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Graph Convergence For η-Subdifferential Mapping With Application *

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Abstract

In this paper, we introduce the concept of graph convergence for η-subdifferential mapping of a nonconvex, proper, lower semi-continuous and subdifferential functional on Banach space and discuss its existence and Lipschitz continuity. Further, we prove equivalence between graph convergence and resolvent operator convergence. We propose a new iterative algorithm for solving the system of generalized implicit variational-like inclusions. Furthermore, we prove the existence of solution for the system of generalized implicit variational-like inclusions and discuss the convergence of iterative sequences generated by proposed algorithm.

1 Introduction

Variational inequality theory has become a very effective and powerful tool in pure and applied sciences and has been used in a large class of problems arising in differential equations, mechanics, optimization and control, contact problems in elasticity and general equilibrium problems, see, [1, 3, 5, 8, 9, 10, 14, 15, 16]. Variational inclusion is an important and useful generalization of the variational inequality. One of the most important and interesting problem in the theory of variational inequality is the development of an efficient and implementable iterative algorithm for solving the variational inequalities. Variational inclusions include variational, quasi-variational, variational-like inequalities as special cases. In 1994, Hassouni and Moudafi [17] introduced and studied a class of variational inclusions. Later, Adly [1], Huang [18], Ding [8, 11], Ding and Luo [9] and Ding and Feng [12] have obtained some important generalizations of the results in [17].

Recently, many authors have studied the perturbed algorithms for variational inequalities involving monotone mappings in Hilbert spaces. Using the concept of graph convergence for maximal monotone mappings, Attouch [2] showed the equivalence between graph convergence and resolvent operator convergence, they constructed some perturbed algorithm for variational inequality and proved the convergence of sequences generated by perturbed algorithm under some suitable conditions. Further Li and Huang [25] generalized the concept of graph convergence for $H(\cdot,\cdot)$-accretive mapping in Banach space.

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In recent past, Ding and Xia [13] introduced the concept of P-proximal mapping for a nonconvex, proper, lower semi-continuous and subdifferentiable functional on Banach space and prove the existence and Lipschitz continuity. Sun et al. [28], Kazmi and Bhat [21] and Kazmi et al. [22, 23] generalized the concept of M-proximal mappings.

Motivated and inspired by the research works going on in this direction, in this paper, we introduce a new concept of graph convergence for η-subdifferential mapping of a nonconvex, proper, lower semi-continuous and subdifferential functional on Banach space and shown its existence and Lipschitz continuity. Further, we prove equivalence between graph convergence and resolvent operator convergence. We propose a new iterative algorithm for solving the system of generalized implicit variational-like inclusions. Furthermore, we prove the existence of the solution for the system of generalized implicit variational-like inclusions and discuss the convergence of iterative sequences generated by proposed algorithm.

2 Preliminaries

Let E be a real Banach space equipped with norm $\|\cdot\|, E^*$ be the topological dual of E and $\langle\cdot,\cdot\rangle$, be the duality pairing between E and E^*. Let 2^E, (respectively, $CB(E)$) be the family of all nonempty (respectively, closed and bounded) subsets of E, let $\mathcal{D}(\cdot,\cdot)$ be the Hausdorff metric on $CB(E)$ defined by

$$\mathcal{D}(A,B) = \max\left\{\sup_{x\in A}\ d(x,B),\ \sup_{y\in B}\ d(A,y)\right\},$$

where

$$A,B\in CB(E),\quad d(x,B)=\inf_{y\in B}\ d(x,y)\ \text{ and }\ d(A,y)=\inf_{x\in A}\ d(x,y).$$

The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \left\{f\in E^* : \langle x,f\rangle = \|x\|^2,\ \|f\|_{E^*} = \|x\|\right\},\quad \forall x\in E.$$

It is well known that if E is smooth, then J is single-valued and if $E \equiv H$, a Hilbert space, then J is the identity mapping.

DEFINITION 2.1 ([7]). A Banach space E is called smooth, if for every $x \in E$ with $\|x\| = 1$, there exists a unique $f \in E^*$ such that $\|f\| = f(x) = 1$. The modulus of smoothness of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(t) = \sup\left\{\frac{\|x+y\|+\|x-y\|}{2} - 1 : \|x\|\le 1 \text{ and } \|y\|\le t\right\}.$$

A Banach space E is called uniformly smooth, if

$$\lim_{t\to 0}\frac{\rho_E(t)}{t} = 0.$$

LEMMA 2.1 ([4]). Let E be a uniformly smooth Banach space and $J : E \to E^*$ be the normalized duality mapping. Then for all $x, y \in E$, we have (*i*) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y)\rangle$;

(*ii*) $\langle x-y, J(x)-J(y)\rangle \leq 2d^2\rho_E\left(\frac{4\|x-y\|}{d}\right)$, where $d = \sqrt{(\|x\|^2+\|y\|^2)/2}$.

LEMMA 2.2 ([26]). Let E be a complete metric space with metric d, and let $T: E \to CB(E)$ be a multi-valued mapping. Then for any $\epsilon > 0$ and for any $x, y \in E$, $u \in T(x)$, there exists $v \in T(y)$ such that $d(u,v) \leq \mathcal{D}(Tx, Ty)$.

LEMMA 2.3 ([27]). Let E be a real Banach space and $J: E \to 2^{E^*}$ be the normalized duality mapping. Then for any $x, y \in E$,

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \ \forall j(x+y) \in J(x+y).$$

DEFINITION 2.3 ([31]). A functional $f: E \times E \to \mathbb{R} \cup \{+\infty\}$ is said to be 0-diagonally quasi-concave (in short, 0-DQCV) in x, if for any finite set $\{x_1, x_2, \cdots, x_n\} \subset E$ and for any $y = \sum_{i=1}^{n} \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1, \min_{1\leq i\leq n} f(x_i, y) \leq 0$ holds.

