Applied Mathematics E-Notes, 19(2019), 46-54 © ISSN 1607-2510
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Characterization Of Best Approximation Of Closed Convex Subsets In $C_b(X, Y)$*

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Received 20 February 2018

Abstract

The purpose of this paper is to prove various Kolmogorov type criteria for $C_b(X, Y)$, the space of bounded continuous functions supplied with the sup-norm from a topological space X into a Banach space Y. We introduce the maximum-numerical range f with respect to g for each f, g to characterize best approximation points in closed convex subsets of $C_b(X, Y)$.

1 Introduction

The approximation problem for space $C(X, H)$ of continuous functions from a compact metric space X into a unitary space H, supplied with the sup-norm, was first introduced by Brosowski [4]. His main result states that the best approximation of subset G to element f can be characterized by means of a Kolmogorov criterion if and only if G satisfies some regularity property. Later Poleunis and Van Devel extend this characterization theorem to the case that X is an arbitrary topological space in which also noncompact spaces are involved [8]. The case where X is compact Hausdorff space and H is the complex number has been studied and developed by many authors [9, 11, 12, 13]. There exist many generalizations of the Kolmogorov theorem (see, e.g., the survey [3]). In this paper, an attempt is made to extend this characterization theorem for the case that X is an arbitrary topological space and Y is Banach space by using the concept of numerical range. This concept goes back to Toeplitz, who defined in 1918 the field of values of a matrix, a concept easily extensible to bounded linear operators on a Hilbert space. Later, Lumer and Bauer gave independent but related extensions of Toeplitz's numerical range to bounded linear operators on Banach spaces which do not use the algebraic structure of the space of all bounded linear operators. These works have been extended to the more general setting, as bounded uniformly continuous functions from the unit sphere of a Banach space to space, but it is not possible to be extended to all bounded functions[1, 2, 10]. For more information and background, we refer the reader to the paper [3]. We define maximum numerical range for operators between different Banach spaces, which can easily be extended to all bounded functions and which is never empty. As the main result, we give some results

*Mathematics Subject Classifications: 41A50, 26B25.

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to characterize the best approximation of convex sets in $C_b(X,Y)$ by using the concept of numerical range.

2 Characterization of Approximation Points

Let X be a topological space and Y a Banach space. We denote by $C_b(X,Y)$, the space of bounded continuous function $f : X \to Y$ equipped with the norm

$$\|f\| = \sup_{x \in X} \|f(x)\|.$$

Let W be a nonempty subset of the normed space $C_b(X,Y)$ and $f \in C_b(X,Y)$. The set of all best approximations to f from W is denoted by $\mathbf{P}_W(f)$. Thus we define

$$\mathbf{P}_W(f) := \{g_0 \in W | \ \|f - g_0\| = \inf_{g \in W} \|f - g\|\}.$$

We are interested in the problem of finding and describing the function g_0 which may be the best approximation for the operator f. In 1948, Kolmogorov generalized the characterization of the best approximation for the polynomial of a fixed continuous function.

THEOREM 1. Let U be a linear subspace of the space $C(X)(X$ is a compact set) and let $f \notin U$. Then $g_0 \in \mathbf{P}_U(f)$, if and only if, for each g,

$$Re \max\{(f(q) - g_0(q))g(q)\} > 0,$$

where $|x(q) - g_0(q)| = \max |x(t) - g_0(t)|$.

For the case of non-compact metric X and arbitrary unitary space Y. Poleunis and Van de Vel extend Brosowskis main result states that the best approximation of convex set U can be characterized by means of a Kolmogorov criterion if and only if U satisfies some regularity property.

THEOREM 2. Let U be a closed convex subset of $C_b(X,Y)$. If G is regular, then $g_0 \in \mathbf{P}_U(f)$, if for $\varepsilon > 0$ and $h \in U$,

$$\sup Re\langle (f - g_0)(x), (g_0 - h)(x)\rangle \geq 0,$$

where $x \in \mathbb{O}(\varepsilon) := \{x \in X : ||f(x) - g_0(x)|| > \|f(x) - g_0(x)|\| - \varepsilon\}$.

In [6], Harris by using the numerical range of f with respect to g gives a condition it which equivalent to an extension of non-compact spaces of Kolmogoroff 's characterization of functions of best approximation.

