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On Existence And Uniqueness Results For Iterative Fractional Integrodifferential Equation With Deviating Arguments*

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Abstract

The objective of the given paper is to study the local existence, uniqueness, stability and other properties of solutions of an iterative fractional integrodifferential equation with deviating arguments. The Successive Approximation Method is applied to the numerical solution of the iterative fractional integrodifferential equation with deviating arguments.

1 Introduction

The study of iterative differential and integrodifferential equations is linked to the wide applications of calculus in mathematical sciences. These equations are vital in the study of infection models. They are related to the study of the motion of charged particles with retarded interaction see [2, 24]. The development in the theory of iterative differential equation begins with the work of E. Eder [6]. In 1984, his studies revealed that a solution of the functional differential equation $x' = x \circ x$ is a function $x : A \longrightarrow \mathbb{R}$ from an interval $A \in \mathbb{R}$ (i.e. a connected subset of $\mathbb{R}$) into $\mathbb{R}$ such that

$$\begin{aligned} x'(t) &= x(x(t)), \\ x(t_0) &= x_0, \quad \forall\ t_0, x_0 \in A \end{aligned}$$

and proved the existence, uniqueness, analyticity and analytic dependence of solutions on initial data.

Sui Sun Cheng et al. [3, 13, 21, 22], investigated analytic and exact solutions of an iterative functional differential equation

$$\begin{aligned} y'(x) &= f\left(x, y(h(x) + g(y(x)))\right), \\ y(x_0) &= x_0, \end{aligned}$$

and its variants.

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M. Lauran [12], investigated the existence and uniqueness results for first order differential and iterative differential equations with deviating argument of the type

$$x'(t) = f\left(t, x(t), x(\lambda t)\right),$$

and

$$x'(t) = f\left(t, x(t), x(x(t)), x(\lambda x(t))\right),$$

with the initial condition

$$x(t_0) = x_0,$$

where $t_0, x_0 \in [a, b]$, $\lambda \in (0, 1)$.

In [7], Ibrahim investigated the existence and uniqueness of

$$\begin{aligned} D^\alpha u(t) &= f(t, u(u(t))), \\ u(0) &= u_0. \end{aligned}$$

The differential and integral equations, in which the deviating arguments depend on both the state variable x and t, are of importance in theory and practice see for example [2, 24] and references therein.

Many papers have dealt with the existence, uniqueness and other properties of solutions of special forms of the iterative integrodifferential equations (1)–(2), see [8, 9, 10, 14, 16, 18, 20, 23, 25] and some of the references cited therein. In an interesting paper [17] M. Podisuk has investigated the numerical solution of simple iterative ordinary differential equations which inspired us to study the iterative fractional integrodifferential equation with deviating argument of the type

$$D^\alpha u(x) = f(x) + \int_0^x K(x, s) u(\lambda u(s)) ds, \quad x, s \in J = [0, T], \quad 0 < \alpha < 1, \tag{1}$$

$$u(0) = u_0, \tag{2}$$

where D^α indicates the α-th Caputo fractional derivative, $f(x)$ and $K(x, s)$ are given continuous functions, $u(x)$ is the unknown function to be determined and $u_0 \in J$, $\lambda \in (0, 1)$.

The main tool employed in our analysis is based on an application of the Banach contraction principle, theory of fractional calculus and Gronwall-Bellman's integral inequality.

The paper is organized as follows: Section 2, is dedicated to the preliminaries and definitions. Section 3, presents the existence and uniqueness results for iterative fractional integrodifferential equation with deviating argument. Section 4, focuses on some examples to illustrate the theory.

2 Preliminaries and Definitions

In this section, we shall recall some definitions and properties of fractional derivatives and fractional integrals, which will be used later. For more details see [11, 15, 19].

