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Riesz Idempotent And Weyl's Theorem For k-Quasi-$*$-Paranormal Operators*

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Abstract

An operator T on H is called a k-quasi-$*$-paranormal operator if

$$\|T^*T^kx\|^2 \leq \|T^{k+2}x\|\|T^kx\|,$$

for all $x \in H$, where k is a natural number. First, there will be seen some spectral properties for k-quasi-$*$-paranormal operator, examples and inclusions. It will also be seen if T is algebraically k-quasi-$*$-paranormal then T has finite ascent and T is polaroid operator. Second, it will be shown that the Riesz idempotent P_μ of every k-quasi-$*$-paranormal T with respect to each isolated point $\mu \neq 0$ of its spectrum $\sigma(T)$ is self-adjoint and satisfies $P_\mu(H) = \ker(T-\mu) = \ker(T-\mu)^*$, and if $\mu = 0$, then $P_\mu(H) = \ker(T^{k+1})$. Finally, it will be proved the generalized Weyl's theorem for $f(T)$ for every $f \in Hol(\sigma(T))$, if T is an algebraically k-quasi-$*$-paranormal operator. If T^* is an algebraically k-quasi-$*$-paranormal then $f(T)$ satisfies a-Weyl's theorem for every $f \in Hol(\sigma(T))$. Moreover, we show that if T is an algebraically k-quasi-$*$-paranormal operator, F is algebraic with $TF = FT$, then $f(T+F)$ satisfies the generalized Weyl's theorem for all $f \in Hol(\sigma(T+F))$.

1 Introduction

Throughout this paper, let H and K be infinite dimensional complex Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$. We denote by $L(H,K)$ the set of all bounded operators from H into K. To simplify, we put $L(H) := L(H,H)$. For $T \in L(H)$, we denote by $\ker T$ the null space and by $T(H)$ the range of T. The closure of a set M will be denoted by $\overline{M}$ and we shall henceforth shorten $T - \lambda I$ to $T - \lambda$. We shall denote the set of all complex numbers by $\mathbb{C}$, the set of all real numbers by $\mathbb{R}$ and the set of all non$-$negative integers by $\mathbb{N}$. An operator $T \in L(H)$, is a positive operator, $T \geq O$, if $\langle Tx, x \rangle \geq 0$ for all $x \in H$.

Let $T \in L(H)$. For an operator T, as usual, by T^* we mean the adjoint of T and $|T| = (T^*T)^{\frac{1}{2}}$. An operator T is said to be a hyponormal, if $|T|^2 \geq |T^*|^2$. An operator T

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is said to be a paranormal, if $\|T^2x\| \geq \|Tx\|^2$ for any unit vector x in H, [19]. Further, T is said to be a $*$-paranormal, if $\|T^2x\| \geq \|T^*x\|^2$ for any unit vector x in H, [7].

T. Furuta, M. Ito and T. Yamazaki [20] introduced a very interesting class of bounded linear Hilbert space operators: class $\mathcal{A}$ defined by $|T^2| \geq |T|^2$, and they showed that the class $\mathcal{A}$ is a subclass of paranormal operators. I. H. Jeon and I. H. Kim [23] introduced quasi$-$class $\mathcal{A}$ (i.e., $T^*|T^2|T \geq T^*|T|^2T$) operators as an extension of the notion of a class $\mathcal{A}$ operators. B. P. Dugall, I. H. Jeon, and I. H. Kim [17], introduced $*$-class $\mathcal{A}$ operator. An operator T is said to be a $*$-class $\mathcal{A}$ operator, if

$$|T^2| \geq |T^*|^2.$$

A $*$-class $\mathcal{A}$ is a generalization of a hyponormal operator, [17, Theorem 1.2], and $*$-class $\mathcal{A}$ is a subclass of the class of $*$-paranormal operators, [17, Theorem 1.3]. We denote the set of $*$-class $\mathcal{A}$ by $\mathcal{A}^*$. J. L. Shen, F. Zuo and S. C. Yang, in [28] introduced a quasi-$*$-class $\mathcal{A}$ operator: An operator T is said to be a quasi-$*$-class $\mathcal{A}$ operator, if

$$T^*|T^2|T \geq T^*|T^*|^2T.$$

We denote the set of quasi-$*$-class $\mathcal{A}$ by $\mathcal{Q}(\mathcal{A}^*)$.

2 Definition and Example

DEFINITION 2.1 ([21]). An operator $T \in L(H)$ is called a k-quasi-$*$-paranormal operator if

$$\|T^*T^kx\|^2 \leq \|T^{k+2}x\|\|T^kx\|,$$

for all $x \in H$, where k is a natural number.

This class of the operators, also is defined in paper [25]. If T is k-quasi-$*$-paranormal operator then T is a $(k+1)$-quasi-$*$-paranormal operator. The inverse is not true, as it can be seen below.

EXAMPLE 2.2. Consider the unilateral weighted shift operators as an infinite dimensional Hilbert space operator. Recall that given a bounded sequence of a positive numbers $\alpha : \alpha_1, \alpha_2, \alpha_3, \alpha_4, ...$ (called weights) the unilateral weighted shift W_α associated with weight α is the operator on $H = l_2$ defined by $W_\alpha e_n = \alpha_n e_{n+1}$ for all $n \geq 1$, where $\{e_n\}_{n=1}^\infty$ is the canonical orthonormal basis on l_2.

$$\mathbf{W}_{\boldsymbol{\alpha}} = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ \alpha_1 & 0 & 0 & 0 & \dots \\ 0 & \alpha_2 & 0 & 0 & \dots \\ 0 & 0 & \alpha_3 & 0 & \dots \\ 0 & 0 & 0 & \alpha_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

It is well known that the following assertions are equivalent:

1. W_α is a $*$-paranormal operator,
2. W_α is a $*$-class $\mathcal{A}$ operator,
3. $\alpha_n^2 \leq \alpha_{n+1}\alpha_{n+2}$ for all $n \geq 1$.

From [21, Proposition 2.1], W_α is a k-quasi-$*$-paranormal operator, if and only if,

$$W_\alpha^{*k}(W_\alpha^{*2}W_\alpha^2 - 2\lambda W_\alpha W_\alpha^* + \lambda^2)W_\alpha^k \geq O \text{ for all } \lambda \in \mathbb{R}.$$

Let $\operatorname{diag}(\{\alpha_n\}_{n=1}^\infty) = \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3, ...)$ denote an infinite diagonal matrix on l_2. Then,

$$\begin{aligned} & W_\alpha^{*k}\left(W_\alpha^{*2}W_\alpha^2 - 2\lambda W_\alpha W_\alpha^* + \lambda^2\right)W_\alpha^k \\ = \;& \operatorname{diag}(\{\alpha_n^2\alpha_{n+1}^2 \cdot ... \cdot \alpha_{n+k-1}^2\alpha_{n+k}^2\alpha_{n+k+1}^2\}_{n=1}^\infty) \\ - \;& 2\lambda\operatorname{diag}(\{\alpha_n^2\alpha_{n+1}^2 \cdot ... \cdot \alpha_{n+k-2}^2\alpha_{n+k-1}^2\alpha_{n+k-1}^2\}_{n=1}^\infty) \\ + \;& \lambda^2\operatorname{diag}(\{\alpha_n^2\alpha_{n+1}^2 \cdot ... \cdot \alpha_{n+k-1}^2\}_{n=1}^\infty) \end{aligned}$$

Thus, W_α is a k-quasi-$*$-paranormal operator, if and only if,

$$\alpha_n^2\alpha_{n+1}^2 \cdot ... \cdot \alpha_{n+k-1}^2\left(\alpha_{n+k}^2\alpha_{n+k+1}^2 - 2\lambda\alpha_{n+k-1}^2 + \lambda^2\right) \geq 0,$$

for all $\lambda \in \mathbb{R}$, and $n \geq 1$. Equivalently

$$\alpha_{n+k-1}^2 \leq \alpha_{n+k}\alpha_{n+k+1} \text{ for all } n \geq 1.$$

If $\alpha_{k+1} \leq \alpha_{k+2} \leq \alpha_{k+3} \leq \alpha_{k+4} \leq ...$ and $\alpha_k > \alpha_{k+1}$, then W_α is a $(k+1)$-quasi-$*$-paranormal but is not a k-quasi-$*$-paranormal operator.