DEFINITION 2.4 ([8]). Let $\eta: E \times E \to E$ be a single-valued mapping. A proper functional $\phi: E \to \mathbb{R} \cup \{+\infty\}$ is said to be η-subdifferentiable at point $x \in E$ if there exists a point $f^* \in E^*$ such that

$$\phi(y) - \phi(x) \geq \langle f^*, \eta(y,x)\rangle, \ \forall y \in E,$$

where f^* is called η-subdgradient of ϕ at x. The set of all η-subgradients of ϕ at x is denoted by $\partial\phi(x)$. The mapping $\partial\phi: E \to 2^{E^*}$ is defined by

$$\partial\phi(x) = \{f^* \in E^* : \phi(y) - \phi(x) \geq \langle f^*, \eta(y,x)\rangle, \ \forall y \in E\}$$

is said to be η-subdifferential of ϕ at x.

DEFINITION 2.5. Let $\eta: E \times E \to E$ and $A, B: E \to E$ be single-valued mappings and let $M: E \times E \to E^*$ be a nonlinear mapping. Then

(*i*) $M(A,\cdot)$ is said to be α-strongly η-monotone with respect to A if there exists a constant $\alpha > 0$ such that

$$\langle M(Ax,u) - M(Ay,u), \eta(x,y)\rangle \geq \alpha\|x-y\|^2, \ \forall x, y, u \in E;$$

(*ii*) $M(\cdot,B)$ is said to be β-relaxed η-monotone with respect to B if there exists a constant $\beta > 0$ such that

$$\langle M(u,Bx) - M(u,By), \eta(x,y)\rangle \geq (-\beta)\|x-y\|^2, \ \forall x, y, u \in E;$$

(*iii*) $M(A,B)$ is said to be $\alpha\beta$-symmetric η-monotone with respect to A and B if $M(A,\cdot)$ is α-strongly η-monotone with respect to A and $M(\cdot,B)$ is β-relaxed η-monotone with respect to B;

(iv) $M(\cdot,\cdot)$ is said to be (ξ_1,ξ_2)-mixed Lipschitz continuous if there exist constants $\xi_1,\xi_2>0$ satisfying

$$\|M(x,u)-M(y,v)\| \leq \xi_1\|x-y\|+\xi_2\|u-v\|,\ \forall x,y,u,v\in E.$$

DEFINITION 2.6. Let $\eta : E\times E\to E$ and $A,B:E\to E$ be single-valued mappings. Let $\phi : E\to \mathbb{R}\cup\{+\infty\}$ be a proper, lower semicontinuous and η-subdifferentiable (may not be convex) functional and $M : E\times E\to E^*$ be a nonlinear mapping. If for any given point $x^*\in E^*$ and $\rho>0$, there exists a unique point $x\in E$ satisfying

$$\langle M(Ax,Bx)-x^*,\eta(y,x)\rangle+\rho\phi(y)-\rho\phi(x)\geq 0,\ \forall y\in E,$$

then the mapping $x^*\to x$, denoted by $R^{\partial\phi}_{\rho,\eta}(x^*)$ is called resolvent operator of ϕ. Then, we have
$x^*-M(Ax,Bx)\in\rho\partial\phi(x)$ and it follows that $R^{\partial\phi}_{\rho,\eta}(x^*)=[M(A,B)+\rho\partial\phi]^{-1}(x^*)$.

LEMMA 2.4 ([23]). Let E be a reflexive Banach space. Let $\eta : E\times E\to E$ be a continuous mapping such that $\eta(y,y')+\eta(y',y)=0$ for all $y,y'\in E$; $M : E\times E\to E^*$ be $\alpha\beta$-symmetric η-monotone continuous with respect to A and B; let for any $x^*\in E^*$, the function $h(y,x)=\langle x^*-M(Ax,Bx),\eta(y,x)\rangle$ be 0-DQCV in y and $\phi : E\to\mathbb{R}\cup\{+\infty\}$ be a proper, lower semicontinuous and η-subdifferentiable (may not be convex) functional. Then for any given constant $\rho>0$ and $x^*\in E^*$, there exists a unique $x\in E$ such that

$$\langle M(Ax,Bx)-x^*,\eta(y,x)\rangle\geq\rho\phi(x)-\rho\phi(y),\ \forall y\in E, \tag{1}$$

that is, $x=R^{\partial\phi}_{\rho,\eta}(x^*)$.

LEMMA 2.5 ([23]). Let $\eta : E\times E\to E$ be τ-Lipschitz continuous such that $\eta(y,y')+\eta(y',y)=0$ for all $y,y'\in E$; $M : E\times E\to E^*$ be $\alpha\beta$-symmetric η-monotone continuous with respect to A and B; let for any $x^*\in E^*$, the function $h(y,x)=\langle x^*-M(Ax,Bx),\eta(y,x)\rangle$ be 0-DQCV in y and $\phi : E\to\mathbb{R}\cup\{+\infty\}$ be a proper, lower semicontinuous and η-subdifferentiable functional and let $\rho>0$ be any given constant. Then the resolvent operator $R^{\partial\phi}_{\rho,M(\cdot,\cdot)}$ of ϕ is $\frac{\tau}{\alpha-\beta}$-Lipschitz continuous, that is, for any $x_1^*,x_2^*\in E^*$,

$$\|R^{\partial\phi}_{\rho,M(\cdot,\cdot)}(x_1^*)-R^{\partial\phi}_{\rho,M(\cdot,\cdot)}(x_2^*)\|\leq\frac{\tau}{\alpha-\beta}\|x_1^*-x_2^*\|.$$

3 Graph Convergence for η-Subdifferential Mapping

Let $\eta : E\times E\to E$ be a single-valued mapping. Let $\phi : E\to\mathbb{R}\cup\{+\infty\}$ be a proper, lower semicontinuous and η-subdifferentiable (may not be convex) functional and let $\partial\phi : E\to 2^{E^*}$ be a η-subdifferential mapping of ϕ. The graph of the η-subdifferential mapping $\partial\phi$ is defined by

$$graph(\partial\phi)=\{(x,y^*)\in E\times E^*: y^*\in\partial\phi(x)\}.$$

In this section, we introduce the notion of graph convergence for η-subdifferential mapping.