DEFINITION 1. (Intrinsic numerical range). Let Y be a Banach space and let X be a non-empty set and $g \in C_b(X;Y)$ with $\|g\| = 1$. For every $f \in C_b(X;Y)$, the intrinsic numerical range of f relative to g is given by

$$V_g(f) := \{\phi(f) : \ \phi \in C_b(X;Y)^*, \|\phi\| = \phi(g) = 1\}.$$

DEFINITION 2. (Spatial numerical range). Let Y be a Banach space and let X be a non-empty set and $g \in C_b(X;Y)$ with $\|g\| = 1$. For every $f \in C_b(X;Y)$, the spatial numerical range of f relative to g is given by

$$\Phi_g(f) := \{\phi(f(t)) : \ \phi \in B_{Y^*}, \phi(g(t)) = 1\}.$$

THEOREM 3. Let $f, g \in C_b(X, Y)$ where X is a topological space and Y is a Hilbert space. Let U be a closed convex subset of $C_b(X, Y)$ such that $f \notin U$ and $f - g_0$ is a scalar multiple of g. Then the following are equivalent:

(1) $g_0 \in \mathbf{P}_U(f)$ and $\sup Re\Phi_g(f) \leq \sup ReV_g(f)$.

(2) For each $h \in U$,

$$\sup Re\langle (f - g_0)(x), (g_0 - h)(x)\rangle \geq 0,$$

where the sup is taken over all those x for which $\|(f - g_0)(x)\| = \|f - g_0\|$.

By using arguments similar to those of Kolmogorov and Harris, it can be shown that the characterization of best approximation theorem holds for arbitrary topological and arbitrary Banach spaces. Now, we define maximum numerical range of f with respect to g by

$$W_g(f) := \{\lambda \in \mathbb{C} : \lambda = \lim_{n\to\infty} \phi(f(x_n)), (x_n, \phi)_{n\in\mathbb{N}} \in Z_g\},$$

where

$$Z_g = \{(x_n, \phi)_{n\in\mathbb{N}} \in X \times B_{Y^*}, \ \lim_{n\to\infty} \phi(g(x_n)) = \|g\|\}. \tag{1}$$

PROPOSITION 1. Let $f, g \in C_b(X, Y)$. Then $W_g(f)$ is a non-empty compact subset of the complex plane.

PROOF. By definition of the norm on $C_b(X, Y)$, there exists $\{x_n\}_{n\in\mathbb{N}}$ such that

$$\|g\| = \sup_{x\in X} \|g(x)\| = \lim_{n\to\infty} \|g(x_n)\| = \lim_{n\to\infty} \sup_{\phi\in B_{Y^*}} |\phi(g(x_n))|.$$

As B_{Y^*} is w^*-compact set then there is a ϕ_0 such that

$$\lim_{n\to\infty} \sup_{\phi\in B_{Y^*}} |\phi(g(x_n))| = \lim_{n\to\infty} |\phi_0(g(x_n))|.$$

Put $\lambda_n = \phi_0(f(x_n))$. Since $\{\lambda_n\}$ is bounded sequence, there is a convergence subsequence $\{\lambda_{n_i}\}$. Let $\lambda = \lim_{n_i\to\infty} \phi_0(f(x_{n_i}))$. Then $\lambda \in W_g(f)$ and it a non-empty set. It is trivial that $W_g(f)$ is a closed set. Since f is a bounded function, for $\lambda \in W_g(f)$, we have

$$|\lambda| = |\lim_{n\to\infty} \phi(f(x_n))| \leq \|\phi\|\|f(\lim_{n\to\infty} x_n)\| \leq \|f\| < \infty.$$

Thus $W_g(f)$ is bounded. Then by Heine-Borel theorem it is a compact set.

PROPOSITION 2. Let X be a compact topological space. For every continuous function g with norm one, we have $W_g(f) = \Phi_g(f)$, for each $f \in C_b(X, Y)$.

PROOF. It is trivial.

For $x, y \in$ in X, the norm of X is Gateaux differentiable at $x \neq 0$, if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}, \text{exists for each } y \in X.$$

The norm of X is called Frechet differentiable at x if the convergence to the limit is uniform for all $y \in X$. In general, the norm is not Gateaux differentiable at $x \neq 0$ in X. Nevertheless, it is known [5] that the limits always exist. Norm derivatives of the spaces $L^p(d\mu), 1 \leq p < \infty$, and $C(X)$, the space of real continuous functions on a compact Hausdorff space X, have been studied extensively (see for example [7]). Recall the norm on X is uniformly Gateaux differentiable if and only if whenever $\{\phi_n\}$ and $\{\psi_n\}$ are sequences in B_{Y^*} such that $\|\phi_n + \psi_n\| \to 2$ it follows that $\|\psi_n - \phi_n\| \to 0$ (in the case X is called uniform smooth).