We write $r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ for the spectral radius. It is well known that $r(T) \leq \|T\|$, for every $T \in L(H)$. The operator T is called normaloid operator if $r(T) = \|T\|$. It is well known that a $*$-paranormal operator is normaloid [7, Theorem 1.1], and a quasi-$*$-paranormal is normaloid, but a k-quasi-$*$-paranormal operator for $k \geq 2$ is not normaloid operator: if $\alpha_1 > \alpha_2$ and $\alpha_2 = \alpha_3 = ... = \alpha_k = \alpha_{k+1} = ...$, then

$$\|T\| = \alpha_1 \text{ and } r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = \alpha_2.$$

THEOREM 2.3. Let $T \in L(H)$ be a k-quasi-$*$-paranormal operator for a positive integer k. Then the following assertions hold.

1. $\|T^*T^n\|^2 \leq \|T^{n+2}\|\|T^n\|$ for all positive integers $n \geq k$,
2. If $\|T^n\| = \|T\|^n$ for some positive $n \geq k$, then T is normaloid operator.

PROOF. 1). Since k-quasi-$*$-paranormal operators are $(k+1)$-quasi-$*$-paranormal operators, we only need to prove the case $n = k$. It is clear by the definition of k-quasi-$*$-paranormal operators.

2). Let $n \geq k$. From 1) and using $\|T^n\| = \|T\|^n$ we have

$$\begin{aligned} \|T\|^{4n} &= \|T^n\|^4 = \|T^{*n}T^n\|^2 = \|T^{*(n-1)}T^*T^n\|^2 \\ &\leq \|T^{*(n-1)}\|^2\|T^*T^n\|^2 \leq \|T\|^{2(n-1)}\|T^{n+2}\|\|T^n\| \\ &= \|T\|^{2(n-1)}\|T^{n+2}\|\|T\|^n. \end{aligned}$$

Therefore, $\|T\|^{n+2} \leq \|T^{n+2}\|$ so $\|T\|^{n+1} = \|T^{n+1}\|$. Thus by induction we have $\|T\|^n = \|T^n\|$, for all $n \in \mathbb{N}$, hence T is normaloid operator.

DEFINITION 2.4. An operator T is called an algebraically k-quasi-$*$-paranormal operator, if there exists a nonconstant complex polynomial $h(z)$ such that $h(T)$ is a k-quasi-$*$-paranormal.

If T is a k-quasi-$*$-paranormal operator, then T is an algebraically k-quasi-$*$-paranormal operator. But the inverse is not true, as shown by the example below.

EXAMPLE 2.5. Let

$$T = \begin{pmatrix} I & O \\ I & I \end{pmatrix} \in L(l_2 \oplus l_2).$$

Since $T^* = \begin{pmatrix} I & I \\ O & I \end{pmatrix}$,

$$T^{*2}\left(T^{*2}T^2 - 2\lambda TT^* + \lambda^2\right)T^2 = \begin{pmatrix} (17 - 26\lambda + 5\lambda^2)I & (4 - 10\lambda + 2\lambda^2)I \\ (4 - 10\lambda + 2\lambda^2)I & (1 - 4\lambda + \lambda^2)I \end{pmatrix}.$$

For $\lambda = 1$, $(17 - 26\lambda + 5\lambda^2)I$ is not a positive operator, thus

$$T^{*2}\left(T^{*2}T^2 - 2\lambda TT^* + \lambda^2\right)T^2 \ngeq O$$

for all $\lambda \in \mathbb{R}$. Therefore T is not a 2-quasi-$*$-paranormal operator.

On the other hand, consider the non constant complex polynomial $h(z) = (z-1)^2$. Then $h(T) = O$, and hence T is an algebraically 2-quasi-$*$-paranormal operator.

LEMMA 2.6 ([13, Holder-McCarthy inequality]). Let T be a positive operator. Then, the following inequalities hold for all $x \in H$:

1. $\langle T^r x, x\rangle \leq \langle Tx, x\rangle^r \|x\|^{2(1-r)}$ for $0 < r < 1$,
2. $\langle T^r x, x\rangle \geq \langle Tx, x\rangle^r \|x\|^{2(1-r)}$ for $r \geq 1$.

LEMMA 2.7. If T is a k-quasi-$*$-class $\mathcal{A}$ operator, then T is a k-quasi-$*$-paranormal operator.

PROOF. Let T be a k-quasi-$*$-class $\mathcal{A}$ operator. From Holder-McCarthy inequality we have

$$\begin{aligned}\left\|T^*T^kx\right\|^2 &= \left\langle T^{*k}|T^*|^2T^kx, x\right\rangle \leq \left\langle |T^2|T^kx, T^kx\right\rangle \\ &\leq \left\langle |T^2|^2T^kx, T^kx\right\rangle^{\frac{1}{2}} \left\|T^kx\right\| = \|T^{k+2}x\|\|T^kx\|.\end{aligned}$$

So T is a k-quasi-$*$-paranormal operator.

LEMMA 2.8. Let $S = \oplus_{n=1}^{\infty} H_n$, where $H_n \cong \mathbb{R}^2$. For given positive operators A, B on $\mathbb{R}^2$ and for any fixed $n \in \mathbb{N}$, the operator $T = T_{A,B}$ on S is defined as follows:

$$T(x_1, x_2, ...) = (0, Ax_1, Bx_2, Bx_3, Bx_4, ...),$$

and the adjoint operator of T is

$$T^*(x_1, x_2, ...) = (Ax_2, Bx_3, Bx_4, Bx_5, ...).$$

Then

1. The operator $T_{A,B}$ is a quasi-$*$-class $\mathcal{A}$ operator, if and only if,
$$AB^2A \geq A^4,$$
2. The operator $T_{A,B}$ is a quasi-$*$-paranormal operator, if and only if,
$$A(B^4 - 2\lambda A^2 + \lambda^2)A \geq O, \text{ for all } \lambda \in \mathbb{R}.$$

EXAMPLE 2.9. A non quasi-$*$-class $\mathcal{A}$ operator, quasi-$*$-paranormal operator.

Take A and B as

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{\frac{1}{2}} \text{ and } B = \begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix}^{\frac{1}{4}}.$$

Then

$$A(B^2 - A^2)A = \begin{pmatrix} -0.3359... & -0.2265... \\ -0.2265... & 0.8244... \end{pmatrix} \ngeq O,$$

hence $T_{A,B}$ is not a quasi-$*$-class $\mathcal{A}$ operator. But,

$$A(B^4 - 2\lambda A^2 + \lambda^2)A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{\frac{1}{2}} \begin{pmatrix} (1-\lambda)^2 & 2(1-\lambda) \\ 2(1-\lambda) & \lambda^2 - 4\lambda + 8 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{\frac{1}{2}} \geq O,$$

so, $T_{A,B}$ is a quasi-$*$-paranormal operator.

3 Spectral Properties

We write $\sigma(T)$, $\sigma_p(T)$ and $\sigma_a(T)$ for the spectrum, point spectrum and approximate point spectrum, respectively. Sets of isolated points and accumulation points of $\sigma(T)$ are denoted by iso$\sigma(T)$ and acc$\sigma(T)$, respectively.

A complex number μ is said to be in the point spectrum $\sigma_p(T)$ of T if there is a nonzero $x \in H$ such that $(T-\mu)x = 0$. If in addition, $(T-\mu)^*x = 0$, then μ is said to be in the joint point spectrum $\sigma_{jp}(T)$ of T. Clearly $\sigma_{jp}(T) \subseteq \sigma_p(T)$. In general $\sigma_{jp}(T) \neq \sigma_p(T)$.

LEMMA 3.1 ([21, Proposition 3.1]). If T is a k-quasi-$*$-paranormal operator and $(T-\mu)x = 0$, then $(T-\mu)^*x = 0$ for all $\mu \neq 0$.

A complex number μ is said to be in the approximate point spectrum $\sigma_a(T)$ of T if there is a sequence $\{x_m\}_{m=1}^{\infty} \subset H$ of unit vectors satisfying $(T-\mu)x_m \to 0$, as $m \to \infty$. If in additions $(T-\mu)^*x_m \to 0$, as $m \to \infty$, then μ is said to be in the joint approximate point spectrum $\sigma_{ja}(T)$ of operator T. Clearly $\sigma_{ja}(T) \subseteq \sigma_a(T)$. In general $\sigma_{ja}(T) \neq \sigma_a(T)$.

