligned}$$

$$
\begin{aligned}
&\le \left(\int_0^1 (u(x)+v(x))^2 + (\alpha u(1)+\eta)^2\right)^{\frac{1}{2}} \left(\int_0^1 (h(x))^2 + (h(1))^2\right)^{\frac{1}{2}} \\
&\quad + \left(\int_0^1 (\phi(x) - \kappa u_x(x))^2\right)^{\frac{1}{2}} \left(\int_0^1 (\sigma(x))^2\right)^{\frac{1}{2}} \\
&\le C_1 \left(\int_0^1 [|h(x)|^2 + |h_x(x)|^2] dx\right)^{\frac{1}{2}} + C_2 \left(\int_0^1 |\sigma(x)|^2 dx\right)^{\frac{1}{2}} \\
&= C_1 ||h||_{H^1_*(0,1)} + C_2 ||\sigma(x)||_{H^1_0(0,1)}.
\end{aligned}
$$

The Lax-Milgram theorem on the space $H^1_*(0,1) \times H^1_0(0,1)$ for the functionals $\mathcal{M}$ and $\mathcal{J}$, yields that (13) has a unique solution $(f,\tau) \in H^1_*(0,1) \times H^1_0(0,1)$. Since u, v and ϕ are in $L^2(0,1)$, so $f \in H^2(0,1) \cap H^1_*(0,1)$ and $\tau \in H^1_0(0,1)$. Hence, $\mathcal{R}(I - \mathcal{A}) = \mathcal{H}$.

Thus, we have proved that $-\mathcal{A}$ is a maximal monotone operator and $\mathcal{A}$ generates a C_0 nonlinear semigroup, see [18]. Then for any $Y_0 \in \overline{\mathcal{D}(\mathcal{A})}$ and $s \in L^1(0,1:\mathcal{H})$, the system (8) has a unique solution, see [1]. This ends the proof.

4 Stability of the Closed-Loop System

In this section, we shall discuss the exponential stability of the closed-loop system (8). To prove the exponential stability of (8), we first need the following inequalities and lemmas as:

I. For any real number $\gamma > 0$, we have **Schwartz's inequality** (cf. [13])

$$\int_0^1 u\,v \le \int_0^1 |u\,v| \le \frac{1}{2}\left(\gamma \int_0^1 u^2 + \frac{1}{\gamma}\int_0^1 v^2\right).$$

II. For any $u(x,t)$, $x \in (0,1)$, $t > 0$ satisfying the boundary conditions $u(0,t) = 0 = u(1,t)$, we have **Poincaré's Inequality** (cf. [13])

$$\int_0^1 u^2 \le \frac{1}{\pi^2}\int_0^1 {u_x}^2.$$

Now, we define the energy functional of the system (8) as

$$\mathcal{E}(t) = \frac{1}{2}\int_0^1 [{y_t}^2 + {y_x}^2 + \theta^2] dx + \frac{K}{2}\eta^2(t).$$

Also, we define another functional associated with energy of the system (8) as

$$V(t) = \mathcal{E}_1(t) + \mathcal{E}_2(t) + G(t), \tag{14}$$

where

$$\mathcal{E}_1(t) = \frac{1}{2}\int_0^1 [{y_t}^2 + {y_x}^2 + \theta^2] dx, \tag{15}$$

$$\mathcal{E}_2(t) = \frac{K}{2}\eta^2(t),$$

$$G(t) = \rho \int_0^1 x\, y_t(x,t)\, y_x(x,t) dx, \quad 0 < \rho < 1.$$

Applying Schwartz's inequality and Poincaré's inequality, we have the following lemma.

LEMMA 1. Let $V(t)$, $\mathcal{E}_1(t)$ be defined as in (14) and (15). Then there exist positive constants c_1, c_2 such that

$$c_1 \left(\mathcal{E}_1(t) + \eta^2(t)\right) < V(t) < c_2 \left(\mathcal{E}_1(t) + \eta^2(t)\right). \tag{16}$$

PROOF. Applying Schwartz's inequality, we have the estimate

$$|G(t)| \le \frac{\rho}{2} \int_0^1 [y_t^2(x,t) + y_x^2(x,t)] dx \le \rho \mathcal{E}_1(t).$$

Seting

$$c_1 = \min\left\{1 - \rho, \frac{K}{2}\right\}, \qquad c_2 = \max\left\{1 + \rho, \frac{K}{2}\right\},$$

we have

$$c_1 \left(\mathcal{E}_1(t) + \eta^2(t)\right) < V(t) < c_2 \left(\mathcal{E}_1(t) + \eta^2(t)\right).$$

This ends the proof.