DEFINITION 3.1. Let $\eta : E \times E \to E; A, B : E \to E$ be single-valued mappings. Let $\phi : E \to R \cup \{+\infty\}$ be a proper, lower semicontinuous and η-subdifferentiable (may not be convex) functional; let $M : E \times E \to E^*$ be a nonlinear mapping. Let $\partial\phi_n, \partial\phi : E \to 2^{E^*}$ be the η-subdifferential mappings of ϕ for $n = 0, 1, 2, \ldots$. The sequence $\{\partial\phi_n\}$ is said to be graph convergence to $\partial\phi$, denoted by $\partial\phi_n \underrightarrow{G} \partial\phi$, if for every $(x, y^*) \in graph(\partial\phi)$ there exists a sequence $(x_n, y_n^*) \in graph(\partial\phi_n)$ such that

$$x_n \to x,\ y_n^* \to y^* \text{ as } n \to \infty.$$

THEOREM 3.1. Let $\eta : E \times E \to E$ be τ-Lipschitz continuous such that $\eta(y, y') + \eta(y', y) = 0$ for all $y, y' \in E$; let $M : E \times E \to E^*$ be $\alpha\beta$-symmetric η-monotone continuous with respect to A and B such that M is γ_1-Lipschitz continuous with respect to A and γ_2-Lipschitz continuous with respect to B. Let for any $x^* \in E^*$, the function $h(y, x) = \langle x^* - M(Ax, Bx), \eta(y, x)\rangle$ be 0-DQCV in y and let $\phi : E \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and η-subdifferentiable functional and let $\rho > 0$ be any given constant. Then $\partial\phi_n \underrightarrow{G} \partial\phi$ if and only if

$$R^{\partial\phi_n}_{\rho,M(\cdot,\cdot)}(x^*) \to R^{\partial\phi}_{\rho,M(\cdot,\cdot)}(x^*),\ \forall x^* \in E^*.$$

PROOF. Suppose that $\partial\phi_n \underrightarrow{G} \partial\phi$. For any $x^* \in E^*$, let

$$z_n = R^{\partial\phi_n}_{\rho,M(\cdot,\cdot)}(x^*),\ z = R^{\partial\phi}_{\rho,M(\cdot,\cdot)}(x^*).$$

It follows that $z = [M(A, B) + \rho\partial\phi]^{-1}(x^*)$,

$$\text{then,}\quad \frac{1}{\rho}[x^* - M(Az, Bz] \in \partial\phi(z),$$

that is, $(z, \frac{1}{\rho}[x^* - M(Az, Bz)]) \in graph(\partial\phi)$. It follows from the definition of the graph convergence that there exists a sequence $(z_n^{'}, y_n^{*'}) \in graph(\partial\phi_n)$ such that

$$z_n' \to z \text{ and } y_n^{*'} \to \frac{1}{\rho}[x^* - M(Az, Bz)] \text{ as } n \to \infty. \tag{2}$$

Since $y_n^{*'} \in \partial\phi_n(z_n^{'})$, we have

$$M(Az_n^{'}, Bz_n^{'}) + \rho y_n^{*'} \in [M(A, B) + \rho\partial\phi_n](z_n^{'})$$

that is, $z_n^{'} = R^{\partial\phi_n}_{\rho,M(\cdot,\cdot)}[M(Az_n^{'}, Bz_n^{'}) + \rho y_n^{*'}]$. Now,

$$\begin{aligned} \|z_n - z\| &\leq \|z_n - z_n^{'}\| + \|z_n^{'} - z\| \\ &= \|R^{\partial\phi_n}_{\rho,M(\cdot,\cdot)}(x^*) - R^{\partial\phi_n}_{\rho,M(\cdot,\cdot)}[M(Az_n^{'}, Bz_n^{'}) + \rho y_n^{*'}]\| \\ &\quad + \|z_n^{'} - z\|. \end{aligned}$$

By using the Lipschitz continuity of the resolvent operator $R^{\partial\phi_n}_{\rho,M(\cdot,\cdot)}$, we have

$$\begin{aligned}\|z_n - z\| &\leq \frac{\tau}{\alpha-\beta}\|x^* - [M(Az_n^{'}, Bz_n^{'}) + \rho y_n^{*'}]\| + \|z_n^{'} - z\| \\ &\leq \frac{\tau}{\alpha-\beta}\|x^* - [M(Az, Bz) + \rho y_n^{*'}]\| \\ &\quad + \frac{\tau}{\alpha-\beta}\|M(Az, Bz) - M(Az_n^{'}, Bz_n^{'}\| + \|z_n^{'} - z\|.\end{aligned}$$

Since M is γ_1-Lipschitz continuous with respect to A and γ_2-Lipchitz continuous with respect to B, we have

$$\begin{aligned}\|z_n - z\| &\leq \frac{\tau}{\alpha-\beta}\left\|x^* - [M(Az, Bz) + \rho y_n^{*'}]\right\| + \frac{\tau(\gamma_1+\gamma_2)}{\alpha-\beta}\|z - z_n^{'}\| + \|z_n^{'} - z\| \\ &= \frac{\tau}{\alpha-\beta}\|x^* - [M(Az, Bz) + \rho y_n^{*'}]\| + [1 + \frac{\tau(\gamma_1+\gamma_2)}{\alpha-\beta}]\|z_n^{'} - z\|.\end{aligned}$$

By (2), we have

$$\|z_n^{'} - z\| \to 0,\ \frac{1}{\rho}\|x^* - [M(Az, Bz) + \rho y_n^{*'}]\| \to 0,\ \text{as } n \to \infty,$$

hence $\|z_n - z\| \to 0$ as $n \to \infty$, that is,

$$R^{\partial\phi_n}_{\rho,M(\cdot,\cdot)}(x^*) \to R^{\partial\phi}_{\rho,M(\cdot,\cdot)}(x^*),\ \forall x^* \in E^*.$$

