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Generalized Quasi Power Increasing Sequences*

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Abstract

In this paper, we prove a theorem dealing with $|A, p_n|_k$ summability method of infinite series by using the concept of quasi β-power increasing sequence instead of almost increasing sequence.

1 Introduction

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Obviously, every increasing sequence is almost increasing. However, the converse need not be true as can be seen by taking an example, say, $b_n = ne^{(-1)^n}$. A positive sequence (γ_n) is said to be a quasi β-power increasing sequence if there exists a constant $K = K(\beta, \gamma) \geq 1$ such that $Kn^\beta \gamma_n \geq m^\beta \gamma_m$ holds for all $n \geq m \geq 1$ (see [8]). It should be noted that every almost increasing sequence is quasi β-power increasing for any nonnegative β, but the converse need not be true as can be seen by taking the example, say $\gamma_n = n^{-\beta}$ for $\beta > 0$. A sequence (λ_n) is said to be of bounded variation, denoted by $(\lambda_n) \in \mathcal{BV}$, if $\sum_{n=1}^{\infty} |\Delta\lambda_n| = \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$. Let $\sum a_n$ be a given infinite series with the partial sums (s_n). By (u_n) and (t_n) we denote the n-th $(C, 1)$ mean of the sequences (s_n) and (na_n), respectively. The series $\sum a_n$ is said to be summable $|C, 1|_k$, $k \geq 1$, if (see [5, 7])

$$\sum_{n=1}^{\infty} n^{k-1} |u_n - u_{n-1}|^k = \sum_{n=1}^{\infty} \frac{1}{n} |t_n|^k < \infty.$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad (n \to \infty), \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1).$$

The sequence-to-sequence transformation

$$\omega_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

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defines the sequence (ω_n) of the Riesz mean or simply the $\left(\bar{N},p_n\right)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [6]). The series $\sum a_n$ is said to be summable $\left|\bar{N},p_n\right|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\Delta\omega_{n-1}\right|^k < \infty,$$

where

$$\Delta\omega_{n-1} = -\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}a_v, \quad n \geq 1.$$

In the special case, when $p_n = 1$ for all values of n, $|\bar{N},p_n|_k$ summability reduces to $|C,1|_k$ summability.

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n}a_{nv}s_v, \quad n = 0,1,....$$

The series $\sum a_n$ is said to be summable $\left|A,p_n\right|_k$, $k \geq 1$, if (see [12])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|\bar{\Delta}A_n(s)|^k < \infty,$$

where

$$\bar{\Delta}A_n(s) = A_n(s) - A_{n-1}(s).$$

If we take $a_{nv} = \frac{p_v}{P_n}$, then $|A,p_n|_k$ summability reduces to $|\bar{N},p_n|_k$ summability. If we take $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then $|A,p_n|_k$ summability reduces to $|C,1|_k$ summability. Also, if we take $p_n = 1$ for all values of n, then $|A,p_n|_k$ summability reduces to $|A|_k$ summability (see [13]). If we take $a_{nv} = \frac{p_v}{P_n}$, then $|A|_k$ summability reduces to $|R,p_n|_k$ summability (see [3]).

Before stating the main theorem we must first introduce some further notations. Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n}a_{ni}, \quad n,v = 0,1,... \tag{1}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1,2,... \tag{2}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n\left(s\right) = \sum_{v=0}^{n}a_{nv}s_v = \sum_{v=0}^{n}\bar{a}_{nv}a_v \tag{3}$$

and

$$\bar{\Delta}A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{4}$$

2 Known Result

In [4], Bor has proved the following theorem for $\left|\bar{N}, p_n\right|_k$ summability factors of infinite series.

THEOREM 1. Let (X_n) be an almost increasing sequence and let there be sequences (β_n) and (λ_n) such that

$$|\Delta\lambda_n| \leq \beta_n, \tag{5}$$

$$\beta_n \to 0 \quad as \quad n \to \infty,$$

$$\sum_{n=1}^{\infty} n\,|\Delta\beta_n|\,X_n < \infty, \tag{6}$$

$$|\lambda_n| X_n = O(1). \tag{7}$$

If

$$\sum_{n=1}^{m} \frac{|\lambda_n|}{n} = O(1) \quad \text{as} \quad m \to \infty,$$

$$\sum_{n=1}^{m} \frac{1}{n} |t_n|^k = O(X_m) \quad \text{as} \quad m \to \infty, \tag{8}$$

and (p_n) is a sequence such that

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m) \quad \text{as} \quad m \to \infty, \tag{9}$$

then the series $\sum a_n \lambda_n$ is summable $\left|\bar{N}, p_n\right|_k, k \geq 1$.

