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A Study Of Fixed Points Of Mappings Satisfying E.A Like Property On Dislocated Quasi b-Metric Spaces*

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Abstract

In this paper, we introduce the concepts of E.A property and E.A like property in dislocated quasi b-metric spaces. We establish fixed point theorems for mappings satisfying E.A like property in dislocated quasi b-metric spaces which extend results of Kastriot Zoto, Arben Isufati, Panda Sumati Kumari ([5]). We also present some examples which support our results.

1 Introduction

Chakkrid and Cholatis [2] introduced the concept of dislocated quasi b-metric space and established fixed point theorems for cyclic contractions. Rahman et al. [8] studied dislocated quasi b-metric spaces and gained fixed point theorems for Kannan and Chetterjea type contractive mappings. Cholatis et al. [3] proved fixed point theorems for cyclic weakly contractive mappings in dislocated quasi b-metric spaces. Also they have discussed some topological properties of dislocated quasi b-metric spaces.

M. Aamri and D. El Moutawakil [6] introduced new concept called E.A property. Kastriot Zoto et al. [5] introduced the concept of E.A like property in dislocated and dislocated quasi-metric spaces. They have adopted the definition of K. Wadhwa, H. Dubey, R. Jain [4] to define E.A like property.

In this paper, we introduce the concept of E.A property and E.A like property in dislocated quasi b-metric spaces. We establish some fixed point theorems for mappings satisfying E.A property and E.A like property in dislocated quasi b-metric spaces which extend results of Zoto et al. [5]. We also present some examples which support our results.

DEFINITION 1. ([2]). Let X be a non-empty set. Let the mapping $d : X \times X \to [0, \infty)$ and constant $k \geq 1$ satisfy the following conditions:

(i) $d(x, y) = 0 = d(y, x) \Rightarrow x = y$, $\forall x, y \in X$.

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(ii) $d(x,y) \leq k[d(x,z)+d(z,y)]$, $\forall x,y,z \in X$.

Then the pair (X,d) is called a dislocated quasi-b-metric space or in short dqb-metric space. The constant k is called the coefficient of dislocated quasi-b-metric space (X,d).

EXAMPLE 1. Consider $X=[1,\infty)$ with $d(x,y)=|x-y|+2|x-1|+|y-1|$. Then (X,d) is a dqb-metric space with coefficient $k=2$.

EXAMPLE 2 ([8]). Let $X=R^+$, $p>1$, $d: X\times X \to [0,\infty)$ be defined as

$$d(x,y)=|x-y|^p+|x|^p, \quad \forall x,y \in X.$$

Then (X,d) is a dqb-metric space with $k=2^p>1$. But (X,d) is not a b-metric space and also not dislocated quasi metric space.

EXAMPLE 3 ([2]). Let $X=R$ and suppose

$$d(x,y)=|2x-y|^2+|2x+y|^2.$$

Then (X,d) is a dqb-metric space with coefficient $k=2$ but (X,d) is not a quasi-b-metric space. Also (X,d) is not a dislocated quasi metric space.

DEFINITION 2 ([2]). A sequence $\{x_n\}$ in a dqb-metric space (X,d), dqb-converges to $x \in X$ if

$$\lim_{n\to\infty} d(x_n,x)=0=\lim_{n\to\infty} d(x,x_n).$$

In this case x is called the dqb-limit of $\{x_n\}$ and $\{x_n\}$ is said to be dqb-convergent to x, written as $x_n \to x$.

DEFINITION 3. ([2]). A sequence $\{x_n\}$ in a dqb-metric space (X,d) is called a dqb-Cauchy sequence if

$$\lim_{n,m\to\infty} d(x_n,x_m)=0=\lim_{n,m\to\infty} d(x_m,x_n).$$

DEFINITION 4 ([2]). A dqb-metric space (X,d) is said to be dqb-complete if every dqb-Cauchy sequence in it is dqb-convergent in X.

LEMMA 1 ([3]). The limit of a dqb-convergent sequence in a dqb-metric space is unique.

PROPOSITION 1. Let (X,d) be a dqb-metric space with coefficient k and u be the dqb-limit of a nonconstant sequence in X. Then $d(u,u)=0$.

PROOF. We see that

$$\begin{aligned} d(u,u) &\leq k[d(u,x_n)+d(x_n,u)] \leq \lim k[d(u,x_n)+d(x_n,u)] \\ &= k[\lim d(u,x_n)+\lim d(x_n,u)]=0. \end{aligned}$$

The proof is complete.

We have observed the following result in Rahman and Sarwar [8].

THEOREM 1 ([8]). Let (X, d) be a *dqb*-complete metric space with coefficient $k \geq 1$. Let $T : X \to X$ be a continuous mapping satisfying

$$\forall x, y \in X, \quad d(Tx, Ty) \leq \alpha d(x, y) \quad \text{where } 0 \leq \alpha < 1 \text{ and } 0 \leq k\alpha < 1.$$

Then T has a unique fixed point in X.

Aamri et al. [6]. introduced the following concept of E.A property in metric spaces.

DEFINITION 5 ([6]). Let S and T be two self mappings of a metric space (X, d). We say that T and S satisfy the property (E.A) if there exists a sequence (x_n) such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

EXAMPLE 4. Let $X = [0, \infty)$. Define mappings T and S as $Tx = \frac{x}{7}$ and $Sx = \frac{3x}{7}$. Now if we take the sequence $\{x_n\} = \{\frac{1}{n}\}$, then it is obvious that $\lim_{n\to\infty} Tx_n = 0 = \lim_{n\to\infty} Sx_n$. And thus T and S satisfy property (E.A).