THEOREM 3.2. Let T be a k-quasi-$*$-paranormal operator, and $(T-\mu)x_m \to 0$, as $m \to \infty$ for $\mu \neq 0$. Then $(T-\mu)^*x_m \to 0$, as $m \to \infty$.

PROOF. Let T be a k-quasi-$*$-paranormal operator and $(T-\mu)x_m \to 0$, as $m \to \infty$. We may assume that $\|x_m\| = 1$. By the assumption and using

$$T^k = (T-\mu+\mu)^k = \sum_{i=1}^{k} \binom{k}{i} \mu^{k-i}(T-\mu)^i + \mu^k, \text{ for } k \in \mathbb{N},$$

we have $(T^k - \mu^k)x_m \to 0$, as $m \to \infty$. By

$$|\|T^k x_m\| - |\mu|^k| \leq \|(T^k - \mu^k)x_m\|$$

hence

$$\|T^k x_m\| \to |\mu|^k, \text{ as } m \to \infty. \tag{1}$$

Moreover

$$\big|\|T^*\mu^k x_m\| - \|T^*(T^k-\mu^k)x_m\|\big| \leq \|T^*T^k x_m\|. \tag{2}$$

Since T is a k-quasi-$*$-paranormal operator, we have

$$\|T^*T^k x_m\| \leq \|T^{2+k}x_m\|^{\frac{1}{2}}\|T^k x_m\|^{\frac{1}{2}}. \tag{3}$$

Then it follows from (1), (2) and (3) that

$$\limsup_{m\to\infty} \|T^* x_m\| \leq |\mu|.$$

Since

$$\begin{aligned} & \|(T-\mu)^* x_m\|^2 \\ = & \|T^* x_m\|^2 - 2\mathrm{Re}\langle T^* x_m, \overline{\mu} x_m\rangle + |\mu|^2\|x_m\|^2 \\ = & \|T^* x_m\|^2 - 2\mathrm{Re}\langle x_m, \overline{\mu} T x_m\rangle + |\mu|^2\|x_m\|^2, \end{aligned}$$

we see that

$$\limsup_{m\to\infty} \|(T-\mu)^* x_m\|^2 \leq |\mu|^2 - |\mu|^2 = 0.$$

This implies $(T-\mu)^* x_m \to 0$, as $m \to \infty$.

COROLLARY 3.3. If T is a k-quasi-$*$-paranormal operator, then $\sigma_{ja}(T) \setminus \{0\} = \sigma_a(T) \setminus \{0\}$.

LEMMA 3.4 ([5, Corollary 2]). Let $T = U|T|$ be the polar decomposition of T, $\mu = |\mu| e^{i\theta} \neq 0$ and $\{x_m\}$ a sequence of vectors. Then the following assertions are equivalent:

1. $(T-\mu)x_m \to 0$ and $(T^* - \overline{\mu})x_m \to 0$, as $m \to \infty$,
2. $(|T| - |\mu|)x_m \to 0$ and $(U - e^{i\theta})x_m \to 0$, as $m \to \infty$,
3. $(|T^*| - |\mu|)x_m \to 0$ and $(U^* - e^{-i\theta})x_m \to 0$, as $m \to \infty$.

COROLLARY 3.5. If T is a k-quasi-$*$-paranormal operator and $\mu \in \sigma_a(T) \setminus \{0\}$, then $|\mu| \in \sigma_a(|T|) \cap \sigma_a(|T^*|)$.

PROOF. If $\mu \in \sigma_a(T) \setminus \{0\}$, then by Theorem 3, there exists a sequence of unit vectors $\{x_m\}$ such that $(T-\mu)x_m \to 0$ and $(T-\mu)^* x_m \to 0$, as $m \to \infty$. Hence, from Lemma 3 we have $|\mu| \in \sigma_a(|T|) \cap \sigma_a(|T^*|)$.

COROLLARY 3.6. Let T be a k-quasi-$*$-paranormal operator and $T = U|T|$ is the polar decomposition of T. If $\mu = |\mu| e^{i\theta} \neq 0$ and $\mu \in \sigma_a(T)$, then $e^{i\theta} \in \sigma_{ja}(U)$.

PROOF. Let $\mu \in \sigma_a(T)$. From Corollary 3, $\mu \in \sigma_{ja}(T)$. Then there exists a sequence of unit vectors $\{x_m\}$ such that $(T-\mu)x_m \to 0$ and $(T-\mu)^* x_m \to 0$, as $m \to \infty$. From Lemma 3 we have $(U - e^{i\theta})x_m \to 0$ and $(U^* - e^{-i\theta})x_m \to 0$, as $m \to \infty$. Thus $e^{i\theta} \in \sigma_{ja}(U)$.

LEMMA 3.7 ([21, Proposition 2.4]). Let $T \in L(H)$ be a k-quasi-$*$-paranormal operator, the range of T^k not to be dense, and

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \quad \text{on} \quad H = \overline{T^k(H)} \oplus \ker T^{*k}.$$

Then, A is a $*$-paranormal on $\overline{T^k(H)}$, $C^k = 0$ and $\sigma(T) = \sigma(A) \cup \{0\}$.

THEOREM 3.8. Let T be a k-quasi-$*$-paranormal operator and $\sigma(T) = \{\mu\}$. Then $T = \mu$ if $\mu \neq 0$, and $T^{k+1} = O$ if $\mu = 0$.

PROOF. Let's suppose that T is a k-quasi-$*$-paranormal operator. We can consider two cases:

Case I: If $\mu \neq 0$, the range of T^k is dense, then it is a $*$-paranormal operator. Hence by [30, Corollary 1], $T = \mu$.

Case II: If $\mu = 0$, T does not have dense range, by Lemma 3 we can represent T as the upper triangular matrix

$$T = \begin{pmatrix} A & B \\ O & C \end{pmatrix} \quad \text{on } H = \overline{T^k(H)} \oplus \ker T^{*k}.$$

From the assumption, $\sigma(T) = \{0\}$ and from Lemma 3 we have $\sigma(A) = \{0\}$. Since A is a $*$-paranormal operator, $A = O$ and we have

$$T^{k+1} = \begin{pmatrix} O & BC^k \\ O & C^{k+1} \end{pmatrix} = O.$$

THEOREM 3.9. If T is a quasinilpotent algebraically k-quasi-$*$-paranormal operator, then T is a nilpotent operator.

PROOF. Let T be an algebraically k-quasi-$*$-paranormal operator. Then, there exists a nonconstant polynomial $h(z)$ such that $h(T)$ is a k-quasi-$*$-paranormal operator. If $h(T)^k(H)$ is dense, $h(T)$ is a $*$-paranormal operator. Therefore T is an algebraically $*$-paranormal operator and by [35, Theorem 2.6] T is a nilpotent operator. If $h(T)^k(H)$ is not dense, by Lemma 3 we have

$$h(T) = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \quad \text{on } \; H = \overline{h(T)^k(H)} \oplus \ker h(T)^{*k},$$

where A is a $*$-paranormal operator on $\overline{h(T)^k(H)}$, $C^k = O$ and $\sigma(h(T)) = \sigma(A) \cup \{0\}$. Since T is a quasinilpotent operator, $\sigma(h(T)) = h(\sigma(T)) = h(0)$. Therefore $\sigma(A) = \{0\}$, thus $\sigma(h(T)) = \{0\}$. Since $h(0) = 0$, we have $h(T) = aT^k \prod_{i=1}^{n}(T - \mu_i)$ for some natural number k and a complex number μ_i, $i = 1, 2, ..., n$. By Theorem 3 we have

$$a^{k+1}T^{k(k+1)} \prod_{i=1}^{n}(T - \mu_i)^{k+1} = O.$$

Since $\sigma(T) = \{0\}$, $T - \mu_i$ is an invertible for all $i = 1, 2, ..., n$, we see that

$$T^{k(k+1)} = O.$$

For $T \in L(H)$, the smallest nonnegative integer p such that $\ker T^p = \ker T^{p+1}$ is called the ascent of T and is denoted by $p(T)$. If no such integer exists, we set $p(T) = \infty$. We say that $T \in L(H)$ is of finite ascent if $p(T - \mu) < \infty$, for all $\mu \in \mathbb{C}$. For $T \in L(H)$, the smallest nonnegative integer q, such that $T^q(H) = T^{q+1}(H)$, is called the descent of T and is denoted by $q(T)$. If no such integer exists, we set $q(T) = \infty$. We say that $T \in L(H)$ is of finite descent if $q(T - \mu) < \infty$, for all $\mu \in \mathbb{C}$.