3 Main Result

Many works dealing with absolute matrix summability factors of infinite series have been done (see [9]-[11]). The purpose of this paper is to generalize Theorem 1 to $|A, p_n|_k$ summability by using quasi β-power increasing sequences instead of almost increasing sequences.

Now, we shall prove the following theorem.

THEOREM 2. Let $A = (a_{nv})$ be a positive normal matrix such that

$$\overline{a}_{n0} = 1, \;\; n = 0, 1, ..., \tag{10}$$

$$a_{n-1,v} \geq a_{nv}, \;\; \text{for } n \geq v + 1, \tag{11}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{12}$$

and (X_n) be a quasi β-power increasing sequence for some $0<\beta<1$. If $(\lambda_n) \in \mathcal{BV}$ and all the conditions of Theorem 1 are satisfied, then the series $\sum a_n\lambda_n$ is summable $|A, p_n|_k$, $k \geq 1$.

REMARK 1. If we take $a_{nv} = \frac{p_v}{P_n}$ and (X_n) as an almost increasing sequence, then we get Theorem 1. In this case the condition $(\lambda_n) \in \mathcal{BV}$ is not needed.

We need the following lemma for the proof of Theorem 2.

LEMMA 1 ([8]). Under the conditions on (X_n), (β_n) and (λ_n) as taken in the statement of Theorem 2, the following conditions hold;

$$n\beta_n X_n = O(1) \quad \text{as} \quad n \to \infty, \tag{13}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{14}$$

4 Proof of Theorem 2

Let (I_n) denote A-transform of the series $\sum a_n\lambda_n$. Then, by (3) and (4), we have

$$\begin{aligned} \bar{\Delta} I_n &= \sum_{v=1}^{n} \hat{a}_{nv} a_v \lambda_v \\ &= \sum_{v=1}^{n} \frac{\hat{a}_{nv}\lambda_v}{v} v a_v. \end{aligned}$$

Using Abel's transformation, we have that

$$\begin{aligned} \bar{\Delta} I_n &= \sum_{v=1}^{n-1} \Delta_v\left(\frac{\hat{a}_{nv}\lambda_v}{v}\right) \sum_{r=1}^{v} r a_r + \frac{\hat{a}_{nn}\lambda_n}{n} \sum_{r=1}^{n} r a_r \\ &= \sum_{v=1}^{n-1} \frac{v+1}{v} \Delta_v(\hat{a}_{nv}) \lambda_v t_v + \sum_{v=1}^{n-1} \frac{1}{v} \hat{a}_{n,v+1} \lambda_{v+1} t_v \\ &+ \sum_{v=1}^{n-1} \frac{v+1}{v} \hat{a}_{n,v+1} \Delta\lambda_v t_v + \frac{n+1}{n} a_{nn} \lambda_n t_n \\ &= I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}. \end{aligned}$$

To complete the proof of Theorem 2, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |I_{n,r}|^k < \infty, \quad for \quad r = 1, 2, 3, 4.$$

First, when $k>1$, applying Hölder's inequality with indices k and $k^{'}$, where $\frac{1}{k}+\frac{1}{k'}=1$, we have that

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|I_{n,1}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v||t_v|\right)^k\\ &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|t_v|^k\right)\\ &\times \left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right)^{k-1}.\end{aligned}$$

By (1) and (2), we have that

$$\Delta_v(\hat{a}_{nv})=\hat{a}_{nv}-\hat{a}_{n,v+1}=\bar{a}_{nv}-\bar{a}_{n-1,v}-\bar{a}_{n,v+1}+\bar{a}_{n-1,v+1}=a_{nv}-a_{n-1,v}.$$

Thus using (1), (10) and (11)

$$\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|=\sum_{v=1}^{n-1}(a_{n-1,v}-a_{nv})\leq a_{nn}.$$