DEFINITION 1. The fractional integral of order α with the lower limit zero for a function f is defined as

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt, \qquad x > 0,\ \alpha > 0, \tag{3}$$

provided the right side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

DEFINITION 2. The Riemann-Liouville derivative of order α with the lower limit zero for a function $f : [0, \infty) \to R$ can be written as

$${}^L D^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{f(t)}{(x-t)^{\alpha}}\, dt,\ x > 0,\ 0 < \alpha < 1. \tag{4}$$

DEFINITION 3. The Caputo derivative of order α for a function $f : [0, \infty) \to R$ can be written as

$$D^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{f'(t)}{(x-t)^{\alpha}}\, dt, \quad x > 0,\ 0 < \alpha < 1.$$

LEMMA 2.1 (Gronwall-Bellman's Inequality [1]). Let $u(x), f(x)$ be nonnegative continuous functions defined on $J = [\alpha, \alpha + h]$ and c be a nonnegative constant. If

$$u(x) \le c + \int_\alpha^x f(s)u(s)ds, \quad x \in J,$$

then

$$u(x) \le c \ \ exp\left(\int_\alpha^x f(s)ds\right), \quad x \in J.$$

LEMMA 2.2. If a function $u \in C[0, T]$ satisfies (1)–(2) in the closed interval $[0, T]$, then the problems (1)–(2) are equivalent to the problem of finding a continuous solution of the integral equation

$$u(x) = u_0 + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \left(f(t) + \int_0^t K(t, s)u(\lambda u(s))ds\right) dt.$$

PROOF**.** Applying I^α on both sides of equation (1) and using initial condition, we get

$$u(x) - u_0 = I^\alpha \left(f(t) + \int_0^t K(t, s)u(\lambda u(s))ds\right),$$

$$u(x) = u_0 + I^\alpha \left(f(t) + \int_0^t K(t, s)u(\lambda u(s))ds\right),$$

$$u(x) = u_0 + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \left(f(t) + \int_0^t K(t, s)u(\lambda u(s))ds\right) dt.$$

Let $B = C(J, J)$ be the Banach space equipped with the norm $\|u\| = \max_{x\in[0,T]} |u(x)|$. For convenience, we are listing the following hypotheses used in our further discussion.

(H_1) There exists a constant k_T such that $k_T = \sup\{|K(t,s)| : 0 \le s \le t \le T\}$.

(H_2) There exists a constant $M > 0$ such that $|u(x_1) - u(x_2)| \le M|x_1 - x_2|^\alpha$ for $u \in B, x_1, x_2 \in J,\ x_1 \le x_2$ and $0 < \alpha < 1$.

(H_3) There exists a constant $L > 0$ such that $L = sup\{|f(t)| : 0 \le t \le T\}$.

(H_4) Let $\rho = u_0 + \frac{T^\alpha(L+T^3k_T)}{\Gamma(\alpha+1)} \le T$ and $T \le M$.

3 Local Existence and Uniqueness Result

Now, we intend to state and prove results related to iterative fractional integrodifferential equation with deviating arguments.

THEOREM 3.1. Suppose that the hypotheses (H_1)–(H_4) are satisfied and

$$\frac{T^{\alpha+1}\lambda k_T(M+1)}{\Gamma(\alpha+1)} < 1.$$

Then there is a unique solution to the problems (1)–(2).

PROOF. Let

$$S(\rho) = \{u \in B : 0 \le u \le \rho \text{ and } |u(x_1) - u(x_2)| \le M|x_1 - x_2|^\alpha, \quad x_1, x_2 \in J,\ x_1 \le x_2\}.$$

To apply Banach contraction principle, we define an operator $P : S(\rho) \to S(\rho)$ by

$$(Pu)(x) = u_0 + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \left(f(t) + \int_0^t K(t,s)u(\lambda u(s))ds \right) dt.$$

Now, we have

$$\begin{aligned}
0 \le (P(u)) &= \left| u_0 + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \left(f(t) + \int_0^t K(t,s)u(\lambda u(s))ds \right) dt \right| \\
&\le u_0 + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \left(|f(t) + \int_0^t K(t,s)u(\lambda u(s))ds| \right) dt \\
&\le u_0 + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \left(|f(t)| + \int_0^t |K(t,s)||u(\lambda u(s))|ds \right) dt \\
&\le u_0 + L\frac{T^\alpha}{\Gamma(\alpha+1)} + k_T\frac{T^\alpha T^3}{\Gamma(\alpha+1)} \\
&\le u_0 + \frac{T^\alpha(L+T^3k_T)}{\Gamma(\alpha+1)} = \rho.
\end{aligned}$$