Conversely, suppose that $R^{\partial\phi_n}_{\rho,M(\cdot,\cdot)}(x^*) \to R^{\partial\phi}_{\rho,M(\cdot,\cdot)}(x^*),\ \forall x^* \in E^*, \rho > 0$. For any $(x, y^*) \in graph(\partial\phi)$, we have, $y^* \in \partial\phi(x)$, that is,

$$M(Ax, Bx) + \rho y^* \in [M(A, B) + \rho\partial\phi](x),$$

and so $x = R^{\partial\phi}_{\rho,M(\cdot,\cdot)}[M(Ax, Bx) + \rho y^*]$. Let $x_n = R^{\partial\phi_n}_{\rho,M(\cdot,\cdot)}[M(Ax, Bx) + \rho y^*]$, then

$$\frac{1}{\rho}[M(Ax, Bx) - M(Ax_n, Bx_n) + \rho y^*] \in \partial\phi_n(x_n).$$

Suppose that $y_n^* = \frac{1}{\rho}[M(Ax, Bx) - M(Ax_n, Bx_n) + \rho y^*]$. Now,

$$\begin{aligned}\|y_n^* - y^*\| &= \left\|\frac{1}{\rho}[M(Ax, Bx) - M(Ax_n, Bx_n) + \rho y^*] - y^*\right\| \\ &= \frac{1}{\rho}\|M(Ax, Bx) - M(Ax_n, Bx_n)\| \\ &\leq \frac{(\gamma_1+\gamma_2)}{\rho}\|x_n - x\|. \qquad (3)\end{aligned}$$

Since $R^{\partial\phi_n}_{\rho,M(\cdot,\cdot)}(x^*) \to R^{\partial\phi}_{\rho,M(\cdot,\cdot)}(x^*)$ for any $x^* \in E^*$, we have $\|x_n - x\| \to 0$ as $n \to \infty$. It follows from (3) that $\|y_n^* - y^*\| \to 0$ as $n \to \infty$. Hence $\partial\phi_n \underrightarrow{G} \partial\phi$.

4 System of Generalized Implicit Variational-like Inclusions

Let for each $i \in \{1,2\}$, E_i be a real Banach space with norm $\|\cdot\|_i$ and E_i^* be its dual space with norm $\|\cdot\|_{*i}$. Let $\langle\cdot,\cdot\rangle_i$ denotes the duality pairing between E_i and E_i^*; let $\eta_i : E_i \times E_i \to E_i$, $N_i : E_1^* \times E_2^* \to E_i^*$ and $S_i : E_i \to E_i^*$ be single-valued mappings; let $g_1 : E_2 \to CB(E_2^*)$ and $g_2 : E_1 \to CB(E_1^*)$ be multi-valued mappings. Let $\phi_i : E_i \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and η_i-subdifferentiable functional. We consider the following system of generalized implicit variational-like inclusions (in short, SGIVLI).

Find (x, y, u, v) such that $x \in E_1, y \in E_2, u \in g_1(y), v \in g_2(x)$ and

$$\begin{cases} \langle N_1(S_1(x), u), \eta_1(a, x)\rangle \geq \rho_1[\phi_1(x) - \phi_1(a)], \ \forall a \in E_1, \\ \langle N_2(v, S_2(y)), \eta_2(b, y)\rangle \geq \rho_2[\phi_2(y) - \phi_2(b)], \ \forall b \in E_2, \end{cases} \tag{4}$$

where $\rho_1, \rho_2 > 0$ are some constants.

REMARK 4.1. For suitable choices of mappings $A_i, B_i, N_i, g_i, S_i, M_i, \eta_i, \phi_i$ and underlying spaces E_i, SGIVLI (4) reduces to various known classes of systems of variational inclusions and variational inequalities, see for examples, [6, 19, 20, 24, 29, 30].

LEMMA 4.1. For each $i \in \{1,2\}$, let E_i be a reflexive Banach space; let $\eta_i : E_i \times E_i \to E_i$ be a continuous mapping such that $\eta_i(y_i, y_i') + \eta_i(y_i', y_i) = 0$, for all $y_i, y_i' \in E_i$. Let $A_i, B_i : E_i \to E_i$ be single-valued mappings; let the mappings $M_i : E_i \times E_i \to E_i^*$ be $\alpha_i\beta_i$-symmetric η_i-monotone continuous with respect to A_i and B_i; let for any $x_i^* \in E_i^*$, the function $h_i(y_i, x_i) = \langle x_i^* - M_i(A_i x_i, B_i x_i), \eta_i(y_i, x_i)\rangle$ be 0-DQCV in y_i and let $\phi_i : E_i \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and η_i-subdifferentiable functional. Then for (x, y, u, v), where $x \in E_1$, $y \in E_2$, $u \in g_1(y)$, $v \in g_2(x)$ is a solution of SGIVLI (4), if and only if (x, y, u, v) satisfies the relation

$$x = R^{\partial\phi_1}_{\rho_1, M_1(\cdot,\cdot)}[M_1(A_1x, B_1x) - N_1(S_1x, u)],$$

$$y = R^{\partial\phi_2}_{\rho_2, M_2(\cdot,\cdot)}[M_2(A_2y, B_2y) - N_2(v, S_2y)],$$

where ρ_1, ρ_2 are some constants, $R^{\partial\phi_1}_{\rho_1, M_1(\cdot,\cdot)}(x^*) = [M_1(A_1, B_1) + \rho_1\partial\phi_1]^{-1}(x^*)$ and $R^{\partial\phi_2}_{\rho_2, M_2(\cdot,\cdot)}(y^*) = [M_2(A_2, B_2) + \rho_2\partial\phi_2]^{-1}(y^*)$.

PROOF. The conclusion follows directly from the definition of resolvent operators $R^{\partial\phi_1}_{\rho_1, M_1(\cdot,\cdot)}$ and $R^{\partial\phi_2}_{\rho_2, M_2(\cdot,\cdot)}$.

