Applied Mathematics E-Notes, 19(2019), 27-37 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

On Approximate Solutions Of Impulsive Fractional Differential Equations*

Rupali Shikharchnad Jain†, Buruju Surendranath Reddy‡,
Sandhyatai Digambarrao Kadam§

Received 15 February 2018

Abstract

In this paper, we investigate bounds on difference between two approximate solutions of fractional impulsive differential equation. Also some qualitative properties of solutions has been studied using fractional impulsive inequality.

1 Introduction

Complex and dynamic behaviour of every real life phenomena may not be modelled using integer order derivatives. Such a problem may instead be better modelled by using fractional order derivatives. Fractional order derivatives are the generalization of integer order derivatives which gives numerous applications in engineering, biological sciences, Fluid mechanics, control theory, signal and image processing etc. For more details see [3], [13]–[15], [18], [21], [22].

In last two decades, impulsive differential equations gained much attention of many researchers as it is useful for modeling real life phenomena and physical process in science and engineering. These type of equations are mainly featured by the sudden change in their states at particular moments over time of negligible duration. For more details on the impulsive differential equations and their applications, we refer to [16], [17].

The method of approximations of solutions gives the beneficial information of solutions without knowing the solutions of differential equations explicitly. Also it establishes the bound on the difference between two ϵ_i-approximate solutions. For more references see [4], [5], [8], [9], [11], [12].

Many authors studied existence, uniqueness of impulsive fractional differential equations, see [1], [6], [7], [10], [19], [24]. For example, the authors, T. Lian et al. [6] studied

*Mathematics Subject Classifications: 34A12, 34A08, 26A33.

†School of Mathematical Sciences, Swami Ramanand Teerth Marathwada University, Nanded, Maharashtra, India

‡School of Mathematical Sciences, Swami Ramanand Teerth Marathwada University, Nanded, Maharashtra, India

§School of Mathematical Sciences, Swami Ramanand Teerth Marathwada University, Nanded, Maharashtra, India

the existence and uniqueness of the fractional differential equation

$$\begin{aligned} &{}^cD_t^q x(t) = f(t, x(t)), \quad t \in (0, b], \quad 0 < \alpha \leq 1, \\ &x(0) = x_0, \end{aligned}$$

by using Picard's iterative method.

In [10], A. Yarkar et al. studied the existence of

$$\begin{aligned} &{}^cD_t^q x(t) = f(t, x), \quad t \in (t_0, T], \quad 0 < \alpha < 1, \\ &x(t)(t - t_0)^{1-q}|_{t=t_o} = x_0, \end{aligned}$$

by using the method of upper and lower solutions.

Motivated by the works of [4], [5], [8], [9], [11], [12], in this paper, we study the following impulsive fractional differential equation:

$$ {}^cD_t^q x(t) = f(t, x(t)), \quad t \in (0, T], \quad t \neq \tau_k, k = 1, 2, .., m, \tag{1}$$
$$ x(0) = x_0, \tag{2}$$
$$ \Delta x(\tau_k) = I_k x(\tau_k), \quad k = 1, 2, ..., m, \tag{3}$$

where, ${}^cD_t^q$ is the classical Caputo frational derivative of order $q \in (0, 1)$ with lower limit zero, $x_0 \in \mathbb{R}$, $f : [0, T] \times X \to X$ is continuous, $I_k : X \to X$. The impulsive moments τ_k are such that $0 = \tau_0 < \tau_1 < \tau_2 < ... < \tau_m < \tau_{m+1} = T$, $m \in \mathbb{N}$, $x(\tau_k^+) = \lim\limits_{\epsilon \to 0^+} x(\tau_k + \epsilon)$ and $x(\tau_k^-) = \lim\limits_{\epsilon \to 0^-} x(\tau_k + \epsilon)$ represent the right and the left limits of x at τ_k.

The paper is organised as follows: Section 2 consists of preliminaries and hypotheses. In section 3, we establish the bound on the difference between two approximate solutions. Section 4 deals with nearness and convergence properties of solutions and finally, we give continuous dependence of solutions on parameters in section 5.

2 Preliminaries and Hypotheses

Let X be a Banach space with the norm $\| \cdot \|$. Let $PC([0, T], X)$ denote the set $\{x : [0, T] \to X : x(t)$ is piecewise continuous at $t \neq \tau_k$, left continuous at $t = \tau_k$, and the right limit $x(\tau_k + 0)$ exists for $k = 1, 2, ..., m\}$. Clearly, $PC([0, T], X)$ is a Banach space with the supremum norm

$$\|x\|_{PC([0,T],X)} = \sup\{\|x(t)\| : t \in [0, T] \setminus \{\tau_1, \tau_2, ..., \tau_m\}\}.$$

DEFINITION 2.1 ([21]). Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function on $\mathbb{R}^+$ and $\alpha > 0$. Then the expression

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha - 1} ds, \quad t > 0$$

is called the Riemman-Liouville integral of order α.