We have extended this property to *dqb*-metric spaces as follows:

DEFINITION 6. Let f and g be two self mappings of a *dqb*-metric space (X, d). We say that f and g satisfy the E.A property if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$.

Note that the limit in the above definition is *dqb*-limit.

EXAMPLE 5. Let $X = [0, \infty)$ and $d(x, y) = |2x - y|^2 + |2x + y|^2$. Then (X, d) is a *dqb*-metric space with coefficient $k = 2$. Let $fx = 3x$ and $gx = x^2$. Note that for the sequence $\{x_n\} = 1/n, n \in N$, we get $\lim fx_n = \lim gx_n = 0$. In other words f and g satisfy E.A like property.

Zoto et al.([5]) have defined E.A like property in dislocated metric spaces as follows:

DEFINITION 7 ([5]). Let S and T be two self mappings of a dislocated metric space (X, d). We say that S and T satisfy the E.A like property if there exists a sequence (x_n) such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in S(X)$ or $t \in T(X)$, i.e. $t \in S(X) \cup T(X)$.

EXAMPLE 6 ([5]). Let $X = R^+$. Define $d : X \times X \to [0, \infty)$ by $d(x, y) = x + 2y$ for all $x, y \in X$. Define $Tx = \frac{x}{5}$ and $Sx = \frac{x}{4}$ for all $x \in X$. Then for the sequence $x_n = \frac{1}{n}, n \in N$, we have

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = 0 \in T(X) \cup S(X).$$

Thus T and S satisfy E.A like property.

We have adopted this definition in dqb-metric spaces as follows:

DEFINITION 8. Let f and g be two self mappings of a dqb-metric space (X, d). We say that f and g satisfy the E.A like property if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in f(X) \cup g(X)$.

Note that the limit in the above definition is dqb-limit.

EXAMPLE 7. Consider $X = [0, \infty)$ with $d(x, y) = |x-y|^2+2|x|+|y|$. Then (X, d) is a dqb-metric space with coefficient $k = 2$. Let $Sx = 2x$ and $Tx = x^4$. Note that for the sequence $\{x_n\} = 1/n$, $n \in N$ we get $\lim Sx_n = \lim Tx_n = 0$ where $0 \in S(X) \cup T(X)$. And thus S and T satisfy E.A like property.

DEFINITION 9 ([7]). Let f and g be self maps of a set X. If $w = fx = gx$ for some x in X then x is called a coincidence point of f and g and w is called a point of coincidence of f and g.

DEFINITION 10 ([7]). Let f and g be self maps of a set X. Then f and g are said to be weakly compatible if they commute at their coincidence point.

2 Main Results

Amri et al. [6] have proved the following theorem.

THEOREM 2. Let S and T be two weakly compatible selfmappings of a metric space (X, d) such that

(i) T and S satisfy the property (E.A),

(ii) $\forall x \neq y \in X$,

$$d(Tx, Ty) < \max\left\{d(Sx, Sy), \frac{[d(Tx, Sx) + d(Ty, Sy)}{2}, \frac{[d(Ty, Sx) + d(Tx, Sy)]}{2}\right\},$$

(iii) $TX \subset SX$.

If SX or TX is complete subspace of X, then T and S have a unique common fixed point.

We have extended this result to dislocated quasi b-metric spaces in following manner.

THEOREM 3. Let f and g be two self maps of a dqb-metric space (X, d), $f(X) \subset g(X)$ and $g(X)$ is dqb-complete, satisfying the following conditions:

(i)

$$d(fx, fy) \leq \max\left\{d(gx, gy), \frac{d(fx, gx) + d(gy, fy)}{2}, \frac{d(gx, fy) + d(fx, gy)}{2}\right\},$$

(ii) f and g are weakly compatible,

(iii) f and g satisfy E.A like property.

Then f and g have a unique common fixed point.

PROOF. In view of assumption (iii), there exists a sequence $\{x_n\}$ in X and $v \in X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = v.$$

Since $g(X)$ is dqb-complete, there exists $u \in X$ such that $\lim_{n\to\infty} gx_n = gu$. Note that $\lim_{n\to\infty} fx_n = gu$. We claim that $fu = gu$. On the contrary assume that $fu \neq gu$ i.e. at least one of $d(fu, gu)$ and $d(gu, fu)$ is greater than 0. We first assume that $d(gu, fu) > 0$. Then in view of assumption (i) with $x = x_n$ and $y = u$ we can write

$$d(fx_n, fu) \leq \max\left\{d(gx_n, gu), \frac{d(fx_n, gx_n) + d(gu, fu)}{2}, \frac{d(gx_n, fu) + d(fx_n, gu)}{2}\right\}.$$

Letting $n \to \infty$ in the above inequality, we get

$$d(gu, fu) \leq \max\left\{d(gu, gu), \frac{d(gu, gu) + d(gu, fu)}{2}, \frac{d(gu, fu) + d(gu, gu)}{2}\right\}.$$