We note that $(E_1 \times E_2, \|\cdot\|_*)$ is a Banach space with norm $\|\cdot\|_*$ defined as

$$\|(x, y)\|_* = \|x\|_1 + \|y\|_2, \ \forall (x, y) \in E_1 \times E_2.$$

Next, we prove existence and uniqueness for SGIVLI (4).

THEOREM 4.1. For each $i \in \{1, 2\}$, let E_i be a uniformly smooth Banach space with $\rho_{E_i}(t) \le C_i t^2$ for some $C_i > 0$; let $\eta_i : E_i \times E_i \to E_i$ be a continuous mapping such that $\eta_i(y_i, y_i^{'}) + \eta_i(y_i^{'}, y_i) = 0$, for all $y_i, y_i^{'} \in E_i$; let $A_i, B_i : E_i \to E_i$ be nonlinear mappings; let $M_i : E_i \times E_i \to E_i^*$ be $\alpha_i\beta_i$-symmetric η_i-monotone continuous with respect to A_i, B_i; let for any given $x_i^* \in E_i^*$, the function $h_i(y_i, x_i) = \langle x_i^* - M_i(A_i x_i, B_i x_i), \eta_i(y_i, x_i)\rangle$ be 0-DQCV in y_i. Let $\phi_i : E_i \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and η_i-subdifferentiable functional. Let $N_i : E_1^* \times E_2^* \to E_i^*$ be (δ_i, r_i)-mixed Lipschitz continuous; let $g_1 : E_2 \to CB(E_2^*)$ and $g_2 : E_1 \to CB(E_1^*)$ be $\lambda_{\mathcal{D}_{g_1}}$ and $\lambda_{\mathcal{D}_{g_2}}$-Lipschitz continuous with respect to second and first argument, respectively; let $N_1(S_1(\cdot), u_1)$ be ϵ_1-strongly accretive with respect to $M_1(A_1, B_1)$ and $N_2(v_1, S_2(\cdot))$ is ϵ_2-strongly accretive with respect to $M_2(A_2, B_2)$; let $M_i(A_i, B_i)$ is λ_{M_i}-Lipschitz continuous with respect to A_i and B_i. Suppose that there exist constants $\rho_1, \rho_2 > 0$ such that

$$\begin{cases} G_1 = u_1 + L_2\delta_2\mathcal{D}_{g_2} < 1, \\ G_2 = u_2 + L_1 r_1 \lambda_{\mathcal{D}_{g_1}} < 1, \end{cases} \tag{5}$$

where

$$u_1 = L_1\sqrt{\lambda_{M_1}^2 - 2\epsilon_1 + 64C_1\delta_1^2}, \quad u_2 = L_2\sqrt{\lambda_{M_2}^2 - 2\epsilon_2 + 64C_2 r_2^2},$$
$$L_1 = \frac{\tau_1}{\alpha_1 - \beta_1}, \quad L_2 = \frac{\tau_2}{\alpha_2 - \beta_2}.$$

Then SGIVLI (4) has a unique solution.

PROOF. It follows that for $(x, y) \in E_1 \times E_2$, the resolvent operators $R^{\partial\phi_1}_{\rho_1, M_1(\cdot,\cdot)}$ and $R^{\partial\phi_2}_{\rho_2, M_2(\cdot,\cdot)}$ are L_1 and L_2-Lipschitz continuous, respectively.

Now, we define a mapping $Q : E_1 \times E_2 \to E_1 \times E_2$ by

$$Q(x, y) = (T(x, y), P(x, y)), \ \forall (x, y) \in E_1 \times E_2; \tag{6}$$

where $T : E_1 \times E_2 \to E_1$ and $P : E_1 \times E_2 \to E_2$ are defined by

$$T(x, y) = R^{\partial\phi_1}_{\rho_1, M_1(\cdot,\cdot)}[M_1(A_1x, B_1x) - N_1(S_1x, u)], \tag{7}$$

$$P(x, y) = R^{\partial\phi_2}_{\rho_2, M_2(\cdot,\cdot)}[M_2(A_2y, B_2y) - N_2(v, S_2y)]. \tag{8}$$

For any $(x_1, y_1), (x_2, y_2) \in E_1 \times E_2$, using (7), (8) and Lipschitz continuity of $R^{\partial\phi_1}_{\rho_1, M_1(\cdot,\cdot)}$ and $R^{\partial\phi_2}_{\rho_2, M_2(\cdot,\cdot)}$, we have

$$\begin{aligned} \|T(x_1, y_1) - T(x_2, y_2)\|_1 &= \|R^{\partial\phi_1}_{\rho_1, M_1(\cdot,\cdot)}[M_1(A_1x_1, B_1x_1) - N_1(S_1x_1, u_1)] \\ &\quad - R^{\partial\phi_1}_{\rho_1, M_1(\cdot,\cdot)}[M_1(A_1x_2, B_1x_2) - N_1(S_1x_2, u_2)]\|_1 \\ &\le L_1\|[M_1(A_1x_1, B_1x_1) - M_1(A_1x_2, B_1x_2) \\ &\quad -(N_1(S_1x_1, u_1) - N_1(S_1x_2, u_2))\|_{*1} \\ &\le L_1\|[M_1(A_1x_1, B_1x_1) - M_1(A_1x_2, B_1x_2) \\ &\quad -(N_1(S_1x_1, u_1) - N_1(S_1x_2, u_1))\|_{*1} \\ &\quad + L_1\|N_1(S_1x_2, u_1)) - N_1(S_1x_2, u_2))\|_{*1}, \end{aligned} \tag{9}$$

$$\begin{aligned}
& \|M_1(A_1x_1,B_1x_1)-M_1(A_1x_2,B_1x_2)-(N_1(S_1x_1,u_1)-N_1(S_1x_2,u_1))\|_{*1}^2 \\
\leq\ & \|M_1(A_1x_1,B_1x_1)-M_1(A_1x_2,B_1x_2)\|_{*1}^2 \\
& -2\langle N_1(S_1x_1,u_1)-N_1(S_1x_2,u_1),J_1^*(M_1(A_1x_1,B_1x_1)-M_1(A_1x_2,B_1x_2))\rangle_1 \\
& +2\langle N_1(S_1x_1,u_1)-N_1(S_1x_2,u_1),J_1^*(M_1(A_1x_1,B_1x_1)-M_1(A_1x_2,B_1x_2)) \\
& -J_1^*(M_1(A_1x_1,B_1x_1)-M_1(A_1x_2,B_1x_2))-(N_1(S_1x_1,u_1)-N_1(S_1x_2,u_1))\rangle_1.
\end{aligned}$$