PROPOSITION 3. Let Y be a uniformly smooth Banach space, X a non-empty set and $g \in C_b(X; Y)$ with $\|g\| = 1$. Then $W_g(f) = \overline{\Phi_g(f)}$ for every $f \in C_b(X, Y)$.

PROOF. Let $\lambda \in W_g(f)$, $\{x_n\}_{n \in \mathbb{N}}$ a sequence in X and $\phi \in B_{Y^*}$ such that $\phi(f(x_n)) \longrightarrow 1$ and $\phi(g(x_n)) \longrightarrow 1$. For every $n \in \mathbb{N}$, we take $\psi_n \in B_{Y^*}$ such that $\psi_n(g(x_n)) = 1$. Since $\|\phi + \psi_n\| \to 2$, the uniform smoothness of Y gives that $\|\phi - \psi_n\| \to 0$. Now, we have

$$\begin{aligned} |\psi_n(f(x_n)) - \lambda| &\leq |\phi(f(x_n)) - \lambda| + |\phi(f(x_n)) - \psi_n(f(x_n))| \\ &\leq |\phi(f(x_n)) - \lambda| + \|\phi - \psi\| \|f\| \to 0. \end{aligned}$$

Therefore $\lambda \in \overline{\Phi_g(f)}$. Then $W_g(f) \subseteq \overline{\Phi_g(f)}$, in other hands, $\overline{\Phi_g(f)} \subseteq W_g(f)$. So $W_g(f) = \overline{\Phi_g(f)}$.

In the following, we use the concept of numerical range to characterize best approximation points in closed convex subsets of $C_b(X, Y)$.

We write $Re(.)$ to denote the real part of (.) and define the directional derivative of the norm in point x along y by

$$\tau(x, y) := \limsup_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t}.$$

THEOREM 4. Let U be a closed convex subset of $C_b(X, Y)$, $f \in C_b(X, Y) \setminus U$ and $g_0 \in U$. Then the following statements are equivalent.

i) $g_0 \in \mathbf{P}_U(f)$.

ii) For each $h \in U$,

$$\max ReW_{f-h}(h-g_0) \leq 0. \tag{2}$$

PROOF. $i \to ii$. Let $h \in C_b(X,Y)$ and $(\{x_n\}, \phi^*) \in Z_{f-h}$ be as in (1). Then

$$\begin{aligned}
\|f-h+t(h-g_0)\| &\geq \lim_{n\to\infty} \|(f-h+t(h-g_0)(x_n)\| \\
&= \lim_{n\to\infty} \sup_{\phi\in B_{Y^*}} |\phi((f-h+t(h-g_0))(x_n))| \\
&\geq \lim_{n\to\infty} \sup_{\phi\in B_{Y^*}} Re\, \phi((f-h+t(h-g_0))(x_n)) \\
&\geq \lim_{n\to\infty} Re\, \phi((f-h)(x_n)) + Re\, \phi(t(h-g_0)(x_n)) \\
&= \|f-h\| + Re \lim_{n\to\infty} \phi((h-g_0)(x_n)).
\end{aligned}$$

Hence

$$Re \lim_{n\to\infty} \phi((h-g_0)(x_n)) \leq \frac{\|f-h+t(h-g_0)\| - \|f-h\|}{t}.$$

Setting $t \to 0^+$, and taking lim sup, thus

$$\max ReW_{f-h}(h-g_0) \leq \tau(f-h, h-g_0). \tag{3}$$

Since $g_0 \in \mathbf{P}_U(f)$, for $h \in U$ and $t = 1$, we have

$$\|f-h+t(h-g_0)\| - \|f-h\| \leq 0.$$

As the function φ defined by $\varphi(t) = \frac{\|f-h+t(h-g_0)\|-\|f-h\|}{t}$ is non-decreasing, setting $t \to 0^+$, and taking lim sup, therefore $\tau(f-h, h-g_0) \leq 0$. Now by (3), we get (2).