Hence, we get

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|I_{n,1}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}a_{nn}^{k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|t_v|^k\right)\\ &= O(1)\sum_{v=1}^{m}|\lambda_v|^{k-1}|\lambda_v||t_v|^k\sum_{n=v+1}^{m+1}|\Delta_v(\hat{a}_{nv})|\\ &= O(1)\sum_{v=1}^{m}\frac{p_v}{P_v}|\lambda_v||t_v|^k\\ &= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{r=1}^{v}\frac{p_r}{P_r}|t_r|^k+O(1)|\lambda_m|\sum_{v=1}^{m}\frac{p_v}{P_v}|t_v|^k\\ &= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v+O(1)|\lambda_m|X_m\\ &= O(1)\sum_{v=1}^{m-1}\beta_vX_v+O(1)|\lambda_m|X_m=O(1)\quad as\quad m\to\infty,\end{aligned}$$

by virtue of the hypotheses of Theorem 2 and Lemma 1. Also, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|I_{n,2}|^k &\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|\frac{|t_v|}{v}\right)^k \\
&\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|\frac{|t_v|^k}{v}\right)\times\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\frac{|\lambda_{v+1}|}{v}\right)^{k-1} \\
&\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^{k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|\frac{|t_v|^k}{v}\right)\times\left(\sum_{v=1}^{n-1}\frac{|\lambda_{v+1}|}{v}\right)^{k-1} \\
&= O(1)\sum_{v=1}^{m}\frac{|\lambda_{v+1}|}{v}|t_v|^k\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}|.
\end{aligned}$$

By (1), (2), (10) and (11), we obtain

$$|\hat{a}_{n,v+1}| = \sum_{i=0}^{v}(a_{n-1,i} - a_{ni}).$$

Thus, using (1) and (10), we have

$$\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}| = \sum_{n=v+1}^{m+1}\sum_{i=0}^{v}(a_{n-1,i} - a_{ni}) \leq 1,$$

then we get

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|I_{n,2}|^k &= O(1)\sum_{v=1}^{m}\frac{|\lambda_{v+1}|}{v}|t_v|^k \\
&= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_{v+1}|\sum_{r=1}^{v}\frac{1}{r}|t_r|^k + O(1)|\lambda_{m+1}|\sum_{v=1}^{m}\frac{1}{v}|t_v|^k \\
&= O(1)\sum_{v=1}^{m-1}\beta_{v+1}X_{v+1} + O(1)|\lambda_{m+1}|X_{m+1} = O(1) \quad as \quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 2 and Lemma 1.

Also, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|I_{n,3}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v||t_v|\right)^k \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v||t_v|^k\right)\times\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v|\right)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}a_{nn}^{k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\beta_v|t_v|^k\right) \\
&= O(1)\sum_{v=1}^{m}\beta_v|t_v|^k\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}| \\
&= O(1)\sum_{v=1}^{m}v\beta_v\frac{|t_v|^k}{v} \\
&= O(1)\sum_{v=1}^{m-1}|\Delta\left(v\beta_v\right)|\sum_{r=1}^{v}\frac{1}{r}|t_r|^k + O(1)m\beta_m\sum_{v=1}^{m}\frac{1}{v}|t_v|^k \\
&= O(1)\sum_{v=1}^{m-1}v|\Delta\beta_v|X_v + O(1)\sum_{v=1}^{m-1}\beta_v X_v + O(1)m\beta_m X_m \\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}$$

by (5), (6), (8), (12), (13) and (14).

Finally, as in $I_{n,1}$, we have that

$$\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|I_{n,4}|^k &= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|\lambda_n|^k|t_n|^k a_{nn}^k \\
&= O(1)\sum_{n=1}^{m}\frac{p_n}{P_n}|\lambda_n||t_n|^k = O(1) \quad as \quad m\to\infty,
\end{aligned}$$

by (5), (7), (9), (12) and (14). Hence, the proof of Theorem 2 is completed.

If we take $a_{nv}=\frac{p_v}{P_n}$ and $p_n=1$ for all values of n, then we get a result concerning $|C,1|_k$ summability factors of infinite series.

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