Since M_1 is λ_{M_1}-Lipschitz continuous with respect to A_1 and B_1, $N_1(S_1(\cdot),u_1)$ is ϵ_1-strongly accretive with respect to $M_1(A_1,B_1)$, N_1 is (δ_1,r_1)-mixed Lipschitz continuous and g_1 is $\lambda_{\mathcal{D}_{g_1}}$-Lipschitz continuous in the second argument, we have

$$\begin{aligned}
& \|M_1(A_1x_1,B_1x_1)-M_1(A_1x_2,B_1x_2)-(N_1(S_1x_1,u_1)-N_1(S_1x_2,u_1))\|_{*1}^2 \\
\leq\ & \lambda_{M_1}^2\|x_1-x_2\|_1^2-2\epsilon_1\|x_1-x_2\|_1^2+64C_1\delta_1^2\|x_1-x_2\|_1^2, \qquad (10)
\end{aligned}$$

where $J_1^*:E_1^*\to E_1$ is normalized duality mapping and

$$\begin{aligned}
\|N_1(S_1x_2,u_1)-N_1(S_1x_2,u_2)\|_{*1} &\leq r_1\|u_1-u_2\|_{*2} \\
&\leq r_1\mathcal{D}(g_1(y_1),g_1(y_2)) \\
&\leq r_1\lambda_{\mathcal{D}_{g_1}}\|y_1-y_2\|_2. \qquad (11)
\end{aligned}$$

From (9)–(11), we have

$$\begin{aligned}
& \|T(x_1,y_1)-T(x_2,y_2)\|_1 \\
\leq\ & L_1\sqrt{\lambda_{M_1}^2-2\epsilon_1+64C_1\delta_1^2}\|x_1-x_2\|_1+L_1r_1\lambda_{\mathcal{D}_{g_1}}\|y_1-y_2\|_2. \qquad (12)
\end{aligned}$$

$$\begin{aligned}
\|P(x_1,y_1)-P(x_2,y_2)\|_2 &\leq \|R^{\partial\phi_2}_{\rho_2,M_2(\cdot,\cdot)}[M_2(A_2y_1,B_2y_1)-N_2(v_1,S_2y_1)] \\
&\quad -R^{\partial\phi_2}_{\rho_2,M_2(\cdot,\cdot)}[M_2(A_2y_2,B_2y_2)-N_2(v_2,S_2y_2)]\|_2 \\
&\leq L_2\|[M_2(A_2y_1,B_2y_1)-M_2(A_2y_2,B_2y_2) \\
&\quad -(N_2(v_1,S_2y_1)-N_2(v_2,S_2y_2))\|_{*2} \\
&\leq L_2\|[M_2(A_2y_1,B_2y_1)-M_2(A_2y_2,B_2y_2) \\
&\quad -(N_2(v_1,S_2y_1)-N_2(v_1,S_2y_2))\|_{*2} \qquad (13) \\
&\quad +L_2\|N_2(v_1,S_2y_2))-N_2(v_2,S_2y_2))\|_{*2},
\end{aligned}$$

$$\begin{aligned}
& \|M_2(A_2y_1,B_2y_1)-M_2(A_2y_2,B_2y_2)-(N_2(v_1,S_2y_1)-N_2(v_1,S_2y_2))\|_{*2}^2 \\
\leq\ & \|M_2(A_2y_1,B_2y_1)-M_2(A_2y_2,B_2y_2)\|_{*2}^2 \\
& -2\langle N_2(v_1,S_2y_1)-N_2(v_1,S_2y_2),J_2^*(M_2(A_2y_1,B_2y_1)-M_2(A_2y_2,B_2y_2))\rangle_2 \\
& +2\langle N_2(v_1,S_2y_1)-N_2(v_1,S_2y_2),J_2^*(M_2(A_2y_1,B_2y_1)-M_2(A_2y_2,B_2y_2)) \\
& -J_2^*(M_2(A_2y_1,B_2y_1)-M_2(A_2y_2,B_2y_2))-(N_2(v_1,S_2y_1)-N_2(v_1,S_2y_2))\rangle_2.
\end{aligned}$$

Since M_2 is λ_{M_2}-Lipschitz continuous with respect to A_2 and B_2, $N_2(v_1,S_2(\cdot))$ is ϵ_2-strongly accretive with respect to $M_2(A_2,B_2)$, N_2 is (δ_2,r_2)-mixed Lipschitz continuous and g_2 is $\lambda_{\mathcal{D}_{g_2}}$-Lipschitz continuous in the first argument, we have

$$\begin{aligned}
& \|M_2(A_2y_1, B_2y_1) - M_2(A_2y_2, B_2y_2) - (N_2(v_1, S_2y_1) - N_2(v_1, S_2y_2))\|_{*2}^2 \\
\leq \ & \lambda_{M_2}^2\|y_1 - y_2\|_2^2 - 2\epsilon_2\|y_1 - y_2\|_2^2 + 64C_2r_2^2\|y_1 - y_2\|_2^2, \qquad (14)
\end{aligned}$$

where $J_2^* : E_2^* \to E_2$ is normalized duality mapping and

$$\begin{aligned}
\|N_2(v_1, S_2x_2) - N_2(v_2, S_2x_2)\| &\leq \delta_2\|v_1 - v_2\|_{*1} \\
&\leq \delta_2\mathcal{D}(g_2(x_1), g_2(x_2)) \\
&\leq \delta_2\lambda_{\mathcal{D}_{g_2}}\|x_1 - x_2\|_1. \qquad (15)
\end{aligned}$$