Also, for each $0 \leq x_1 \leq x_2 \leq T$, we have

$$
\begin{aligned}
& Pu(x_2) - Pu(x_1) \\
= \; & \int_0^{x_1} \frac{(x_2-t)^{\alpha-1} - (x_1-t)^{\alpha-1}}{\Gamma(\alpha)} \left(f(t) + \int_0^t K(t,s)u(\lambda u(s))ds \right) dt \\
& + \int_{x_1}^{x_2} \frac{(x_2-t)^{\alpha-1}}{\Gamma(\alpha)} \left(f(t) + \int_0^t K(t,s)u(\lambda u(s))ds \right) dt \\
= \; & \frac{-1}{\Gamma(\alpha)} \int_0^{x_1} \left[(x_1-t)^{\alpha-1} - (x_2-t)^{\alpha-1} \right] \left(f(t) + \int_0^t K(t,s)u(\lambda u(s))ds \right) dt \\
& + \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x_2} (x_2-t)^{\alpha-1} \left(f(t) + \int_0^t K(t,s)u(\lambda u(s))ds \right) dt.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& |Pu(x_2) - Pu(x_1)| \\
\leq \; & \left| \frac{1}{\Gamma(\alpha)} \int_0^{x_1} \left[(x_1-t)^{\alpha-1} - (x_2-t)^{\alpha-1} \right] \left(f(t) + \int_0^t K(t,s)u(\lambda u(s))ds \right) dt \right| \\
& + \left| \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x_2} (x_2-t)^{\alpha-1} \left(f(t) + \int_0^t K(t,s)u(\lambda u(s))ds \right) dt \right| \\
\leq \; & \frac{1}{\Gamma(\alpha)} \int_0^{x_1} \left[(x_1-t)^{\alpha-1} - (x_2-t)^{\alpha-1} \right] \left(|f(t)| + \int_0^t |K(t,s)||u(\lambda u(s))|ds \right) dt \\
& + \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x_2} (x_2-t)^{\alpha-1} \left(|f(t)| + \int_0^t |K(t,s)||u(\lambda u(s))|ds \right) dt \\
\leq \; & \frac{1}{\Gamma(\alpha)} \int_0^{x_1} \left[(x_1-t)^{\alpha-1} - (x_2-t)^{\alpha-1} \right] (L + k_T T^3) dt \\
& + \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x_2} (x_2-t)^{\alpha-1} (L + k_T T^3) dt \\
\leq \; & \frac{(L + k_T T^3)}{\Gamma(\alpha)} \left(\int_0^{x_1} (x_1-t)^{\alpha-1} dt - \int_0^{x_1} (x_2-t)^{\alpha-1} dt + \int_{x_1}^{x_2} (x_2-t)^{\alpha-1} dt \right) \\
\leq \; & \frac{(L + k_T T^3)}{\Gamma(\alpha+1)} [x_1^\alpha - x_2^\alpha + 2(x_2 - x_1)^\alpha] \\
\leq \; & \frac{2(L + k_T T^3)}{\Gamma(\alpha+1)} |x_2 - x_1|^\alpha.
\end{aligned}
$$

This shows that P maps from $S(\rho)$ to $S(\rho)$. Now, for all $u, v \in S(\rho)$ we have

$$\begin{aligned}
&|(Pu)(x) - (Pv)(x)| \\
&\leq \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \int_0^t |K(t,s)||u(\lambda u(s) - v(\lambda v(s))|dsdt \\
&\leq k_T \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \int_0^t (|u(\lambda u(s) - u(\lambda v(s))| + |u(\lambda v(s)) - v(\lambda v(s))|)dsdt \\
&\leq \lambda k_T \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \int_0^t (M|u(s) - v(s)| + |u(s)) - v(s)|)dsdt \\
&\leq \lambda k_T \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \int_0^t (M+1)|u(s) - v(s)|dsdt \\
&\leq T\lambda k_T(M+1)\|u-v\| \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} dt \\
&\leq \frac{T^{\alpha+1}\lambda k_T(M+1)}{\Gamma(\alpha+1)}\|u-v\|.
\end{aligned}$$

Hence we obtain

$$\|(Pu)(x) - (Pv)(x)\| \leq \frac{T^{\alpha+1}\lambda k_T(M+1)}{\Gamma(\alpha+1)}\|u-v\|.$$

Since

$$\frac{T^{\alpha+1}\lambda k_T(M+1)}{\Gamma(\alpha+1)} < 1,$$

by the Banach Contraction Principle, P has a unique fixed point. This means that the equation (1)–(2) has unique solution.