THEOREM 3.10. If T is an algebraically k-quasi-$*$-paranormal operator, then $T - \mu$ has finite ascent for all $\mu \in \mathbb{C}$.

PROOF. Let T be an algebraically k-quasi-$*$-paranormal operator. Then, there exists a nonconstant polynomial $h(z)$ such that $h(T)$ is a k-quasi-$*$-paranormal operator and we have

$$h(T) - h(\mu) = a(T - \mu)^k \prod_{i=1}^{n}(T - \mu_i),$$

where $a \neq 0$, $\mu_i \neq \mu$ and integers k and n. Let $x \neq 0$. We consider two cases:

I. If $x \in \ker(T - \mu)^{k+1}$ and $h(\mu) \neq 0$, we have

$$(h(T) - h(\mu))x = a(T - \mu)^k \prod_{i=1}^{n}(T - \mu + \mu - \mu_i)x = a \prod_{i=1}^{n}(\mu - \mu_i)(T - \mu)^k x. \tag{4}$$

Hence

$$(h(T) - h(\mu))^2 x = a^2 \prod_{i=1}^{n}(\mu - \mu_i)^2 (T - \mu)^{2k} x = 0.$$

From [21, Proposition 3.1] we have $x \in \ker(h(T) - h(\mu))^2 = \ker(h(T) - h(\mu))$. Hence $(h(T) - h(\mu))x = 0$, and from relation (4) we have $(T - \mu)^k x = 0$, so $x \in \ker(T - \mu)^k$.

II. If $h(\mu) = 0$ we have

$$h(T)^k x = a^k \prod_{i=1}^{n}(\mu - \mu_i)^k (T - \mu)^{k^2} x = b^{-k}(T - \mu)^{k^2} x. \tag{5}$$

and

$$\begin{aligned}
& \|(T - \mu)^{k^2} x\|^4 \\
= \ & \langle (T - \mu)^{k^2} x, (T - \mu)^{k^2} x \rangle^2 \\
= \ & \langle b^k h(T)^k x, b^k h(T)^k x \rangle^2 \\
= \ & |b|^{4k} \langle h(T)^* h(T)^k x, h(T)^{k-1} x \rangle^2 \\
\leq \ & |b|^{4k} \|h(T)^* h(T)^k x\|^2 \|h(T)^{k-1} x\|^2 \\
\leq \ & |b|^{4k} \|h(T)^{k+2} x\| \|h(T)^{k-1} x\|^2 \|h(T)^k x\| \\
= \ & |b|^{k-1} \|(T - \mu)^{k^2+2k} x\| \|(T - \mu)^{k^2-k} x\| \|(T - \mu)^{k^2} x\| = 0.
\end{aligned}$$

So,

$$\|(T - \mu)^{k^2} x\|^3 \leq |b|^{k-1} \|(T - \mu)^{k^2+2k} x\| \|(T - \mu)^{k^2-k} x\|.$$

If $x \in \ker(T - \mu)^{k^2+1}$, therefore $\ker(T - \mu)^{k^2} = \ker(T - \mu)^{k^2+1}$.

Let $Hol(\sigma(T))$ be the space of all analytic functions in an open neighborhood of $\sigma(T)$. We say that $T \in L(H)$ has the single valued extension property at $\mu \in \mathbb{C}$, if for every open neighborhood U of μ the only analytic function $f : U \to \mathbb{C}$ which satisfies the equation $(T-\mu)f(\mu) = 0$, is the constant function $f \equiv 0$. The operator T is said to have SVEP if T has SVEP at every $\mu \in \mathbb{C}$.

COROLLARY 3.11. If $T \in L(H)$ is an algebraically k-quasi-$*$-paranormal operator, then T has SVEP.

PROOF. The proof of the corollary follows directly from Theorem 3 and [1, Theorem 3.39].

The quasinilpotent part $\mathcal{H}_0(T-\mu)$ and analytic core $K(T-\mu)$ of $T-\mu$ are defined by

$$\mathcal{H}_0(T-\mu) = \{x \in H : \lim_{n\to\infty} \|(T-\mu)^n x\|^{\frac{1}{n}} = 0\},$$

and

$$\begin{aligned} & K(T-\mu) \\ = \ & \{x \in H : \text{ there exists a sequence } \{x_n\} \subset H \text{ and } \delta > 0 \text{ for which} \\ & x = x_0, (T-\mu)x_{n+1} = x_n \text{ and } \|x_n\| \leq \delta^n \|x\| \text{ for all } n = 1, 2, ...\}. \end{aligned}$$

Clearly $\mathcal{H}_0(T-\mu)$ and $K(T-\mu)$ are linear subspaces of H, in general $\mathcal{H}_0(T-\mu)$ and $K(T-\mu)$ are non-closed hyperinvariant subspaces of $T-\mu$, such that $\ker(T-\mu) \subseteq \mathcal{H}_0(T-\mu)$.

An operator T is said to be a semi-regular if $T(H)$ is a closed subspace and $\ker T \subseteq \cap_{n\in\mathbb{N}} T^n(H)$. An operator T admits a generalized Kato decomposition, if there exists a pair of T-invariant closed subspaces (M, N) such that $H = M \oplus N$, the restriction $T|_M$ is a quasinilpotent and $T|_N$ is a semi-regular operator. If $T|_M$ is a nilpotent, we say T is a Kato type.

An operator T is said to be isoloid operator if every isolated point of $\sigma(T)$ is an eigenvalue of T, while an operator T is said to be polaroid if every isolated point of $\sigma(T)$ is a pole of the resolvent of T. In general, if T is polaroid operator, then T is isoloid operator. However, the converse is not true.

THEOREM 3.12. If T is an algebraically k-quasi-$*$-paranormal operator, then T and T^* are polaroid operator.

PROOF. Let $\mu \in \text{iso}\sigma(T)$. From [2, Theorem 3.76] we have $H = \mathcal{H}_0(T-\mu) \oplus K(T-\mu)$, where $\mathcal{H}_0(T-\mu)$ and $K(T-\mu)$ are closed subspaces. By [1, Theorem 1.28], $(T-\mu)(K(T-\mu)) = K(T-\mu)$ is a closed subspace and $\ker(T-\mu) \subseteq \cap_{n\in\mathbb{N}}(T-\mu)^n(K(T-\mu))$, thus $(T-\mu)|_{K(T-\mu)}$ is a semi$-$regular operator. We have $\sigma(T|_{\mathcal{H}_0(T-\mu)}) = \{\mu\}$, then $\sigma((T-\mu)|_{\mathcal{H}_0(T-\mu)}) = \{0\}$, thus $(T-\mu)|_{\mathcal{H}_0(T-\mu)}$ is quasinilpotent operator. Therefore $T-\mu$ admits a generalized Kato decomposition. But, $T-\mu$ is an algebraically k-quasi-$*$-paranormal operator, by Theorem 3 $(T-\mu)|_{\mathcal{H}_0(T-\mu)}$ is a nilpotent operator, thus $T-\mu$ admits a Kato type. Since $\sigma(T)$ does not cluster at μ, then T and T^* have the SVEP in μ. From [1, Theorem 2.45 and Theorem 2.46] we have $p(T-\mu) < \infty$ and $q(T-\mu) < \infty$. Hence μ is a pole of the resolvent of T, so T is polaroid operator, therefore T is isoloid operator. From [5, Theorem 2.5], T^* is polaroid operator.

An operator T is called a-isoloid if every isolated point of $\sigma_a(T)$ is an eigenvalue of T. An operator T is called a-polaroid if every isolated point of $\sigma_a(T)$ is a pole of the resolvent of T. Clearly, if T is a-polaroid, then T is $a-$isoloid. However, the converse is not true

LEMMA 3.13. Suppose T^* is an algebraically k-quasi-$*$-paranormal operator. Then T is a-polaroid.

PROOF. Let μ be an isolated point of $\sigma_a(T)$. Since T^* has SVEP, by [1, Corollary 2.28] μ is an isolated point of $\sigma(T)$. But, if T^* is polaroid, then T is also polaroid. Therefore, T is a-polaroid operator.