Now, we shall discuss the stability of the system (8) with the help of $V(t)$ as defined in (14).

THEOREM 2. Let us assume that $|d(t)| < M,\ M > 0$. If β in control law (5) satisfies the inequality $\beta^2 < 2mK$ and $\rho \le \min\{1, \alpha, 1/\alpha\}$, then the system (8) is exponentially stable.

PROOF. Let $y(x,t)$ and $\theta(x,t)$ be the solutions of (8) and $V(t)$ be defined in (14). Noting that $\alpha\,\beta = m$, a simple calculation gives

$$\dot{V}(t) = \dot{\mathcal{E}}_1(t) + \dot{\mathcal{E}}_2 + \dot{G}(t),$$

where

$$\begin{aligned}
\dot{\mathcal{E}}_1(t) &= y_x(1,t)\, y_t(1,t) - \int_0^1 \theta_x^2\, dx \\
&= \frac{1}{2\alpha}\eta^2(t) - \frac{\alpha}{2} y_t^2(1,t) - \frac{1}{2\alpha} y_x^2(1,t) - \int_0^1 \theta_x^2\, dx,
\end{aligned}$$

$$\dot{\mathcal{E}}_2(t) = K\, \eta(t)\, \dot{\eta}(t) = -\frac{K}{\beta}\eta^2(t) - \frac{K\,M}{\beta}|\eta(t)| + \frac{K}{\beta}\eta(t)\, d(t),$$

and

$$\dot{G}(t) = \rho \int_0^1 x\, y_{xt}\, y_t\, dx + \rho \int_0^1 x\, y_x\, y_{tt}\, dx$$

$$
\begin{aligned}
&= \rho\int_0^1 x\,y_{xt}\,y_t\,dx + \rho\int_0^1 x\,y_x[y_{xx}+\kappa\theta_x]dx \\
&= \frac{\rho}{2}[y_t^2(1)+y_x^2(1)] - \frac{\rho}{2}\int_0^1 [y_t^2+y_x^2]dx + \rho\kappa\int_0^1 x\,y_x\,\theta_x\,dx \\
&\le \frac{\rho}{2}[y_t^2(1)+y_x^2(1)] - \frac{\rho}{2}\int_0^1 [y_t^2+y_x^2]dx + \frac{\rho\kappa}{2}\int_0^1 [y_x^2+\theta_x^2]dx \\
&= \frac{\rho}{2}[y_t^2(1)+y_x^2(1)] - \frac{\rho}{2}\int_0^1 y_t^2\,dx + \frac{\rho\kappa}{2}\int_0^1 \theta_x^2\,dx - \frac{(1-\kappa)\rho}{2}\int_0^1 y_x^2\,dx.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\dot V(t) \le\ & \frac{1}{2\alpha}\eta^2(t) - \frac{\alpha}{2}y_t^2(1,t) - \frac{1}{2\alpha}y_x^2(1,t) - \int_0^1 \theta_x^2\,dx - \frac{K}{\beta}\eta^2(t) \\
& - \frac{K\,M}{\beta}|\eta(t)| + \frac{K}{\beta}\eta(t)\,d(t) + \frac{\rho}{2}[y_t^2(1)+y_x^2(1)] - \frac{\rho}{2}\int_0^1 y_t^2\,dx \\
& + \frac{\rho\kappa}{2}\int_0^1 \theta_x^2\,dx - \frac{(1-\kappa)\rho}{2}\int_0^1 y_x^2\,dx \\
\le\ & \left(\frac{1}{2\alpha}-\frac{K}{\beta}\right)\eta^2(t) + \left(\frac{\rho}{2}-\frac{\alpha}{2}\right)y_t^2(1) + \left(\frac{\rho}{2}-\frac{1}{2\alpha}\right)y_x^2(1) \\
& - \left(1-\frac{\rho\kappa}{2}\right)\int_0^1 \theta_x^2\,dx - \frac{\rho}{2}\int_0^1 y_t^2\,dx - \frac{K\,M}{\beta}|\eta(t)| + \frac{K}{\beta}\eta(t)\,d(t) \\
& - \frac{(1-\kappa)\rho}{2}\int_0^1 y_x^2\,dx \\
\le\ & \left(\frac{1}{2\alpha}-\frac{K}{\beta}\right)\eta^2(t) + \left(\frac{\rho}{2}-\frac{\alpha}{2}\right)y_t^2(1) + \left(\frac{\rho}{2}-\frac{1}{2\alpha}\right)y_x^2(1) \\
& - \left(1-\frac{\rho\kappa}{2}\right)\pi^2\int_0^1 \theta^2\,dx - \frac{\rho}{2}\int_0^1 y_t^2\,dx - \frac{(1-\kappa)\rho}{2}\int_0^1 y_x^2\,dx.
\end{aligned}
$$