It follows that

$$d(gu, fu) \leq \frac{d(gu, fu)}{2},$$

which is clearly a contradiction unless $d(gu, fu) = 0$. Now, assume that $d(fu, gu) > 0$. Again as above taking $x = u$ and $y = x_n$ in assumption (i), we can write

$$d(fu, fx_n) \leq \max\left\{d(gu, gx_n), \frac{d(fu, gu) + d(gx_n, fx_n)}{2}, \frac{d(gu, fx_n) + d(fu, gx_n)}{2}\right\}.$$

Letting $n \to \infty$ in the above inequality, we get

$$d(fu, gu) \leq \max\left\{d(gu, gu), \frac{d(fu, gu) + d(gu, gu)}{2}, \frac{d(gu, gu) + d(fu, gu)}{2}\right\}.$$

It follows that

$$d(fu, gu) \leq \frac{d(fu, gu)}{2}.$$

Which is again clearly a contradiction unless $d(fu, gu) = 0$. Thus $d(fu, gu) = 0 = d(gu, fu)$ which means $fu = gu$. As f and g are weakly compatible, we have, $fgu = gfu$ and hence $f^2u = fgu = gfu = g^2u$. Now we claim that $fu = f^2u$ i.e. $fu = ffu$ i.e. fu is fixed point of f. On the contrary we assume that $ffu \neq fu$ i.e. $d(fu, ffu) > 0$

and/or $d(ffu, fu) > 0$. We first assume that $d(fu, ffu) > 0$. Now taking $x = u$ and $y = fu$ in assumption (i), we get

$$\begin{aligned}
d(fu, ffu) &\leq \max\left\{d(gu, gfu), \frac{d(fu, gu) + d(gfu, ffu)}{2}, \frac{d(gu, ffu) + d(fu, gfu)}{2}\right\} \\
&= \max\left\{d(fu, ffu), \frac{d(fu, fu) + d(ffu, ffu)}{2}, \frac{d(fu, ffu) + d(fu, ffu)}{2}\right\} \\
&= \max\left\{d(fu, ffu), \frac{d(ffu, ffu)}{2}\right\} \\
&\leq \max\left\{d(fu, ffu), \frac{k}{2}\Big[d(fu, ffu) + d(ffu, fu)\Big]\right\} \\
&= \frac{k}{2}\Big[d(fu, ffu) + d(ffu, fu)\Big].
\end{aligned}$$

This gives

$$d(fu, ffu) \leq \frac{\frac{k}{2} d(ffu, fu)}{1 - \frac{k}{2}} < 0,$$

which is a contradiction. Hence $d(fu, ffu) = 0$. Now assume that $d(ffu, fu) > 0$. Taking $x = fu$ and $y = u$ in assumption (i), we get

$$\begin{aligned}
d(ffu, fu) &\leq \max\left\{d(gfu, gu), \frac{d(ffu, gfu) + d(gu, fu)}{2}, \frac{d(gfu, fu) + d(ffu, gu)}{2}\right\} \\
&= \max\left\{d(ffu, fu), \frac{d(ffu, ffu) + d(fu, fu)}{2}, \frac{d(ffu, fu) + d(ffu, fu)}{2}\right\} \\
&= \max\left\{d(ffu, fu), \frac{d(ffu, ffu)}{2}\right\} \\
&\leq \max\left\{d(ffu, fu), \frac{k}{2}\Big[d(fu, ffu) + d(ffu, fu)\Big]\right\} \\
&= \frac{k}{2}\Big[d(fu, ffu) + d(ffu, fu)\Big].
\end{aligned}$$

This gives

$$d(ffu, fu) \leq \frac{\frac{k}{2} d(fu, ffu)}{1 - \frac{k}{2}} < 0,$$

which is again a contradiction. Therefore $d(ffu, fu) = 0$. Thus, we conclude that $d(fu, ffu) = 0 = d(ffu, fu)$ i.e. $ffu = fu$. This shows that fu is fixed point of f. But $gfu = ffu = fu$. That is fu is also a fixed point of g. Hence we conclude that fu is a common fixed point of f and g. Now we prove that fu is unique. Let us assume that t is another common fixed point of f and g i.e. $ft = t = gt$. Consider

$$\begin{aligned}
d(t, fu) &= d(ft, ffu)\\
&\leq \max\left\{d(gt, gfu), \frac{d(ft, gt) + d(ffu, gfu)}{2}, \frac{d(gt, ffu) + d(ft, gfu)}{2}\right\}\\
&= \max\left\{d(t, fu), \frac{d(t, t) + d(fu, fu)}{2}, \frac{d(t, fu) + d(t, fu)}{2}\right\}\\
&= \max\left\{d(t, fu), \frac{d(t, t)}{2}\right\}\\
&\leq \max\left\{d(t, fu), \frac{k}{2}\Big[d(t, fu) + d(fu, t)\Big]\right\}\\
&= \frac{k}{2}\Big[d(t, fu) + d(fu, t)\Big].
\end{aligned}$$

This implies that

$$d(t, fu) \leq \frac{\frac{k}{2} d(fu, t)}{1 - \frac{k}{2}},$$

which is clearly a contradiction unless $d(fu, t) = 0$. Similarly, consider

$$\begin{aligned}
d(fu, t) &= d(ffu, ft)\\
&\leq \max\left\{d(gfu, gt), \frac{d(ffu, gfu) + d(ft, gt)}{2}, \frac{d(gfu, ft) + d(ffu, gt)}{2}\right\}\\
&= \max\left\{d(fu, t), \frac{d(fu, fu) + d(t, t)}{2}, \frac{d(fu, t) + d(fu, t)}{2}\right\}\\
&= \max\left\{d(fu, t), \frac{d(fu, fu)}{2}\right\}\\
&\leq \max\left\{d(fu, t), \frac{k}{2}\left[d(fu, t) + d(t, fu)\right]\right\}\\
&= \frac{k}{2}\Big[d(fu, t) + d(t, fu)\Big].
\end{aligned}$$

This implies that

$$d(fu, t) \leq \frac{\frac{k}{2} d(t, fu)}{1 - \frac{k}{2}},$$

which is clearly a contradiction unless $d(t, fu) = 0$. Thus $t = fu$. Hence fu is a unique common fixed point of f and g. This completes the proof.