4 Riesz Idempotent for k-Quasi-$*$-Paranormal Operator

The Riesz idempotent P_μ of an operator T with respect to an isolated point μ of $\sigma(T)$ is defined by

$$P_\mu = \frac{1}{2\pi i}\int_{\partial D_\mu}(z-T)^{-1}dz,$$

where the integral is taken in the positive direction and D_μ is a closed disk centered at μ with a small enough radius r such as $D_\mu \cap \sigma(T) = \{\mu\}$. Then, it is well known that $P_\mu^2 = P_\mu$, $TP_\mu = P_\mu T$, $\sigma(T|_{P_\mu(H)}) = \{\mu\}$ and $\sigma(T|_{(I-P_\mu)(H)}) = \sigma(T)\setminus\{\mu\}$.

In general, it is well known that the Riesz idempotent P_μ is not an orthogonal projection, and a necessary and sufficient condition for P_μ to be orthogonal is that P_μ is self-adjoint, [15]. For a hyponormal operator in [29], Stampfli has shown that the Riesz idempotent for an isolated point of spectrum of T is self-adjoint and

$$P_\mu(H) = \ker(T-\mu) = \ker(T-\mu)^*.$$

In [31], Uchiyama extended this result for the class $\mathcal{A}$ with respect $\mu \neq 0$ and he proved that in general, the Riesz idempotent of the class $\mathcal{A}$ with respect to 0 is not self-adjoint and $\ker T \neq \ker T^*$. In [22], Jeon and Kim extended this result for $\mu \neq 0$ in quasi-class $\mathcal{A}$. Also, in [24], Mecheri extended this result for $\mu \neq 0$ in k-quasi-$*$-class $\mathcal{A}$ operators. In this paper, we extended this result for k-quasi-$*$-paranormal operator.

THEOREM 4.1. Let $T \in L(H)$ be a k-quasi-$*$-paranormal operator for the positive integer k, and let μ be an isolated point of $\sigma(T)$, and P_μ the Riesz idempotent for μ. Then, the following assertions hold:

1. If $\mu \neq 0$, $P_\mu(H) = \ker(T-\mu) = \ker(T-\mu)^*$, and P_μ is self-adjoint.
2. If $\mu = 0$, then $P_\mu(H) = \ker(T^{k+1})$

PROOF. 1). Let T be a k-quasi-$*$-paranormal operator and $\mu \neq 0 \in \text{iso}\sigma(T)$. From Theorem 3 μ is an eigenvalue of T, thus $(T-\mu)x = 0$, for every $x \neq 0 \in H$. Then $x \in$

$\ker(T-\mu)^m x = P_\mu(H)$, hence $\ker(T-\mu) \subseteq P_\mu(H)$. On the other hand, $\sigma(T|_{P_\mu(H)}) = \{\mu\}$. From [21, Proposition 2.2], $T|_{P_\mu(H)}$ is a k-quasi-$*$-paranormal operator and by Theorem 3, $T|_{P_\mu(H)} = \mu$. If $x \in P_\mu(H)$, then $Tx = \mu x$, hence $x \in \ker(T-\mu)$. Therefore $P_\mu(H) = \ker(T-\mu)$.

Next, we show that $\ker(T-\mu) = \ker(T-\mu)^*$. Since $P_\mu(H) = \ker(T-\mu)$, we have $\ker(T-\mu)$ is a reducing subspace of T and T can be written as follows

$$T = \mu \oplus T_1 \text{ on } H = \ker(T-\mu) \oplus \ker(T-\mu)^\perp,$$

where T_1 is a k-quasi-$*$-paranormal operator and $\sigma(T) = \{\mu\} \cup \sigma(T_1)$.

If $\mu \in \sigma(T_1)$ then μ is an isolated point of $\sigma(T_1)$. Since T_1 is a k-quasi-$*$-paranormal operator, $\mu \in \sigma_p(T_1)$, thus $\ker(T_1-\mu) \neq \{0\}$. From $\ker(T_1-\mu) \subseteq \ker(T-\mu)$, and $\ker(T_1-\mu) \subseteq \ker(T-\mu)^\perp$, we have:

$$\{0\} \neq \ker(T_1-\mu) \subseteq \ker(T-\mu) \cap \ker(T-\mu)^\perp = \{0\},$$

which is a contradiction. Thus $\mu \notin \sigma(T_1)$ and $T_1 - \mu$ is invertible in $\ker(T-\mu)^\perp$. Therefore $(T-\mu)(\ker(T-\mu)^\perp) = \ker(T-\mu)^\perp$, so $\ker(T-\mu)^\perp \subseteq (T-\mu)(H)$. By Lemma 3 we have $\ker(T-\mu) \subseteq \ker(T-\mu)^* = (T-\mu)(H)^\perp$, therefore

$$(T-\mu)(H) \subseteq \ker(T-\mu)^\perp \subseteq (T-\mu)(H).$$

Thus $(T-\mu)(H) = \ker(T-\mu)^\perp$, which implies that

$$\ker(T-\mu)^* = (T-\mu)(H)^\perp = \ker(T-\mu).$$

Finally, we show that P_μ is a self-adjoint operator. From

$$P_\mu(H) = \ker(T-\mu) = \ker(T-\mu)^*,$$

we have $T|_{P_\mu(H)} = \mu$. Thus, $((z-T)^*)^{-1} P_\mu = \overline{(z-\mu)^{-1}} P_\mu$ and we have

$$P_\mu^* P_\mu = -\frac{1}{2\pi i}\int_{\partial D_\mu} ((z-T)^*)^{-1} P_\mu dz$$

$$= -\frac{1}{2\pi i}\int_{\partial D_\mu} \overline{(z-\mu)^{-1}} P_\mu dz = \overline{\frac{1}{2\pi i}\int_{\partial D_\mu} (z-\mu)^{-1} dz P_\mu} = P_\mu.$$

So $P_\mu^* P_\mu = P_\mu = P_\mu^2$, thus $P_\mu^* = P_\mu$.

2). Since $\ker T^k \subseteq P_0(H)$, we have to prove that $P_0(H) \subseteq \ker T^{k+1}$. It is known that $P_0(H)$ is an invariant subspace of T and $\sigma(T|_{P_0(H)}) = \{0\}$. From Theorem 3 we have $(T|_{P_0(H)})^{k+1} = T^{k+1}|_{P_0(H)} = O$. This implies $P_0(H) \subseteq \ker T^{k+1}$.

5 Generalized Weyl's Theorem for k-Quasi-$*$-Paranormal Operator

We write $\alpha(T) = \dim\ker T$ and $\beta(T) = \dim(H/T(H))$. An operator $T \in L(H)$ is called an upper semi-Fredholm, if it has a closed range and $\alpha(T) < \infty$, while T is called a lower semi-Fredholm if $\beta(T) < \infty$. However, T is called a semi-Fredholm operator, if T is either an upper or a lower semi-Fredholm, and T is said to be a Fredholm operator if it is both an upper and a lower semi-Fredholm. If $T \in L(H)$ is semi-Fredholm, then the index is defined by

$$\text{ind}(T) = \alpha(T) - \beta(T).$$

An operator $T \in L(H)$ is said to be an upper semi$-$Weyl operator if it is an upper semi-Fredholm and $\text{ind}(T) \leq 0$, while $T \in L(H)$ is said to be a lower semi$-$Weyl operator if it is a lower semi-Fredholm operator and $\text{ind}(T) \geq 0$. An operator is said to be a Weyl operator if it is a Fredholm operator of index zero. The Weyl spectrum and the essential approximate spectrum are defined by

$$\sigma_w(T) = \{\mu \in \mathbb{C} : T - \mu \text{ is not Weyl}\}$$

and

$$\sigma_{uw}(T) = \{\mu \in \mathbb{C} : T - \mu \text{ is not upper semi-Weyl}\}.$$

For $T \in L(H)$ and a nonnegative integer n we define $T_{[n]}$ to be the restriction of T to $T^n(H)$ viewed as a map from $T^n(H)$ into $T^n(H)$, (in particular $T_{[0]} = T$.)

DEFINITION 5.1 ([11]). We say that $T \in L(H)$

1. is B-Fredholm operator [B-Weyl], if for some integer $n \geq 0$ the range space $T^n(H)$ is a closed and $T_{[n]} = T \mid_{T^n(H)}: T^n(H) \to T^n(H)$ is a Fredholm operator[Weyl operator].
2. is upper(lower) semi-B-Fredholm operator if for some integer $n \geq 0$ the range space $T^n(H)$ is a closed and $T_{[n]} = T \mid_{T^n(H)}: T^n(H) \to T^n(H)$ is upper (resp. lower) semi-Fredholm operator.
3. is upper semi-B-Weyl if T is upper semi-B-Fredholm and $ind(T) \leq 0$.