The above theorem shows that there exists a unique solution to the problems (1)–(2). However, it does not tell us how to find this solution. To find the solution of the problems (1)–(2), we will define the following sequence

$$u_{n+1}(x) = u_0 + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \left(f(t) + \int_0^t K(t,s)u_n(\lambda u_n(s))ds \right) dt \tag{5}$$

where $n = 0, 1, 2, \ldots$ and $u_0(x)$ is fixed functions of the class C^1 mapping from $[0,T]$ to $[0,T]$ such that $|u_0(x)| \leq T$. For this, we have the following theorem,

THEOREM 3.2**.** If the assumptions of the Theorem 3.1 are satisfied then the sequences defined in (5) converges uniformly to the unique solution of the problems (1)–(2).

PROOF. Let $U_k = Max_{x\in J}|u_k(x) - u_{k-1}(x)|$. Then

$$\begin{aligned}
U_1 &= Max_{x\in J}|u_1(x) - u_0(x)| \\
&= Max_{x\in J} \left| u_0 + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \left(f(t) + \int_0^t K(t,s)u_0(\lambda u_0(s))ds \right) dt - u_0(x) \right| \\
&\leq \frac{T^\alpha}{\Gamma(\alpha+1)}(L + T^3 k_T).
\end{aligned}$$

Since u_0 maps from $[0,T]$ to $[0,T]$, we see that we have

$$U_1 \leq \frac{T^\alpha}{\Gamma(\alpha+1)}(L+T^3k_T) \leq T,$$

$$\begin{aligned}
U_2 &= Max_{x\in J}|u_2(x)-u_1(x)| \\
&= Max_{x\in J}\left|\int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\left[\int_0^t K(t,s)\Big(u_1(\lambda u_1(s))-u_0(\lambda u_0(s))\Big)ds\right]dt\right| \\
&\leq Max_{x\in J}\int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\left[\int_0^t |K(t,s)\Big(u_1(\lambda u_1(s))-u_0(\lambda u_0(s))\Big)ds|\right]dt \\
&\leq TU_1 \leq T^2.
\end{aligned}$$

Assume that result is true for n i.e. $U_n \leq TU_{n-1} \leq T^n$. Now, we show that result holds for $n+1$

$$\begin{aligned}
U_{n+1} &= Max_{x\in J}|u_{n+1}(x)-u_n(x)| \\
&= Max_{x\in J}\left|\int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\left[\int_0^t K(t,s)\Big(u_n(\lambda u_n(s))-u_{n-1}(\lambda u_{n-1}(s))\Big)ds\right]dt\right| \\
&\leq Max_{x\in J}\int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\left[\int_0^t |K(t,s)\Big(u_n(\lambda u_n(s))-u_{n-1}(\lambda u_{n-1}(s))\Big)ds|\right]dt \\
&\leq TU_n \leq T^{n+1}.
\end{aligned}$$

Thus by induction, we have $U_k \leq T^k$. Since

$$u_0 + \frac{T^\alpha(L+T^3k_T)}{\Gamma(\alpha+1)} \leq T,$$

we see that $T<1$ when $u_0 \geq 0$.

Hence U_k tends to zero as k tends to infinity. Since the family $\{U_k\}$ is the Arzela-Ascoli family thus for every subsequence $\{u_{kj}\}$ of $\{U_k\}$ there exists a subsequence $\{u_{kj}\}$ uniformly convergent and the limit needs to be a solution of the problem (1)–(2). Thus, the sequence $\{u_k\}$ tends uniformly to the unique solution of the problem (1)–(2).

THEOREM 3.3. Suppose that the hypotheses of the Theorem 3.1 hold. Let u_1 and u_2 satisfy the equation (1) for $0 \leq x \leq T$ with $u_1(0)=u_0^*$ and $u_2(0)=u_0^{**}$ respectively then

$$\|u_1(x)-u_2(x)\| \leq \|u_0^*-u_0^{**}\| \exp\left\{\frac{\lambda k_T(M+1)T^{\alpha+1}}{\Gamma(\alpha+1)}\right\} \quad \text{for } 0\leq x\leq T, \quad M>0.$$