EXAMPLE 8. Let $X = [0, \infty)$ and $d(x, y) = |2x - y|^2 + |2x + y|^2$. Then (X, d) is a dqb-metric space with coefficient $k = 2$. Let $fx = 2x$ and $gx = x^3$. Note that for the sequence $\{x_n\} = 1/n, n \in N$, we get $\lim fx_n = \lim gx_n = 0$. In other words f and g satisfy E.A like property. Also observe that f and g are weakly compatible. Now

$$d(2x,2y) \leq \max\left\{ d(x^3,y^3), \frac{d(2x,x^3)+d(y^3,2y)}{2}, \frac{d(x^3,2y)+d(2x,y^3)}{2} \right\}, \text{ i.e.,}$$

$$(4x-2y)^2+(4x+2y)^2 \leq \max\left\{ (2x^3-y^3)^2+(2x^3+y^3)^2, \frac{(4x-x^3)^2+(4x+x^3)^2+(2y^3-2y)^2+(2y^3+2y)^2}{2}, \frac{(2x^3-2y)^2+(2x^3+2y)^2+(4x-y^3)^2+(4x+y^3)^2}{2} \right\}$$

is true for all $x, y \in [0, \infty)$. Thus f and g satisfy all the conditions of the theorem and hence have unique common fixed point $0 = f0 = g0$.

Kastriot Zoto et al. [5] have proved the following theorem.

THEOREM 4. Let (X, d) be a complete dislocated quasi metric space and $f, g : X \to X$ are two self maps satisfying the conditions:

(i) $d(fx, fy) \leq \alpha d(fx, gy) + \beta d(gx, fy) + \gamma d(gx, gy) + \delta d(gy, fy) + \eta d(gx, fx)$ for all $x, y \in X$, where the constants $\alpha, \beta, \gamma, \delta, \eta \geq 0$ are nonnegative and $0 \leq \alpha + \beta + \gamma + \delta + \eta < \frac{1}{2}$,

(ii) f and g satisfy E.A like property,

(iii) f and g are weakly compatible for all $x, y \in X$, and $0 \leq \alpha + \beta + \gamma + \delta + \eta < \frac{1}{2}$.

Then f and g have a unique common fixed point in X.

We have extended this result to the dislocated quasi b-metric space in the following way.

THEOREM 5. Let (X, d) be dqb-complete metric space with coefficient $k \geq 1$ and S and T be two self maps on X satisfying following conditions:

(i) $d(Sx, Sy) \leq \alpha d(Sx, Ty) + \beta d(Tx, Sy) + \gamma d(Tx, Ty) + \delta d(Ty, Sy) + \eta d(Tx, Sx)$ for all $x, y \in X$ and the constants $\alpha, \beta, \gamma, \delta, \eta \geq 0$ are such that $0 \leq \alpha + \beta + \gamma + \delta + \eta < \frac{1}{2k}$,

(ii) S and T satisfy E.A like property,

(iii) S and T are weakly compatible.

Then, T and S have a unique common fixed point in X.

PROOF. In view of assumption (ii), there exists a sequence $\{x_n\}$ in X and $u \in S(X) \cup T(X)$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = u.$$

Let us assume that $\lim_{n\to\infty} Sx_n = u \in T(X)$. Now we can find $v \in X$ such that $Tv = u$. Now from inequality (i), taking $x = v$ and $y = x_n$, we can write

$$d(Sv, Sx_n) \leq \alpha d(Sv, Tx_n) + \beta d(Tv, Sx_n) + \gamma d(Tv, Tx_n) + \delta d(Tx_n, Sx_n) + \eta d(Tv, Sv).$$

Letting $n \to \infty$ in above inequality, we get

$$\begin{aligned} d(Sv, u) &\leq \alpha d(Sv, u) + \beta d(Tv, u) + \gamma d(Tv, u) + \delta d(u, u) + \eta d(Tv, Sv) \\ &= \alpha d(Sv, u) + \beta d(u, u) + \gamma d(u, u) + \delta d(u, u) + \eta d(u, Sv) \\ &\leq \alpha d(Sv, u) + \eta d(u, Sv) + k\beta[d(u, Sv) + d(Sv, u)] \\ &\quad + k\gamma[d(u, Sv) + d(Sv, u] + k\delta[d(u, Sv) + d(Sv, u)] \\ &= (\alpha + k\beta + k\gamma + k\delta)d(Sv, u) + (\eta + k\beta + k\gamma + k\delta)d(u, Sv). \end{aligned}$$