The B-Weyl spectrum is defined by

$$\sigma_{BW}(T) = \{\mu \in \mathbb{C} : T - \mu \text{ is not B-Weyl}\}$$

while the upper semi-B-Weyl spectrum defined by

$$\sigma_{UBW}(T) = \{\mu \in \mathbb{C} : T - \mu \text{ is not upper semi-B-Weyl}\}.$$

For $T \in L(H)$ we write $\Pi_{00}(T) = \{\mu \in \text{iso}\sigma(T) : 0 < \alpha(T - \mu)\}$ for the set of all eigenvalues of T which are isolated in $\sigma(T)$, and $\pi_{00}(T) = \{\mu \in \text{iso}\sigma(T) : 0 < \alpha(T - \mu) < \infty\}$ for the set of all isolated eigenvalues of finite multiplicity in $\sigma(T)$.

We say that T satisfies the generalized Weyl's theorem [10] if

$$\sigma(T) \setminus \sigma_{BW}(T) = \Pi_{00}(T),$$

and we say that T satisfies Weyl's theorem [14], if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T).$$

In [33], H. Weyl proved that Weyl's theorem holds for hermitian operators. Weyl's theorem has been extended from hermitian operators to hyponormal operators from Coburn in [14]. M. Berkani investigated the generalized Weyl's theorem which extends Weyl's theorem, and proved that the generalized Weyl's theorem holds for normal operators [10] and hypernormal operators [12].

THEOREM 5.2. If $T \in L(H)$ is an algebraically k-quasi-$*$-paranormal operator, then $f(T)$ satisfies the generalized Weyl's theorem for every $f \in Hol(\sigma(T))$.

PROOF. Let $\mu \in \Pi_{00}(T)$. Then μ is an isolated point in the spectrum $\sigma(T)$. Using the spectral projection $P_\mu = \frac{1}{2\pi i}\int_{\partial D_\mu}(T-\mu)^{-1}d\mu$, where D_μ is a closed disk of center μ which contains no other points of $\sigma(T)$, we can represent T as the direct sum

$$T = T_1 \oplus T_2, \text{ where } \sigma(T_1) = \{\mu\} \text{ and } \sigma(T_2) = \sigma(T) \setminus \{\mu\}.$$

From Theorem 3, μ is a pole of the resolvent of T, there exists a positive integer $p = p(\mu)$ such that $T_1 - \mu = (T-\mu)|_{P(H)=\ker(T-\mu)^p}$ and $T_2 - \mu = (T-\mu)|_{\ker P=(T-\mu)^p(H)}$. So $(T-\mu)^p(H)$ is a closed subspace. From Theorem 3, $T-\mu$ has finite ascent for all $\mu \in \mathbb{C}$, then $(T-\mu)^n(H) = (T-\mu)^p(H)$ is a closed for all integers $n \geq p$. By [3, Theorem 2.8] T satisfies the generalized Weyl's theorem. By [34, Theorem 2.1], $f(\sigma_{BW}(T)) = \sigma_{BW}(f(T))$ for all $f \in Hol(\sigma(T))$, since T has SVEP. Since T is an isoloid operator from [16, Lemma 3.3],

$$f(\sigma(T) \setminus \Pi_{00}(T)) = \sigma(f(T)) \setminus \Pi_{00}(f(T)),$$

and

$$\sigma(f(T)) \setminus \Pi_{00}(f(T)) = f(\sigma(T) \setminus \Pi_{00}(T)) = f(\sigma_{BW}(T)) = \sigma_{BW}(f(T)),$$

which implies that $f(T)$ satisfies the generalized Weyl's theorem.

From [9, Theorem 3.9], we know that: generalized Weyl's theorem $\Longrightarrow$ Weyl's theorem.

COROLLARY 5.3. If T is an algebraically k-quasi-$*$-paranormal then $f(T)$ satisfies Weyl's theorem for every $f \in Hol(\sigma(T))$.

THEOREM 5.4. If $T^* \in L(H)$ is an algebraically k-quasi-$*$-paranormal operator, then $f(T)$ satisfies the generalized Weyl's theorem for every $f \in Hol(\sigma(T))$.

PROOF. Let $\mu \in \Pi_{00}(T)$. So μ is an isolated point of $\sigma(T)$. By Theorem 3, T^* is polaroid operator, hence T is polaroid operator. Thus, μ is a pole of the resolvent of T. There exists a positive integer $p = p(\mu)$ such that $p = p(T-\mu) = q(T-\mu)$. Then $(T-\mu)^p(H) = (T-\mu)^{(p+1)}(H)$ and $(T-\mu)^n(H)$ is closed for every $n \geq p$. By [3, Theorem 2.8] T satisfies the generalized Weyl's theorem. Since T^* has SVEP, $f(\sigma_{BW}(T)) = \sigma_{BW}(f(T))$ for all $f \in Hol(\sigma(T))$, from [34, Theorem 2.1]. By Theorem 3, T is polaroid operator, hence T is isoloid operator. From [16, Lemma 3.3],

$$f(\sigma(T) \setminus \Pi_{00}(T)) = \sigma(f(T)) \setminus \Pi_{00}(f(T)),$$

and

$$\sigma(f(T)) \setminus \Pi_{00}(f(T)) = f(\sigma(T) \setminus \Pi_{00}(T)) = f(\sigma_{BW}(T)) = \sigma_{BW}(f(T)),$$

which implies that $f(T)$ satisfies the generalized Weyl's theorem. For $T \in L(H)$ we write $P_{00}(T) = \{\mu \in \sigma(T) : 0 < p(T-\mu) = q(T-\mu) < \infty\}$ for the set of all pole of resolvent of T, and $p_{00}(T) = \{\mu \in P_{00}(T) : \alpha(T-\mu) < \infty\}$ for the set of all pole of finite rank of resolvent of T.

We say that T satisfies the generalized Browder's theorem if

$$\sigma(T) \setminus \sigma_{BW}(T) = P_{00}(T),$$

and we say that T satisfies Browder's theorem, if

$$\sigma(T) \setminus \sigma_w(T) = p_{00}(T).$$

COROLLARY 5.5. If $T \in L(H)$ is an algebraically k-quasi-$*$-paranormal operator, then $f(T)$ satisfies the generalized Browder's theorem for every $f \in Hol(\sigma(T))$.

PROOF. Let T be an algebraically k-quasi-$*$-paranormal operator, then $f(T)$ has SVEP. From [16, Theorem 2.9], it follows $f(T)$ satisfies the generalized Browder's theorem for every $f \in Hol(\sigma(T))$.

From [9, Theorem 3.15], we know that: generalized Browder's theorem $\Longrightarrow$ Browder's theorem.

Let $\Pi_{00}^a(T) = \{\mu \in \text{iso}\sigma_a(T) : 0 < \alpha(T-\mu)\}$ be the set of all eigenvalues of T, which are isolated in the approximate point spectrum, and $\pi_{00}^a(T) = \{\mu \in \text{iso}\sigma_a(T) : 0 < \alpha(T-\mu) < \infty\}$ be the set of all eigenvalues of finite multiplicity, which are isolated in the approximate point spectrum of T.

We say that T satisfies the generalized $a-$Weyl's theorem [9], if

$$\sigma_a(T) \setminus \sigma_{UBW}(T) = \Pi_{00}^a(T),$$

and we say that T satisfies the $a-$Weyl's theorem [27], if

$$\sigma_a(T) \setminus \sigma_{uw}(T) = \pi_{00}^a(T).$$

Let $P_{00}^a(T) = \{\mu \in \sigma_a(T) : p(T-\mu) < \infty \text{ and } (T-\mu)^{p(T-\mu)+1}(H) \text{ is closed }\}$, the set of all left poles of resolvent of T and $p_{00}^a(T) = \{\mu \in P_{00}^a(T) : \alpha(T-\mu) < \infty\}$, the set of all left poles of finite rank of resolvent of T.