PROOF. Making use of Theorem 3.1, we have

$$u_1(x) = u_0^* + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\left(f(t)+\int_0^t K(t,s)u_1(\lambda u_1(s))ds\right)dt$$

and
$$u_2(x) = u_0^{**} + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \left(f(t) + \int_0^t K(t,s) u_2(\lambda u_2(s)) ds \right) dt.$$
Then
$$\begin{aligned}
& |u_1(x) - u_2(x)| \\
\leq\ & |u_0^* - u_0^{**}| + \left| \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \int_0^t K(t,s) \left(u_1(\lambda u_1(s)) - u_2(\lambda u_2(s)) \right) dsdt \right| \\
\leq\ & |u_0^* - u_0^{**}| + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} k_T \int_0^t |u_1(\lambda u_1(s)) - u_2(\lambda u_2(s))| dsdt| \\
\leq\ & |u_0^* - u_0^{**}| \\
& + k_T \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \int_0^t |u_1(\lambda u_1(s)) - u_1(\lambda u_2(s)| + |u_1(\lambda u_2(s)) - u_2(\lambda u_2(s))| dsdt \\
\leq\ & |u_0^* - u_0^{**}| + \lambda k_T \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \int_0^t [M+1] |u_1(s) - u_2(s)| dsdt \\
\leq\ & |u_0^* - u_0^{**}| + \lambda k_T (M+1) \int_0^x \int_0^t \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} |u_1(s) - u_2(s)| dsdt \\
\leq\ & |u_0^* - u_0^{**}| + \lambda k_T (M+1) \int_0^x \int_s^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} |u_1(s) - u_2(s)| dtds \\
\leq\ & |u_0^* - u_0^{**}| + \lambda k_T (M+1) \int_0^x |u_1(s) - u_2(s)| \left(\int_s^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} dt \right) ds \\
\leq\ & |u_0^* - u_0^{**}| + \lambda k_T (M+1) \int_0^x |u_1(s) - u_2(s)| \frac{(x-s)^{\alpha}}{\Gamma(\alpha+1)} ds \\
\leq\ & |u_0^* - u_0^{**}| + \frac{\lambda k_T (M+1) T^{\alpha}}{\Gamma(\alpha+1)} \int_0^x |u_1(s) - u_2(s)| ds.
\end{aligned}$$
Using Gronwall-Bellman's inequality, we get
$$\begin{aligned}
|u_1(x) - u_2(x)| &\leq |u_0^* - u_0^{**}| exp \left(\int_0^x \frac{\lambda k_T (M+1) T^{\alpha}}{\Gamma(\alpha+1)} \right) ds \\
&\leq |u_0^* - u_0^{**}| exp \left(\frac{\lambda k_T (M+1) T^{\alpha+1}}{\Gamma(\alpha+1)} \right).
\end{aligned}$$
Hence, we have
$$\|u_1(x) - u_2(x)\| \leq \|u_0^* - u_0^{**}\| exp \left(\frac{\lambda k_T (M+1) T^{\alpha+1}}{\Gamma(\alpha+1)} \right).$$
This completes the proof of Theorem 3.3.

4 Applications

In this section, we give the applications of our main results established in previous section.

EXAMPLE 4.1. Consider the following nonlinear iterative fractional integrodifferential equation with deviating arguments

$$D^{\alpha}u(x) = 0.4 + \int_0^x u(\frac{1}{3}u(s))ds, \qquad 0 \le s \le x \le 0.5, \tag{6}$$

$$u(0) = 0. \tag{7}$$

Problem (6)–(7) is of the form (1)–(2) with $T = 0.5, L = 0.4, k_T = 1, \lambda = \frac{1}{3}, \alpha = \frac{1}{2}$ and $M = 1$ which satisfies

$$u_0 + \frac{T^{\alpha}(L + T^3 k_T)}{\Gamma(\alpha+1)} = 0 + \frac{0.5^{0.5}(0.4 + (0.5^3))}{\Gamma(3/2)} = \frac{0.71(0.4 + 0.125)}{0.88} = 0.423 < 0.5 = T$$

and

$$\frac{T^{\alpha+1}(M+1)\lambda k_T}{\Gamma(\alpha+1)} = \frac{0.5^{1.5}(1+1)\frac{1}{3}}{\Gamma(3/2)} = \frac{0.355(1+1)\frac{1}{3}}{0.88} = 0.27 < 1.$$

Since all the hypotheses of Theorem 3.1 are satisfied, therefore a unique solution of the equations (6)–(7) exists.