Note that $\alpha\,\beta = m$ and $K > \dfrac{\beta^2}{2m}$. So $\dfrac{K}{\beta} > \dfrac{\beta}{2m} = \dfrac{1}{2\alpha}$. Also, we choose $\rho \in (0,1)$ such that

$$
\rho < \min\left\{\alpha, \frac{1}{\alpha}\right\} \quad \text{i.e.,} \quad \frac{\rho}{2}-\frac{\alpha}{2} < 0, \quad \frac{\rho}{2}-\frac{1}{2\alpha} < 0.
$$

Thus, in view of our assumption $0 < \kappa < 1$, we have

$$
\begin{aligned}
\dot V(t) \le\ & -\left(\frac{K}{\beta}-\frac{1}{2\alpha}\right)\eta^2(t) - \frac{\rho}{2}\int_0^1 y_t^2\,dx - \left(1-\frac{\rho\kappa}{2}\right)\pi^2\int_0^1 \theta^2\,dx \\
& - \frac{(1-\kappa)\rho}{2}\int_0^1 y_x^2\,dx \\
\le\ & -\min\left\{\left(\frac{K}{\beta}-\frac{1}{2\alpha}\right), \rho, (1-\kappa)\rho, (2-\rho\kappa)\,\pi^2\right\}\left(\mathcal{E}_1(t)+\eta^2(t)\right)
\end{aligned}
$$

$$\leq \quad -\frac{\min\left\{\left(\frac{K}{\beta}-\frac{1}{2\alpha}\right), \rho, (1-\kappa)\rho, (2-\rho\kappa)\,\pi^2\right\}}{c_2} V(t). \tag{17}$$

Integrating (17) over $[0,t]$, we get

$$V(t) \leq e^{-\lambda\, t} V(0),$$

where

$$\lambda = \frac{\min\left\{\left(\frac{K}{\beta}-\frac{1}{2\alpha}\right), \rho, (1-\kappa)\rho, (2-\rho\kappa)\,\pi^2\right\}}{c_2}.$$

Finally, we observe that

$$\min\{1,K\}(\mathcal{E}_1(t)+\eta^2(t)) \leq \mathcal{E}(t) \leq \max\{1,K\}(\mathcal{E}_1(t)+\eta^2(t)).$$

So, we have

$$\mathcal{E}(t) \leq \frac{\max\{1,K\}}{c_1} V(t) \leq \frac{\max\{1,K\}}{c_1} e^{-\lambda\, t} V(0).$$

This ends the proof.

5 Conclusions

In this work, we have studied the stabilization problem of a wave equation with a tip mass coupled with Fourier heat conduction, under the assumption that there is a disturbance on the tip mass. In order to stabilize the system, a control force is applied at the tip mass end, which consists of two parts: one part makes the system exponentially stable, if the disturbance is absent; the other part is used to reject the disturbances. Based on the observation of disturbances, we adopt a positive feedback controller with a suitable anti-disturbance term. Designing of such a controller, we have proved the well-posedness of the system and finally established the exponential stable result by constructing suitable Lyapunov functional. Design of such type controller can be extended to other models also.

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