$ii \to i$. It is not restrictive to assume $g_0 = 0$. Let inequality (2) hold but $g_0 \notin \mathbf{P}_U(f)$, then there exists $h_1 \in U \setminus \{0\}$, such that $\|f-h_1\| < \|f\|$. By applying (2) to $h_\lambda = \lambda h_1$, for $0 < \lambda \leq 1$, we get

$$\max ReW_{f-h_\lambda}(h_\lambda) = \max ReW_{f-\lambda h_1}(\lambda h_1) \leq 0.$$

Since $0 < \lambda$, we have

$$\max ReW_{f-\lambda h_1}(h_1) \leq 0,$$

and

$$\begin{aligned}
\max ReW_{f-\lambda h_1}(-h_1) &\geq \min ReW_{f-\lambda h_1}(-h_1) \\
&= -\max ReW_{f-\lambda h_1}(h_1) \geq 0.
\end{aligned}$$

Then

$$\tau(f-\lambda h_1, -h_1) \geq \max ReW_{f-\lambda h_1}(-h_1) \geq 0.$$

Since τ is upper semi-continuous in its arguments, we have

$$\tau(f,-h_1) \geq \limsup_{\lambda\to 0^+} \tau(f-\lambda h_1,-h_1) \geq 0.$$

This implies that there exists ε_1 such that for $t \in (0,\varepsilon_1]$, we have $\frac{\|f-th_1\|-\|f\|}{t} \geq 0$. Again since φ is non-decreasing, we have $\|f\| \leq \|f-h_1\|$, which is a contradiction.

LEMMA 1. Let U be a closed convex subset of $C_b(X,Y)$, $f \in C_b(X,Y) \setminus U$ and $g_0 \in U$. If for each $h \in U$, $\max Re\, W_{f-h}(h-g_0) \geq 0$, then

$$\max ReW_{f-g_0}(g_0-h) \geq 0.$$

PROOF. Suppose, on the contrary, it is possible to find an element $h \in U$ such that

$$\max Re\, W_{f-g_0}(g_0-h) = -\delta < 0.$$

Let (y_n, ϕ_0) be an element that $\lim Re\, \phi_0(g_0-h)(y_n) = \max Re\, W_{f-g_0}(g_0-h)$. Since g_0-h and ϕ_0 are continuous functions on X, there exists an open set $G \subseteq X$ such that

$$\max_{x\in G} Re\, \phi_0(g_0-h)(x) < -\delta.$$

Then there exists $0 < \varepsilon_0$, such that for each $\varepsilon \in (0,\varepsilon_0]$, we have

$$X_{f-g_0}(\varepsilon) = \{\{x_n\}_{n\in\mathbf{N}} \in X : \lim_{n\to\infty} \|(f-g_0)(x_n)\| \geq \|f-g_0\| - \varepsilon\} \subseteq G,$$

and

$$\max Re\, \phi_0(g_0-h)(x_n) < -\frac{\delta}{2},\ where\ \{x_n\} \in X_{f-g_0}(\varepsilon).$$

Put $D = X_{f-g_0}(\varepsilon_0)$, $\varepsilon_1 \leq \frac{1}{2}\varepsilon_0$ and

$$h_t = th + (1-t)g_0, \quad for\ 0 < t < 1.$$

Since $h_t \to g_0$ as $t \to 0$, there exists $t_0 > 0$ such that for any $0 < t \leq t_0$,

$$\|h_t - g_0\| \leq \varepsilon_1. \tag{4}$$

Now

$$\begin{aligned} Re \lim_{n\to\infty} \phi_0((g_0-h_t)(x_n)) &= Re \lim_{n\to\infty} \phi_0(th+(1-t)g_0-g_0)(x_n)) \\ &= Re \lim_{n\to\infty} \phi_0(t(h-g_0)(x_n)) \\ &= -tRe \lim_{n\to\infty} \phi_0((g_0-h)(x_n) \\ &> \frac{\delta}{2}. \end{aligned}$$

Then we have

$$Re \lim_{n\to\infty} \phi_0((g_0-h_t)(x_n)) > 0,\ for\ \{x_n\}_{n\in\mathbf{N}} \in D.$$

On the other hand, by (4), for each $(x_n,\phi)_{n\in\mathbf{N}} \in Z_{f-h_t}$, we have

$$\begin{aligned}\lim_{n\to\infty}\|(f-g_0)(x_n)\| &= \lim_{n\to\infty}\|(f-h_t)(x_n)-(g_0-h_t)(x_n)\|\\ &\geq \lim_{n\to\infty}\|(f-h_t)(x_n)\|-\|(g_0-h_t)(x_n)\|\\ &\geq \lim_{n\to\infty}\phi_0((f-h_t)(x_n))-\|(g_0-h_t)(x_n)\|\\ &\geq \|f-h_t\|-\|g_0-h_t\|\\ &\geq \|f-g_0\|-2\varepsilon_1\\ &\geq \|f-g_0\|-\varepsilon_0.\end{aligned}$$

It follows that $Z_{f-h_t}\subseteq D$. Now by (4), we get

$$\begin{aligned}\max ReW_{f-h_t}(h_t-g_0) &= \max_{(x_n,\phi)\in Z_{f-h_t}} Re\lim_{n\to\infty}\phi(h_t-g_0)(x_n))\\ &\geq \min_{(x_n,\phi)\in Z_{f-h_t}} Re\lim_{n\to\infty}\phi_0(h_t-g_0)(x_n))\\ &\geq \min_{x\in D} Re\,\phi_0(h_t-g_0)(x)) > 0.\end{aligned}$$

This contradiction completes the proof.