We say that T satisfies the generalized $a-$Browders theorem [9], if

$$\sigma_a(T) \setminus \sigma_{UBW}(T) = P_{00}^a(T),$$

and we say that T satisfies the $a-$Browders theorem [27], if

$$\sigma_a(T) \setminus \sigma_{uw}(T) = p_{00}^a(T).$$

THEOREM 5.6. Suppose T^* is an algebraically k-quasi-$*$-paranormal operator. Then the generalized $a-$Browder's theorem holds for $f(T)$ for all $f \in Hol(\sigma(T))$.

PROOF. Since algebraically k-quasi-$*$-paranormal operator has finite ascent, then T^* has SVEP. From [6, Theorem 3.2], $f(T)$ satisfies the generalized $a-$Browders theorem for all $f \in Hol(\sigma(T))$.

THEOREM 5.7. Suppose T^* is an algebraically k-quasi-$*$-paranormal operator. Then the generalized $a-$Weyl's theorem holds for T.

PROOF. Since algebraically k-quasi-$*$-paranormal operator has finite ascent, then T^* has SVEP. Then from Theorem 5, T satisfies the generalized $a-$Browders theorem. So, in view of [4], it is sufficient to show that $\Pi_{00}^a(T) = P_{00}^a(T)$. Since the inclusion $P_{00}^a(T) \subseteq \Pi_{00}^a(T)$ always holds true, then it is sufficient to prove this $\Pi_{00}^a(T) \subseteq P_{00}^a(T)$. Let μ be an arbitrary point of $\Pi_{00}^a(T)$, then μ is an isolated point on $\sigma_a(T)$. From Lemma 3, μ is a pole of the resolvent of T, there exists a positive integer $p = p(\mu)$ such that $p(T-\mu) = q(T-\mu) = p < \infty$. Thus, $(T-\mu)^{p+1}(H) = (T-\mu)^p(H)$ and $(T-\mu)^p(H)$ is closed, since it coincides with the kernel of the spectral projection associated with $\{\mu\}$. Therefore, $\mu \in P_{00}^a(T)$

THEOREM 5.8. Suppose T^* is an algebraically k-quasi-$*$-paranormal operator. Then the generalized a-Weyl's theorem holds for $f(T)$ for all $f \in Hol(\sigma(T))$.

PROOF. Suppose that T^* is an algebraically k-quasi-$*$-paranormal operator. Then T^* has SVEP, thus $f(T)$ satisfies the generalized $a-$Browders theorem. From [4], it is sufficient to show $\Pi_{00}^a(f(T)) \subseteq P_{00}^a(f(T))$. Suppose $\mu \in \Pi_{00}^a(f(T))$. Then μ is an isolated point of $\sigma_a(f(T))$ and $0 < \alpha(f(T)-\mu)$. Then $\mu \in \sigma_a(f(T))$, and it satisfies the equation:

$$f(T) - \mu = c(T-\mu_1)(T-\mu_2)\cdot \ldots \cdot (T-\mu_n)g(T) \tag{6}$$

where $c, \mu_1, \mu_2, ..., \mu_n \in \mathbb{C}$, and $g(T)$ is invertible.

Since μ is an isolated point of $f(\sigma_a(T))$, if $\mu_i \in \sigma_a(T)$, then μ_i is an isolated point of $\sigma_a(T)$ by relation (6). Since T is $a-$isoloid, $0 < \alpha(T-\mu_i)$ for each $i = 1, 2, ..., n$. Then $\mu_i \in \Pi_{00}^a(T)$ for each $i = 1, 2, ..., n$. From Theorem 5, T satisfies the generalized $a-$Weyl's theorem, then $T - \lambda_i$ is upper semi B-Fredholm and $\text{ind}(T-\mu_i) \leq 0$ for each $i = 1, 2, ..., n$. Therefore $f(T) - \mu$ is upper semi-B-Fredholm. Since $\mu \in \text{iso}\sigma_a(f(T))$ then $f(T)$ has SVEP in μ, then by [1, Theorem 2.89], $p(f(T)-\mu) < \infty$. Also, since T^* has SVEP, $f(T)^*$ has SVEP in μ, then by [1, Theorem 2.90] $p(f(T)-\mu) = q(f(T)-\mu) = p < \infty$. Thus, $(f(T)-\mu)^{p+1}(H) = (f(T)-\mu)^p(H)$ and $(f(T)-\mu)^p(H)$ is closed, since it coincides with the kernel of the spectral projection associated with $\{\mu\}$. Therefore $\mu \in P_{00}^a(f(T))$.

From [9, Theorem 3.11], we know that:

generalized $a-$Weyl's theorem $\Longrightarrow$ $a-$Weyl's theorem.

and from [9, Theorem 3.8], we know that:

generalized $a-$Browder's theorem $\Longrightarrow$ $a-$Browder's theorem.

COROLLARY 5.9. If T^* is an algebraically k-quasi-$*$-paranormal then $f(T)$ satisfies $a-$Weyl's theorem for every $f \in Hol(\sigma(T))$.

If T is an algebraically k-quasi-$*$-paranormal, then T not satisfies $a-$Weyl's theorem [1, Example 4.53], consequently T not satisfies generalized $a-$Weyl's theorem, by [9, Theorem 3.11].

A bounded operator $T \in L(H)$ is said to be *hereditarily polaroid*, i.e. any restriction to an invariant closed subspace is polaroid. This class of operators has been first considered in [18].

COROLLARY 5.10. Algebraically k-quasi-$*$-paranormal operators are hereditarily polaroid.

PROOF. Let $T \in L(H)$ be an algebraically k-quasi-$*$-paranormal and M a closed T-invariant subspace of H. By assumption there exists a nontrivial polynomial h such that $h(T)$ is a k-quasi-$*$-paranormal. The restriction of any k-quasi-$*$-paranormal operator to an invariant closed subspace is also k-quasi-$*$-paranormal, so $h(T)|_M$ is a k-quasi-$*$-paranormal. Since $h(T|_M) = h(T)|_M$, $T|_M$ is algebraically k-quasi-$*$-paranormal, hence polaroid, from Theorem 3.

Let $\mathcal{K}(H)$ be the space of all compact operators on H. Note that $\mathcal{K}(H)$ is a closed ideal of $L(H)$. On the quotient space $L(H)/\mathcal{K}(H)$ it is defined the product $[S][T] = [ST]$, where $[S]$ is the coset $S + \mathcal{K}(H)$. The space $L(H)/\mathcal{K}(H)$ with this additional operation is an algebra, which is called the Calkin Algebra. Let $\pi : L(H) \to L(H)/\mathcal{K}(H)$ be the natural mapping (Calkin homomorphism). If $T \geq O$ then $\pi(T) \geq O$. It is well known the Theorem of Atkinson: T is a Fredholm operator if and only if $\pi(T)$ is an invertible operator in Calkin algebra, thus $\sigma(\pi(T)) = \sigma_e(T)$, where

$$\sigma_e(T) = \{\mu \in \mathbb{C} : T - \mu \text{ is not Fredholm}\}.$$

An operator T is said to be a Riesz operator if $T - \mu$ is a Fredholm operator for all $\mu \in \mathbb{C} \setminus \{0\}$. Thus, $\sigma_e(T) = \{0\}$. Compact operators, also quasinilpotent operators, are Riesz operators.

THEOREM 5.11. If $T \in L(H)$ is a k-quasi-$*$-paranormal, $\|T^n\| = \|T\|^n$ for some $n \geq k$, and Riesz operator, then T is a compact operator.

PROOF. Let T be a k-quasi-$*$-paranormal operator. Then

$$\begin{aligned} & \pi(T)^{*k}\left(\pi(T)^{*2}\pi(T)^2 - 2\lambda\pi(T)\pi(T)^* + \lambda^2\right)\pi(T)^k \\ = \; & \pi\left(T^{*k}\left(T^{*2}T^2 - 2\lambda TT^* + \lambda^2\right)T^k\right) \geq O, \end{aligned}$$

which shows that $\pi(T)$ is a k-quasi-$*$-paranormal. Thus $\pi(T)$ is normaloid operator, Theorem 2. Since T is a Riesz operator by West Decomposition Theorem [32], we can write $T = S + Q$ where S is a compact and Q is a quasinilpotent operator. From the definition of homomorphism π we have $\pi(T) = \pi(Q)$, thus $\sigma(\pi(T)) = \sigma(\pi(Q)) = \sigma_e(Q) = \{0\}$, so $\pi(T)$ is a quasinilpotent operator. Therefore, $\|\pi(T)\| = r(\pi(T)) = 0$, thus $\pi(T) = O$. Then T is a compact operator.