This gives

$$\begin{aligned} d(Sv, u) &\leq \frac{\eta + k\beta + k\gamma + k\delta}{1 - (\alpha + k\beta + k\gamma + k\delta)} d(u, Sv) \\ &\leq \frac{k\eta + k\beta + k\gamma + k\delta}{1 - (k\alpha + k\beta + k\gamma + k\delta)} d(u, Sv). \end{aligned} \quad (1)$$

Similarly, taking $x = x_n$ and $y = v$, in inequality (i) we can write

$$d(Sx_n, Sv) \leq \alpha d(Sx_n, Tv) + \beta d(Tx_n, Sv) + \gamma d(Tx_n, Tv) + \delta d(Tv, Sv) + \eta d(Tx_n, Sx_n).$$

Letting $n \to \infty$ in above inequality, we get

$$\begin{aligned} d(u, Sv) &\leq \alpha d(u, Tv) + \beta d(u, Sv) + \gamma d(u, Tv) + \delta d(u, Sv) + \eta d(u, u) \\ &= \alpha d(u, u) + \beta d(u, Sv) + \gamma d(u, u) + \delta d(u, Su) + \eta d(u, u)) \\ &\leq k\alpha[d(u, Sv) + d(Sv, u)] + \beta d(u, Sv) + k\gamma[d(u, Sv) + d(Sv, u)] \\ &\quad + \delta d(u, Sv) + k\eta[d(u, Sv) + d(Sv, u)] \\ &= (k\alpha + k\gamma + k\eta)d(Sv, u) + (k\alpha + \beta + k\gamma + \delta + k\eta)d(u, Sv). \end{aligned}$$

This gives

$$\begin{aligned} d(u, Sv) &\leq \frac{k\alpha + k\gamma + k\eta}{1 - (k\alpha + \beta + k\gamma + \delta + k\eta)} d(Sv, u) \\ &\leq \frac{k\alpha + k\gamma + k\eta}{1 - (k\alpha + k\beta + k\gamma + k\delta + k\eta)} d(Sv, u). \end{aligned} \quad (2)$$

Taking

$$\xi = \max\left\{\frac{k\eta + k\beta + k\gamma + k\delta}{1 - (k\alpha + k\beta + k\gamma + k\delta)}, \frac{k\alpha + k\gamma + k\eta}{1 - (k\alpha + k\beta + k\gamma + k\delta + k\eta)}\right\},$$

from inequalities (1) and (2), we get

$$d(Sv,u) \leq \xi^2 d(Sv,u) \ \text{ and } \ d(u,Sv) \leq \xi^2 d(u,Sv)$$

where $0 \leq \xi < 1$. Thus $d(u,Sv) = 0 = d(Sv,u)$ and hence $Sv = u$. Now, we have $Tv = u = Sv$. As we know that S and T are weakly compatible, we conclude that v is a coincidence point of S and T, so that S and T commute at v i.e. $S(Tv) = T(Sv)$ i.e. $Su = Tu$.

Next, we claim that u is a common fixed point of S and T. For this we consider

$$d(Su,Sx_n) \leq \alpha d(Su,Tx_n)+\beta d(Tu,Sx_n)+\gamma d(Tu,Tx_n)+\delta d(Tx_n,Sx_n)+\eta d(Tu,Su).$$

Letting $n \to \infty$ in the above inequality, we get

$$\begin{aligned} d(Su,u) &\leq \alpha d(Su,u) + \beta d(Tu,u) + \gamma d(Tu,u) + \delta d(u,u) + \eta d(Tu,Su) \\ &= \alpha d(Su,u) + \beta d(Su,u) + \gamma d(Su,u) + \delta d(u,u) + \eta d(Su,Su) \\ &\leq \alpha d(Su,u) + \beta d(Su,u) + \gamma d(Su,u) + k\delta[d(u,Su) + d(Su,u)] \\ &\quad +k\eta[d(Su,u) + d(u,Su)] \\ &= (\alpha + \beta + \gamma + k\delta + k\eta)d(Su,u) + (k\delta + k\eta)d(u,Su). \end{aligned}$$

This gives

$$\begin{aligned} d(Su,u) &\leq \frac{k\delta + k\eta}{1 - (\alpha + \beta + \gamma + k\delta + k\eta)} d(u,Su) \\ &\leq \frac{k\delta + k\eta}{1 - (k\alpha + k\beta + k\gamma + k\delta + k\eta)} d(u,Su). \end{aligned} \tag{3}$$

Similarly, consider

$$d(Sx_n,Su) \leq \alpha d(Sx_n,Tu)+\beta d(Tx_n,Su)+\gamma d(Tx_n,Tu)+\delta d(Tu,Su)+\eta d(Tx_n,Sx_n).$$

Letting $n \to \infty$ in above inequality, we get

$$\begin{aligned} d(u,Su) &\leq \alpha d(u,Tu) + \beta d(u,Su) + \gamma d(u,Tu) + \delta d(Tu,Su) + \eta d(u,u) \\ &= \alpha d(u,Su) + \beta d(u,Su) + \gamma d(u,Su) + \delta d(Su,Su) + \eta d(u,u) \\ &\leq \alpha d(u,Su) + \beta d(u,Su) + \gamma d(u,Su) + k\delta[d(u,Su) + d(Su,u)] \\ &\quad +k\eta[d(Su,u) + d(u,Su)] \\ &= (k\delta + k\eta)d(Su,u) + (\alpha + \beta + \gamma + k\delta + k\eta)d(u,Su). \end{aligned}$$