THEOREM 5. Let U be a closed convex subset of $C_b(X,Y)$, $f\in C_b(X,Y)\setminus U$ and $g_0\in U$. Then the following statements are equivalent.

i) $g_0\in\mathbf{P}_U(f)$.

ii) For each $h\in U$,

$$\max Re\,W_{f-g_0}(g_0-h)\geq 0. \tag{5}$$

PROOF. $i\to ii$. Since $g_0\in\mathbf{P}_U(f)$, by Theorem 4, for each $h\in U$, we have

$$\max Re\,W_{f-h}(h-g_0)\leq 0.$$

Now by Lemma 1 we obtain (5).

$ii\to i$. Let (5) be true and (x_n^h,ϕ) be a sequence of Z_{f-g_0} such that

$$\lim_{n\to\infty} Re\,\phi(g_0-h)(x_n^h)\geq 0.$$

Therefore,

$$\begin{aligned}\|f-h\| &\geq \left\|(f-h)\left(x_n^h\right)\right\|\geq\phi((f-h)\left(x_n^h\right))\\ &= \lim_{n\to\infty}\phi((f-g_0+g_0-h)\left(x_n^h\right))\\ &= \lim_{n\to\infty}\left[\phi((f-g_0)\left(x_n^h\right))+\phi((g_0-h)\left(x_n^h\right))\right]\\ &\geq \lim_{n\to\infty}\phi((f-g_0)\left(x_n^h\right))=\|f-g_0\|.\end{aligned}$$

This completes the proof.

COROLLARY 1. Let U be a closed convex subset of $C_b(X,Y)$, $f \in C_b(X,Y) \setminus U$ and $g_0 \in U$. Then the following statements are equivalent.

i) $g_0 \in P_U(f)$.

ii) For each $h \in U$,

$$\max Re\, W_{f-h}(h-g_0) \leq 0 \leq \max Re W_{f-g_0}(g_0-h).$$

PROOF. It is a consequence of Theorems 4 and 5.

COROLLARY 2. Let $f \in C_b(X,Y)$ where X is a topological metric space and Y is a Hilbert space. Let U be a closed convex subset of $C_b(X,Y)$ with $f \notin U$. Then $g_0 \in P_U(f)$ if and only if

$$\max \lim_{n\to\infty} Re\langle (f-g_0)(x_n^h), (g_0-h)(x_n^h)\rangle \geq 0, \text{ for each } h \in U, \tag{6}$$

where $\lim_{n\to\infty} \|(f-g_0)(x_n^h)\| = \|f-g_0\|$.

PROOF. It is a consequence of the Riesz representation theorem and Theorem 5.

EXAMPLE 1. Let $X = C$, $Y = X$ and $U = co(I)$ be the set of all convex combination of I, where I is the identity operator. Suppose that the function f defined by $(x_1, x_2) \to (-x_2, x_1)$. By Corollary 1, $g_0 = \lambda_0 I \in P_U(f)$ if and only if the inequality (6) holds for every λ. But this inequality holds only if $\lambda_0 = 0$. Then $P_U(f) = \{0\}$. Also, we can show this without applying Corollary 1. For $\lambda \in [0,1]$, we have

$$\|f - \lambda I\| \geq r(f-\lambda I) = \sup\{|-\lambda \pm i|\} \geq 1.$$

Thus $\inf_{\lambda\in[0,1]} \|f-\lambda I\| \geq 1$, in the other hand $\|f-0\| = 1$, therefore $P_U(f) = \{0\}$.

EXAMPLE 2. Let $X = Y$, $Y = \{z : z \in C, |z| \leq 1\}$. Let $f \in C(X,Y)$ be defined by $f(z) = z^n$ and $U = P_{n-1}$ be the set of all polynomials of degree at most $n-1$. By Corollary 1, $P_U(f) = \{0\}$.

Acknowledgment. The authors wish to thank the referees for reading the article carefully and for the comments made on this article.

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