COROLLARY 5.12. If T is a k-quasi-$*$-paranormal operator and if $\sigma_{BW}(T) = \{0\}$, then T is normal operator.

PROOF. From Theorem 5, T satisfies the generalized Weyl's theorem. By assumption, we have $\sigma(T) \setminus \{0\} = \Pi_{00}(T)$. So every nonzero point of $\sigma(T)$ is an isolated point of $\sigma(T)$ and an eigenvalue. Hence $\sigma(T) \setminus \{0\}$ is a finite set or a countably infinite set whose only cluster point is 0. Let $\sigma(T) \setminus \{0\} = \{\mu_n\}$, with $|\mu_1| \geq |\mu_2| \geq ... > 0$. Since μ_n is isolated point of $\sigma(T)$, from Theorem 4, $\ker(T - \mu_n)$ is a reducing subspace of T. Let P_n be the orthogonal projection onto $\ker(T - \mu_n)$. Then $TP_n = P_nT = \mu_n P_n$ and $P_n P_m = 0$ if $n \neq m$. Put $P = \oplus_n P_n$, and we have

$$T = T|_{\ker(I-P)} \oplus T|_{(I-P)(H)} = \oplus_n \mu_n P_n \oplus T|_{(I-P)(H)},$$

with $\sigma(T|_{(I-P)(H)}) = \sigma(T) \setminus \{\mu_n\} = \{0\}$. Since $T|_{(I-P)(H)}$ is also k-quasi-$*$-paranormal operator, $T|_{(I-P)(H)} = O$. Hence $T = \oplus_n \mu_n P_n$, thus T is normal operator.

6 Generalized Weyl's Theorem for Perturbations of Algebraically k-Quasi-$*$-Paranormal Operator

A bounded operator $T \in L(H)$ is said to be *algebraic* if there exists a non-constant polynomial h such that $h(T) = 0$. Trivially, every nilpotent operator is algebraic and it is well-known that if $T^n(H)$ has finite dimension for some $n \in \mathbb{N}$ then T is algebraic.

THEOREM 6.1. If T is an algebraically k-quasi-$*$-paranormal operator, F is algebraic with $TF = FT$, then $T + F$ satisfies generalized Weyl's theorem.

PROOF. Since F is algebraic operator, $\sigma(F) = \{\mu_1, \mu_2, ..., \mu_n\}$. Denote by P_i the spectral projections associated with F and the spectral set $\{\mu_i\}$, $i = 1, 2, ..., n$. We write $F_i = F|_{P_i(H)}$ and $T_i = T|_{P_i(H)}$. Clearly, $\sigma(F_i) = \{\mu_i\}$ for every $i = 1, 2, ..., n$. Let h be a nontrivial complex polynomial such that $h(F) = O$. Then $O = h(F_i) = h(F)|_{P_i(H)}$, and from

$$\{0\} = \sigma(h(F_i)) = h(\sigma(F_i)) = h(\mu_i),$$

we obtain that $h(\mu_i) = 0$. Write $h(\mu) = (\mu - \mu_i)^k g(\mu)$ with $g(\mu_i) \neq 0$. Then $O = h(F_i) = (F_i - \mu_i)^k g(F_i)$, where $g(F_i)$ is invertible. Therefore $(F_i - \mu_i)^k = O$, hence $F_i - \mu_i$ is a nilpotent operator for all $i = 1, 2, ..., n$. Let $\mu \in \Pi_{00}(T + F)$. Then μ is isolated point in the spectrum $\sigma(T + F)$. Since $\sigma(T + F) = \cup_{i=1}^{n} \sigma(T_i + F_i)$, then $\mu \in \sigma(T_i + F_i)$, for some $i = 1, 2, ..., n$ and hence $\mu - \mu_i \in \text{iso}\sigma(T_i + F_i - \mu_i)$. The restriction T_i to a closed invariant subspace $P_i(H)$ is also algebraically k-quasi-$*$-paranormal operator, then T_i is polaroid for all $i = 1, 2, ..., n$. Since $F_i - \mu_i$ is a nilpotent operator for all $i = 1, 2, ..., n$, by [5, Theorem 2.10] $T_i + F_i - \mu_i$ is polaroid for all $i = 1, 2, ..., n$. Then $\mu - \mu_i$ is a pole of the resolvent of $T_i + F_i - \mu_i$. By [2, Theorem 3.74] there exists a positive numbers m_i such that

$$\mathcal{H}_0(T_i + F_i - \mu_i - (\mu - \mu_i)) = \mathcal{H}_0(T_i + F_i - \mu) = \ker(T_i + F_i - \mu)^{m_i},$$

for $i = 1, 2, ..., n$. Taking $\mathcal{H}_0(T_i + F_i - \mu) = \{0\}$ for $\mu \notin \sigma(T_i + F_i)$ and we have

$$\mathcal{H}_0(T + F - \mu) = \oplus_{i=1}^n \mathcal{H}_0(T_i + F_i - \mu) = \oplus_{i=1}^n \ker(Ti + F_i - \mu)^{m_i} = \ker(T + F - \mu)^m,$$

where $m = \max\{m_1, m_2, ..., m_n\}$. Since $\mu \in \mathrm{iso}\sigma(T + F)$, we have

$$H = \mathcal{H}_0(T + F - \mu) \oplus K(T + F - \mu) = \ker(T + F - \mu)^m \oplus K(T + F - \mu).$$

Therefore,

$$(T + F - \mu)^m(H) = K(T + F - \mu) \text{ and } H = \ker(T + F - \mu)^m \oplus (T + F - \mu)^m(H).$$

From [2, Theorem 3.6] $T + F - \mu$ has finite ascent. So, $(T + F - \mu)^m(H) = (T + F - \mu)^{m+1}(H)$ and $(T + F - \mu)^p(H)$ is closed for every $p \geq m$. By [3, Theorem 2.8] $T + F$ satisfies the generalized Weyl's theorem.

THEOREM 6.2. If T is an algebraically k-quasi-$*$-paranormal operator, F is algebraic with $TF = FT$, then $f(T + F)$ satisfies the generalized Weyl's theorem for all $f \in Hol(\sigma(T + F))$.

PROOF. Let F be an algebraic operator. Then, $\sigma(F) = \{\mu_1, \mu_2, ..., \mu_n\}$, and $F_i - \mu_i$ is nilpotent operator for $i = 1, 2, ..., n$. Since T is an algebraically k-quasi-$*$-paranormal, then $T_i + \mu_i$ is also an algebraically k-quasi-$*$-paranormal operator. Then $T_i + \mu_i$ has SVEP for $i = 1, 2, ..., n$ and from [2, theorem 2.12] $T_i + \mu_i + F_i - \mu_i = T_i + F_i$ has SVEP. From [2, theorem 2.9] $T + F = \oplus_{i=1}^n (T_i + F_i)$ has SVEP. By [16, Corollary 2.8], $f(\sigma_{BW}(T + F)) = \sigma_{BW}(f(T + F))$ for all $f \in Hol(\sigma(T + F))$. But, from the above theorem we have that $T + F$ is isoloid operator, then from [16, Lemma 3.3],

$$f(\sigma(T + F) \setminus \Pi_{00}(T + F)) = \sigma(f(T + F)) \setminus \Pi_{00}(f(T + F)),$$

and

$$\begin{aligned} \sigma(f(T + F)) \setminus \Pi_{00}(f(T + F)) &= f(\sigma(T + F) \setminus \Pi_{00}(T + F)) \\ &= f(\sigma_{BW}(T + F)) = \sigma_{BW}(f(T + F)), \end{aligned}$$

which implies that $f(T + F)$ satisfies the generalized Weyl's theorem.

An operator T is said to be finitely-isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T of the finite multiplicity, so: from $\mu \in \mathrm{iso}\sigma(T)$ we have $\mu \in \pi_{00}(T)$.

COROLLARY 6.3. If T is finitely-isoloid and T is an algebraically k-quasi-$*$-paranormal operator, R is Riesz operator with $TR = RT$, then $T + R$ satisfies Weyl's theorem.

PROOF. From Corollary 5, it follows that T satisfies Weyl's theorem and by [26, Theorem 2.7] $T + R$ satisfies Weyl's theorem.

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