This gives

$$\begin{aligned} d(u,Su) &\leq \frac{k\delta + k\eta}{1 - (\alpha + \beta + \gamma + k\delta + k\eta)} d(Su,u) \\ &\leq \frac{k\delta + k\eta}{1 - (k\alpha + k\beta + k\gamma + k\delta + k\eta)} d(Su,u). \end{aligned} \tag{4}$$

Taking

$$\xi' = \max\left\{\frac{k\delta + k\eta}{1-(k\alpha + k\beta + k\gamma + k\delta + k\eta)}, \frac{k\delta + k\eta}{1-(k\alpha + k\beta + k\gamma + k\delta + k\eta)}\right\},$$

from inequalities (3) and (4), we get

$$d(Su, u) \leq \xi'^2 d(Su, u) \text{ and } d(u, Su) \leq \xi'^2 d(u, Su) \text{ where } 0 \leq \xi' < 1.$$

Thus $d(u, Su) = 0 = d(Su, u)$. This means that $Su = u$. Which in turn implies that $Tu = Su = u$ i.e. u is common fixed point of S and T.

Next, we prove that this common fixed point of S and T is unique. Let, if possible, u' be another common fixed point of S and T. Then from inequality (i) we can write

$$\begin{aligned}
d(u, u') &= d(Su, Su') \leq \alpha d(Su, Tu') + \beta d(Tu, Su') + \gamma d(Tu, Tu') + \delta d(Tu', Su') \\
&\quad + \eta d(Tu, Su) \\
&= \alpha d(u, u') + \beta d(u, u') + \gamma d(u, u') + \delta d(u', u') + \eta d(u, u) \\
&\leq \alpha d(u, u') + \beta d(u, u') + \gamma d(u, u') + k\delta[d(u', u) + d(u, u')] \\
&\quad + k\eta[d(u, u') + d(u', u)] \\
&= (\alpha + \beta + \gamma + k\delta + k\eta) d(u, u') + (k\delta + k\eta) d(u', u).
\end{aligned}$$

This gives

$$\begin{aligned}
d(u, u') &\leq \frac{k\delta + k\eta}{1-(\alpha + \beta + \gamma + k\delta + k\eta)} d(u', u) \\
&\leq \frac{k\delta + k\eta}{1-(k\alpha + k\beta + k\gamma + k\delta + k\eta)} d(u', u). \qquad (5)
\end{aligned}$$

Similarly, consider

$$\begin{aligned}
d(u', u) &= d(Su', Su) \\
&\leq \alpha d(Su', Tu') + \beta d(Tu', Su) + \gamma d(Tu', Tu) + \delta d(Tu, Su) + \eta d(Tu', Su') \\
&= \alpha d(u', u) + \beta d(u', u) + \gamma d(u', u) + \delta d(u, u) + \eta d(u', u') \\
&\leq \alpha d(u', u) + \beta d(u', u) + \gamma d(u', u) + k\delta[d(u', u) + d(u, u')] \\
&\quad + k\eta[d(u, u') + d(u', u)] \\
&= (\alpha + \beta + \gamma + k\delta + k\eta) d(u', u) + (k\delta + k\eta) d(u, u').
\end{aligned}$$

This gives

$$\begin{aligned}
d(u', u) &\leq \frac{k\delta + k\eta}{1-(\alpha + \beta + \gamma + k\delta + k\eta)} d(u, u') \\
&\leq \frac{k\delta + k\eta}{1-(k\alpha + k\beta + k\gamma + k\delta + k\eta)} d(u, u'). \qquad (6)
\end{aligned}$$

Taking $\epsilon = \frac{k\delta + k\eta}{1-(k\alpha + k\beta + k\gamma + k\delta + k\eta)}$, from inequalities (5) and (6), we get

$$d(u, u') \leq \epsilon^2 d(u, u') \quad \text{and} \quad d(u', u) \leq \epsilon^2 d(u', u), \quad \text{where } 0 \leq \epsilon < 1.$$

We arrive at the conclusion that $d(u,u') = 0 = d(u',u)$ i.e. $u = u'$. Thus u is a unique common fixed point of S and T. Hence the theorem.

EXAMPLE 9. Consider $X = [1,\infty)$ with $d(x,y) = |x-y| + 2|x-1| + |y-1|$. Then (X,d) is a dqb-metric space with coefficient $k = 2$. Let $Sx = 2x-1$ and $Tx = x^4$. Note that for the sequence $\{x_n\} = 1 + 1/n, n \in N$, we get $\lim Sx_n = \lim Tx_n = 1$ where $1 \in S(X) \cup T(X)$. In other words, S and T satisfy E.A like property. Also we observe that S and T are weakly compatible. Now,

$$\begin{aligned} d(Sx,Sy) &= d(2x-1,2y-1) = |2x-1-2y+1| + 2\,|2x-1-1| + |2y-1-1| \\ &= |2x-2y| + 2\,|2x-2| + |2y-2|, \end{aligned}$$

$$\begin{aligned} d(Sx,Ty) &= d(2x-1,y^4) = \left|2x-1-y^4\right| + 2\,|2x-1-1| + |y^4-1| \\ &= \left|2x-1-y^4\right| + 2|2x-2| + |y^4-1|, \end{aligned}$$

$$\begin{aligned} d(Tx,Sy) &= d(x^4,2y-1) = |x^4-2y-1| + 2|x^4-1| + |2y-1-1| \\ &= |x^4-2y-1| + 2|x^4-1| + |2y-2|, \end{aligned}$$

$$d(Tx,Ty) = d(x^4,y^4) = |x^4-y^4| + 2|x^4-1| + |y^4-1|,$$

$$\begin{aligned} d(Ty,Sy) &= d(y^4,2y-1) = |y^4-2y-1| + 2|y^4-1| + |2y-1-1| \\ &= |y^4-2y-1| + 2|y^4-1| + |2y-2|, \end{aligned}$$

$$\begin{aligned} d(Tx,Sx) &= d(x^4,2x-1) = |x^4-2x-1| + 2|x^4-1| + |2x-1-1| \\ &= |x^4-2x-1| + 2|x^4-1| + |2x-2|. \end{aligned}$$