From (13)–(15), we have

$$\begin{aligned}
& \|P(x_1, y_1) - P(x_2, y_2)\|_2 \\
\leq \ & L_2\sqrt{\lambda_{M_2}^2 - 2\epsilon_2 + 64C_2r_2^2}\|y_1 - y_2\|_2 + L_2\delta_2\lambda_{\mathcal{D}_{g_2}}\|x_1 - x_2\|_1. \qquad (16)
\end{aligned}$$

From (12) and (16), we have

$$\begin{aligned}
& \|T(x_1, y_1) - T(x_2, y_2)\|_1 + \|S(x_1, y_1) - S(x_2, y_2)\|_2 \\
\leq \ & G_1\|x_1 - x_2\|_1 + G_2\|y_1 - y_2\|_2 \\
\leq \ & \max\{G_1, G_2\}(\|x_1 - x_2\|_1 + \|y_1 - y_2\|_2), \qquad (17)
\end{aligned}$$

where

$$\begin{cases} G_1 = u_{1+}L_2\delta_2\lambda_{\mathcal{D}_{g_2}}, \\ G_2 = u_2 + L_1r_1\lambda_{\mathcal{D}_{g_1}}, \end{cases} \qquad (18)$$

and

$$u_1 = L_1\sqrt{\lambda_{M_1}^2 - 2\epsilon_1 + 64C_1\delta_1^2}, \ u_2 = L_2\sqrt{\lambda_{M_2}^2 - 2\epsilon_2 + 64C_2r_2^2}.$$

Now, we define the norm $\|\cdot\|_*$ on $E_1 \times E_2$ by

$$\|(x, y)\|_* = \|x\|_1 + \|y\|_2, \ \forall (x, y) \in E_1 \times E_2. \qquad (19)$$

Since $(E_1 \times E_2, \|\cdot\|_*)$ is a Banach space and hence from (6), (17) and (19), we have

$$\begin{aligned}
\|Q(x_1, y_1) - Q(x_2, y_2)\|_* &= \|T(x_1, y_1) - T(x_2, y_2)\|_1 + \|P(x_1, y_1) - P(x_2, y_2)\|_2 \\
&\leq \max\{G_1, G_2\}\|(x_1, y_1) - (x_2, y_2)\|_*. \qquad (20)
\end{aligned}$$

By condition (5), $\max\{G_1, G_2\} < 1$, hence Q is a contraction mapping. It follows from Banach contraction principle, there exists a point $(x, y) \in E_1 \times E_2$ such that

$$Q(x, y) = (x, y),$$

which implies that

$$x = R_{\rho_1, M_1(\cdot,\cdot)}^{\partial\phi_1}[M_1(A_1x, B_1x) - N_1(S_1x, u)],$$

$$y = R_{\rho_2, M_2(\cdot,\cdot)}^{\partial\phi_2}[M_2(A_2y, B_2y) - N_2(v, S_2y)].$$

Then by Lemma 4.1, (x, y, u, v) is a unique solution of SGIVLI (4).

ALGORITHM 4.1. For any $(x_0, y_0) \in E_1 \times E_2$, compute the sequence $(x_n, y_n) \in E_1 \times E_2, u_0 \in g_1(y_0), v_0 \in g_2(x_0)$ by the following iterative scheme:

$$x_{n+1} = R^{\partial\phi_1}_{\rho_1, M_{1n}(\cdot,\cdot)}[M_1(A_1x_n, B_1x_n) - N_1(S_1x_n, u_n)], \tag{21}$$

$$y_{n+1} = R^{\partial\phi_2}_{\rho_2, M_{2n}(\cdot,\cdot)}[M_2(A_2y_n, B_2y_n) - N_2(v_n, S_2y_n)], \tag{22}$$

$$u_n \in g_1(y_n) : \|u_{n+1} - u_n\| \leq \mathcal{D}(g_1(y_{n+1}), g_1(y_n)),$$

$$v_n \in g_2(x_n) : \|v_{n+1} - v_n\| \leq \mathcal{D}(g_2(x_{n+1}), g_2(x_n)),$$

where $n = 0, 1, 2, \ldots; \rho_1 > 0, \rho_2 > 0$ are some constants.

THEOREM 4.2. For each $i \in \{1, 2\}$, let A_i, B_i, S_i, g_i, N_i, M_i, ϕ_i and η_i be same as in Theorem 4.1. Suppose that $\partial\phi_{in} \underset{\rightarrow}{G} \partial\phi_i$ and the condition (5) holds. Then approximate solution (x_n, y_n) generated by Algorithm 4.1 converges strongly to unique solution (x, y) of SGIVLI (4).

PROOF. It follows from Theorem 4.1 that there exists a unique solution (x, y, u, v) of SGIVLI (4). By Algorithm 4.1 and Lipschitz continuity of the resolvent operators, we have