Making use of Theorem 3.2, we find the approximate solution of (6)–(7) for various values of α. Let $u_0(x) = 0$ be the first approximation, then for $\alpha = 1/2$

$$\begin{aligned}
u_1(x) &= u_0 + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\left(0.4 + \int_0^t u_o(\frac{1}{3}u_0(s))ds\right)dt \\
&= 0 + \int_0^x \frac{0.4+0}{\Gamma\frac{1}{2}\sqrt{x-t}}dt \\
&= 0.451352\sqrt{x}, \\
u_2(x) &= 0 + \int_0^x \frac{0.4 + \int_0^t u_1(\frac{1}{3}u_1(s))ds}{\Gamma\frac{1}{2}\sqrt{x-t}}dt = \frac{0.174874x^{7/4} + 0.8\sqrt{x}}{\sqrt{\pi}}.
\end{aligned}$$

Also, we can verify all the hypotheses of Theorem 3.1 for $\alpha = 1/3$ and we obtain

$$\begin{aligned}
u_1(x) &= u_0 + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\left(0.4 + \int_0^t u_o(\frac{1}{3}u_0(s))ds\right)dt \\
&= 0 + \int_0^x \frac{0.4+0}{\Gamma\frac{1}{3}(x-t)^{\frac{2}{3}}}dt \\
&= 0.447939x^{\frac{1}{3}}, \\
u_2(x) &= 0 + \int_0^x \frac{0.4 + \int_0^t u_1(\frac{1}{3}u_1(s))ds}{\Gamma\frac{1}{3}(x-t)^{\frac{2}{3}}}dt \\
&= \frac{0.471207x^{13/9} + 1.2\sqrt[3]{x}}{\Gamma\left(\frac{1}{3}\right)}.
\end{aligned}$$

which is the approximate solution of (6)–(7) up to third iteration.

EXAMPLE 4.2. Consider the following nonlinear iterative fractional integrodifferential equation with deviating arguments

$$D^{\alpha}u(x) = 0.2 + \int_0^t su(\frac{2}{3}u(s))ds, \qquad 0 \le s \le x \le 0.5, \tag{8}$$

$$u(0) = 0.25. \tag{9}$$

Here $T = 0.5, L = 0.2, M = 1, k_T = 0.5, \lambda = \frac{2}{3}$ and $\alpha = 0.5$, we have

$$\begin{aligned} u_0 + \frac{T^{\alpha}(L + T^3 k_T)}{\Gamma(\alpha+1)} &= 0.25 + \frac{0.5^{0.5}(0.2 + 0.5^3(0.5))}{\Gamma(3/2)} \\ &= 0.25 + \frac{0.71(0.2 + 0.0625)}{0.88} = 0.46 < 0.5 = T. \end{aligned}$$

Also

$$\frac{T^{\alpha+1}(M+1)\lambda k_T}{\Gamma(\alpha+1)} = \frac{0.5^{1.5}(1+1)\frac{2}{3}(0.5)}{\Gamma(3/2)} = \frac{0.355(1+1)(0.33)}{0.88} = 0.2689 < 1.$$

Since all the hypotheses of Theorem 3.1 are satisfied and therefore a unique solution of the equations (8)–(9) exists.

Making use of Theorem 3.2, we find the approximate solution of (8)–(9) for various values of α. Let $u_0(x) = 0.25$ be the first approximation, then for $\alpha = 1/2$

$$\begin{aligned} u_1(x) &= u_0 + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \left(0.2 + \int_0^t su_o(\frac{2}{3}u_0(s))ds\right)dt \\ &= 0.25 + \frac{0.133333x^{5/2} + 0.4\sqrt{x}}{\sqrt{\pi}} \end{aligned}$$

and for $\alpha = 1/3$, we have

$$\begin{aligned} u_1(x) &= u_0 + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \left(0.2 + \int_0^t su_o(\frac{2}{3}u_0(s))ds\right)dt \\ &= 0.25 + \frac{0.241071x^{7/3} + 0.6\sqrt[3]{x}}{\Gamma\left(\frac{1}{3}\right)}. \end{aligned}$$

which is the approximate solution of (8)–(9) up to second iteration.

5 Conclusion

In this study, we investigated the existence and uniqueness results for solution of iterative fractional integrodifferential equation with deviating arguments. The fractional derivatives are considered in the Caputo sense. We also utilized the Banach contraction fixed-point theorem.

All numerical results are obtained by using Mathematica 11.

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