It is easy to verify that for all $x,y \in X$,

$$\begin{aligned} d(2x-1,2y-1) \leq\; & \frac{1}{25}d(2x-1,y^4) + \frac{1}{25}d(x^4,2y-1) + \frac{1}{25}d(x^4,y^4) \\ & + \frac{1}{25}d(y^4,2y-1) + \frac{1}{25}d(x^4,2x-1). \end{aligned}$$

Where

$$\alpha = \frac{1}{25} = \beta = \gamma = \delta = \eta$$

and

$$0 \leq \alpha+\beta+\gamma+\delta+\eta = \frac{1}{25}+\frac{1}{25}+\frac{1}{25}+\frac{1}{25}+\frac{1}{25} = \frac{5}{25} = \frac{1}{5} < \frac{1}{4}.$$

Thus S and T satisfy all the conditions of the theorem and hence have a unique common fixed point 1 in $X = [1,\infty)$. Uniqueness can also be established by observing that $x^4 = 2x-1$ i.e. $x^4 - 2x + 1 = 0$ has only two real roots 1 and other less than 1. Thus it is clear that 1 is the only common fixed point of S and T in $X = [1,\infty)$.

THEOREM 6. Let (X,d) be a dqb-metric space with coefficient $k > 1$ and S and T be two self maps on X satisfying the following conditions:

(i) $d(Sx,Sy) \leq \alpha[d(Sx,Ty)+d(Tx,Sy)]+\beta[d(Sx,Ty)+d(Tx,Ty)]+\gamma[d(Tx,Sy)+d(Tx,Ty)]$ for all $x,y \in X$ and the constants $\alpha,\beta,\gamma \geq 0$ are such that $0 \leq \alpha+\beta+\gamma < \frac{1}{4k}$,

(ii) S and T satisfy E.A like property,

(iii) S and T are weakly compatible.

Then T and S have a unique common fixed point in X.

PROOF. In view of assumption (ii), there exists a sequence $\{x_n\}$ in X and $u \in S(X) \cup T(X)$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = u.$$

Let us assume that $\lim_{n\to\infty} Sx_n = u \in T(X)$. Now we can find $v \in X$ such that $Tv = u$. Now from inequality (i), taking $x = v$ and $y = x_n$, we can write

$$\begin{aligned} d(Sv,Sx_n) \quad &\leq \quad \alpha[d(Sv,Tx_n)+d(Tv,Sx_n)]+\beta[d(Sv,Tx_n)+d(Tv,Tx_n)] \\ &\quad +\gamma[d(Tv,Sx_n)+d(Tv,Tx_n)]. \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we get

$$\begin{aligned} d(Sv,u) &\leq \alpha[d(Sv,u)+d(Tv,u)]+\beta[d(Sv,u)+d(Tv,u)]+\gamma[d(Tv,u)+d(Tv,u)] \\ &= \alpha[d(Sv,u)+d(u,u)]+\beta[d(Sv,u)+d(u,u)]+\gamma[d(u,u)+d(u,u)] \\ &= (\alpha+k\alpha+\beta+k\beta+2k\gamma)d(Sv,u)+(k\alpha+k\beta+2k\gamma)d(u,Sv) \\ &\leq 2k(\alpha+\beta+\gamma)d(Sv,u)+2k(\alpha+\beta+\gamma)d(u,Sv). \end{aligned}$$

This gives

$$d(Sv,u) \leq \frac{2k(\alpha+\beta+\gamma)}{1-2k(\alpha+\beta+\gamma)}d(u,Sv). \tag{7}$$

Similarly, taking $x = x_n$ and $y = v$, in condition (i), we can write

$$\begin{aligned} d(Sx_n,Sv) \quad &\leq \quad \alpha[d(Sx_n,Tv)+d(Tx_n,Sv)]+\beta[d(Sx_n,Tv)+d(Tx_n,Tv)] \\ &\quad +\gamma[d(Tx_n,Sv)+d(Tx_n,Tv)]. \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we get

$$\begin{aligned} d(u,Sv) &\leq \alpha[d(u,Tv)+d(u,Sv)]+\beta[d(u,Tv)+d(u,Tv)]+\gamma[d(u,Sv)+d(u,Tv)] \\ &\leq \alpha[d(u,u)+d(u,Sv)]+\beta[d(u,u)+d(u,u)]+\gamma[d(u,Sv)+d(u,u)] \\ &\leq (\alpha+k\alpha+2k\beta+\gamma+k\gamma)d(u,Sv)+(k\alpha+2k\beta+k\gamma)d(Sv,u) \\ &\leq 2k(\alpha+\beta+\gamma)d(u,Sv)+2k(\alpha+\beta+\gamma)d(Sv,u). \end{aligned}$$