$$\begin{aligned}
\|x_{n+1} - x\|_1 &= \Big\| R^{\partial\phi_1}_{\rho_1, M_{1n}(\cdot,\cdot)}[M_1(A_1x_n, B_1x_n) - N_1(S_1x_n, u_n)] \\
&\quad - R^{\partial\phi_1}_{\rho_1, M_1(\cdot,\cdot)}[M_1(A_1x, B_1x) - N_1(S_1x, u)]\Big\|_1 \\
&\leq \Big\| R^{\partial\phi_1}_{\rho_1, M_{1n}(\cdot,\cdot)}[M_1(A_1x_n, B_1x_n) - N_1(S_1x_n, u_n)] \\
&\quad - R^{\partial\phi_1}_{\rho_1, M_{1n}(\cdot,\cdot)}[M_1(A_1x, B_1x) - N_1(S_1x, u)]\Big\|_1 \\
&\quad + \Big\| R^{\partial\phi_1}_{\rho_1, M_{1n}(\cdot,\cdot)}[M_1(A_1x, B_1x) - N_1(S_1x, u)] \\
&\quad - R^{\partial\phi_1}_{\rho_1, M_1(\cdot,\cdot)}[M_1(A_1x, B_1x) - N_1(S_1x, u)]\Big\|_1
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
\|y_{n+1} - y\|_2 &= \Big\| R^{\partial\phi_2}_{\rho_2, M_{2n}(\cdot,\cdot)}[M_2(A_2y_n, B_2y_n) - N_2(v_n, S_2y_n)] \\
&\quad - R^{\partial\phi_2}_{\rho_2, M_2(\cdot,\cdot)}[M_2(A_2y, B_2y) - N_2(v, S_2y)]\Big\|_2 \\
&\leq \Big\| R^{\partial\phi_2}_{\rho_1, M_{2n}(\cdot,\cdot)}[M_2(A_2y_n, B_2y_n) - N_2(v_n, S_2y_n)] \\
&\quad - R^{\partial\phi_2}_{\rho_2, M_{2n}(\cdot,\cdot)}[M_2(A_2y, B_2y) - N_2(v, S_2y)]\Big\|_2 \\
&\quad + \Big\| R^{\partial\phi_2}_{\rho_2, M_{2n}(\cdot,\cdot)}[M_2(A_2y, B_2y) - N_2(v, S_2y)] \\
&\quad - R^{\partial\phi_2}_{\rho_2, M_2(\cdot,\cdot)}[M_2(A_2y, B_2y) - N_2(v, S_2y)]\Big\|_2 .
\end{aligned} \tag{24}$$

Now, using the same arguments as from (9)–(12), we have

$$\begin{aligned}&\Big\|R^{\partial\phi_1}_{\rho_1,M_{1n}(\cdot,\cdot)}[M_1(A_1x_n,B_1x_n)-N_1(S_1x_n,u_n)]\\&\qquad\qquad -R^{\partial\phi_1}_{\rho_1,M_1(\cdot,\cdot)}[M_1(A_1x,B_1x)-N_1(S_1x,u)]\Big\|\\ \le\ & u_1\|x_n-x\|_1+L_1r_1\lambda_{\mathcal{D}_{g_1}}\|y_n-y\|_2,\end{aligned}\tag{25}$$

and following the same arguments as from (13)–(16), we have

$$\begin{aligned}&\Big\|R^{\partial\phi_2}_{\rho_2,M_{2n}(\cdot,\cdot)}[M_2(A_2y_n,B_2y_n)-N_2(v_n,S_2y_n)]\\&\qquad\qquad -R^{\partial\phi_2}_{\rho_2,M_2(\cdot,\cdot)}[M_2(A_2y,B_2y)-N_2(v,S_2y)]\Big\|_2\\ \le\ & u_2\|y_n-y\|_2+L_2\delta_2\lambda_{\mathcal{D}_{g_2}}\|x_n-x\|_1.\end{aligned}\tag{26}$$

By Theorem 3.1, we have

$$R^{\partial\phi_1}_{\rho_1,M_{1n}(\cdot,\cdot)}[M_1(A_1x,B_1x)-N_1(S_1x,u)]\to R^{\partial\phi_1}_{\rho_1,M_1(\cdot,\cdot)}[M_1(A_1x,B_1x)-N_1(S_1x,u)],$$

and

$$R^{\partial\phi_2}_{\rho_2,M_{2n}(\cdot,\cdot)}[M_2(A_2y,B_2y)-N_2(v,S_2y)]\to R^{\partial\phi_2}_{\rho_2,M_2(\cdot,\cdot)}[M_2(A_2y,B_2y)-N_2(v,S_2y)].$$

Let

$$\begin{aligned}a_n\ =\ & R^{\partial\phi_1}_{\rho_1,M_{1n}(\cdot,\cdot)}[M_1(A_1x,B_1x)-N_1(S_1x,u)]\\ & -R^{\partial\phi_1}_{\rho_1,M_1(\cdot,\cdot)}[M_1(A_1x,B_1x)-N_1(S_1x,u)]\end{aligned}\tag{27}$$

and

$$\begin{aligned}b_n\ =\ & R^{\partial\phi_2}_{\rho_2,M_{2n}(\cdot,\cdot)}[M_2(A_2y,B_2y)-N_2(v,S_2y)]\\ & -R^{\partial\phi_2}_{\rho_2,M_2(\cdot,\cdot)}[M_2(A_2y,B_2y)-N_2(v,S_2y)].\end{aligned}\tag{28}$$

Then

$$a_n,b_n\to 0\quad n\to\infty.\tag{29}$$

From (23)–(26), (27) and (28), we have

$$\begin{aligned}\|x_{n+1}-x\|_1+\|y_{n+1}-y\|_2\ \le\ & G_1\|x_n-x\|_1+G_2\|y_n-y\|_2+a_n+b_n\\ \le\ & \max\{G_1,G_2\}(\|x_n-x\|_1+\|y_n-y\|_2)+a_n+b_n.\end{aligned}$$

It follows from (19) that $(E_1\times E_2,\|\cdot\|_*)$ is a Banach space, we have

$$\begin{aligned}\|(x_{n+1},y_{n+1})-(x,y)\|_*\ =\ & \|x_{n+1}-x\|_1+\|y_{n+1}-y\|_2\\ & \max\{G_1,G_2\}(\|(x_n,y_n)-(x-y)\|_*)+a_n+b_n.\end{aligned}\tag{30}$$

From condition (5) and (29), (30), we have

$$\|(x_{n+1},y_{n+1})-(x,y)\|_*\to 0\ \text{ as } n\to\infty.$$

Thus $\{(x_n, y_n)\}$ converges strongly to the unique solution (x, y) of SGIVLI (4). This completes the proof.

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