This gives

$$d(u,Sv) \leq \frac{2k(\alpha+\beta+\gamma)}{1-2k(\alpha+\beta+\gamma)}d(Sv,u). \tag{8}$$

From inequalities (7) and (8), we get

$$d(u,Sv) \leq \left(\frac{2k(\alpha+\beta+\gamma)}{1-2k(\alpha+\beta+\gamma)}\right)^2 d(Sv,u)$$

and

$$d(Sv,u) \leq \left(\frac{2k(\alpha+\beta+\gamma)}{1-2k(\alpha+\beta+\gamma)}\right)^2 d(u,Sv)$$

where $0 \leq \frac{2k(\alpha+\beta+\gamma)}{1-2k(\alpha+\beta+\gamma)} < 1$. Hence, we conclude that $d(Sv,u) = 0 = d(u,Sv)$ i.e. $Sv = u$. Thus $Tv = u = Sv$. As we know that S and T are weakly compatible, we conclude that v is a coincidence point of S and T, so that $S(Tv) = T(Sv)$ implies that $Su = Tu$.

Now we claim that u is a common fixed point of S and T. For this, we consider

$$\begin{aligned} d(Su,Sx_n) \quad \leq \quad & \alpha[d(Su,Tx_n)+d(Tu,Sx_n)]+\beta[d(Su,Tx_n)+d(Tu,Tx_n)] \\ & +\gamma[d(Tu,Sx_n)+d(Tu,Tx_n)]. \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we get

$$\begin{aligned} d(Su,u) &\leq \alpha[d(Su,u)+d(Tu,u)]+\beta[d(Su,u)+d(Tu,u)]+\gamma[d(Tu,u)+d(Tu,u)] \\ &= \alpha[d(Su,u)+d(Su,u)]+\beta[d(Su,u)+d(Su,u)]+\gamma[d(Su,u)+d(Su,u)] \\ &= (2\alpha+2\beta+2\gamma)d(Su,u). \end{aligned}$$

This gives, since $2\alpha+2\beta+2\gamma < 1$, $d(Su,u) = 0$. Similarly, we can show that $d(u,Su) = 0$. Thus we get $d(Su,u) = 0 = d(u,Su)$ which implies that $Su = u$ and $Su = u = Tu$. Hence we infer that u is a common fixed point of T and S. Next we claim that u is a unique common fixed point of T and S. Let, if possible, u' be another common fixed point of S and T. Then from inequality (i) we can write

$$\begin{aligned} d(u,u') \quad &= \quad d(Su,Su') \\ &\leq \quad \alpha[d(Su,Tu')+d(Tu,Su')]+\beta[d(Su,Tu')+d(Tu,Tu')] \\ & \qquad +\gamma[d(Tu,Su')+d(Tu,Tu')] \\ &= \quad \alpha[d(u,u')+d(u,u')]+\beta[d(u,u')+d(u,u')]+\gamma[d(u,u')+d(u,u')] \\ &= \quad 2(\alpha+\beta+\gamma)d(u,u'). \end{aligned}$$

Since $2\alpha+2\beta+2\gamma < 1$, this gives that $d(u,u') = 0$. Similarly, we show that $d(u',u) = 0$. Thus $d(u,u') = 0 = d(u',u)$ which implies that $u = u'$. Thus u is a unique common fixed point of S and T. Hence the theorem.

EXAMPLE 10. Consider $X = [1,\infty)$ with $d(x,y) = |x-y|^2 + 2|x-1| + |y-1|$. Then (X,d) is a *dqb*-metric space with coefficient $k = 2$. Let $Sx = 2x-1$ and $Tx = x^7$. Note that for the sequence $\{x_n\} = 1+1/n, n \in N$ we get $\lim Sx_n = \lim Tx_n = 1$ where $1 \in S(X) \cup T(X)$. In other words S and T satisfy E.A like property. Also observe that

S and T are weakly compatible. Now

$$\begin{aligned} d(Sx,Sy) &= d(2x-1,2y-1) = |2x-2y|^2 + 2|2x-2| + |2y-2|, \\ d(Sx,Ty) &= d(2x-1,y^7) = |2x-1-y^7|^2 + 2|2x-2| + |y^7-1|, \\ d(Tx,Sy) &= d(x^7,2y-1) = |x^7-2y-1|^2 + 2|x^7-1| + |2y-1|, \\ d(Sx,Ty) &= d(2x-1,y^7) = |2x-1-y^7|^2 + 2|2x-2| + |y^7-1|, \\ d(Tx,Ty) &= d(x^7,y^7) = |x^7-y^7|^2 + 2|x^7-1| + |y^7-1|. \end{aligned}$$

It is easy to verify that, for all $x, y \in X$,

$$\begin{aligned} d(2x-1,2y-1) \quad \le \quad & \alpha[d(2x-1,y^7) + d(x^7,2y-1)] + \beta[d(2x-1,y^7) + d(x^7,y^7)] \\ & +\gamma[d(x^7,2y-1) + d(x^7,y^7)] \end{aligned}$$

taking $\alpha = \beta = \gamma = \frac{1}{27}$ so that $\alpha + \beta + \gamma = \frac{1}{9} < \frac{1}{8}$. Thus S and T satisfy all the conditions of the above theorem and hence have the unique common fixed point 1 in X.

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