DEFINITION 2.2 ([21]). Let $f : \mathbb{R}^+ \to \mathbb{R}$. The Caputo fractional derivative of order α of f is defined by

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds, t > 0$$

where $\alpha \in (n-1, n), n \in N$.

DEFINITION 2.3 ([24]). A function $x \in PC([0,T], X)$ satisfying the equations:

$$\begin{aligned} x(t) &= x_0 + \frac{1}{\Gamma(q)} \sum_{0<\tau_k<t} \int_{\tau_{k-1}}^{\tau_k} (\tau_k - s)^{q-1} f(s, x(s))ds \\ &\quad + \frac{1}{\Gamma(q)} \int_{\tau_k}^{t} (\tau_k - s)^{q-1} f(s, x(s))ds \\ &\quad + \sum_{0<\tau_k<t} I_k x(\tau_k^-), \quad t \in (\tau_k, \tau_{k+1}],\ k = 1, 2, .., m, \\ x(0) &= x_0 \end{aligned}$$

is said to be the solution of the fractional impulsive initial value problem (FIIVP) (1)–(3).

DEFINITION 2.4. For $i = 1, 2$, $x_i \in PC([0,T], X)$ is the function such that $x_i(t)$ exists for $t \in (0, T]$ and satisfies the inequality

$$\|{}^c D_t^q x(t) - f(t, x(t))\| \leq \varepsilon_i$$

for given constant $\varepsilon_i \geq 0$, where it is considered that the initial and impulsive conditions

$$x_i(0) = x_0^i, \tag{4}$$

$$\Delta x_i(\tau_k) = I_k x_i(\tau_k), \tag{5}$$

are satisfied. Then $x_i(t)$ is said to be an ϵ_i-approximate solution to the FIIVP (1)–(3).

LEMMA 2.5 ([20]). Suppose that $p \in C[\mathbb{R}_+, \mathbb{R}_+]$ and for $k = 1, 2, ..., t \geq t_0$,

$$m(t) \leq c + \int_{t_0}^{t} p(s)m(s)ds + \sum_{t_0<t_k<t} \frac{\alpha_k}{\Gamma(\beta_k)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta_k - 1} m(s)ds + \sum_{t_0<t_k<t} \gamma_k m(t_k)$$

where $\alpha_k \geq 0$, $\gamma_k \geq -1$, $\beta_k > 0$, $k = 1, 2, ...$ and c are constants. Then, for $t \geq t_o$, the following assertions hold:

(I) $0 < \beta_k \leq \frac{1}{2}$ for $k = 1, 2, ...$ We see that

$$\begin{aligned} m(t) \leq c \prod_{t_0<t_k<t} \Bigg\{ &(1+\gamma_k) e^{\left(\int_{t_{k-1}}^{t_k} p(\xi)d\xi\right)} + \frac{\alpha_k}{\Gamma(\beta_k)} \left(\frac{e^{\mu_k t_k}}{\mu_k^{\beta_k^2}} \Gamma(\beta_k^2) \right)^{\frac{1}{\mu_k}} \\ &\times \left(\int_{t_{k-1}}^{t_k} e^{v_k\left(\int_{t_{k-1}}^{s} p(\xi)d\xi - s\right)} ds \right)^{\frac{1}{v_k}} \Bigg\} e^{\left(\int_{t_l}^{t} p(\xi)d\xi\right)} \end{aligned}$$

where $\mu_k = \beta_k + 1$ and $v_k = 1 + \frac{1}{\beta_k}$.

(II) $\beta_k > \frac{1}{2}$ for $k = 1, 2, ...$ We see that

$$m(t) \leq c \prod_{t_0<t_k<t} \Bigg\{ (1+\gamma_k) e^{\left(\int\limits_{t_{k-1}}^{t_k} p(\xi)d\xi\right)} + \frac{\alpha_k}{\Gamma(\beta_k)} \left(\frac{e^{2t_k}}{2^{2\beta_k - 1}} \Gamma(2\beta_k - 1) \right)^{\frac{1}{2}} \times \left(\int\limits_{t_{k-1}}^{t_k} e^{2\left(\int\limits_{t_{k-1}}^{s} p(\xi)d\xi - s\right)} ds \right)^{\frac{1}{2}} \Bigg\} e^{\left(\int\limits_{t_l}^{t} p(\xi)d\xi\right)}.$$

Now we introduce the following hypotheses.

(H_1) Let $f : [0,T] \times X \to X$ be a continuous function such that there exists a positive constant $p^* > 0$ satisfying

$$\|f(t, \psi) - f(t, \phi)\| \leq p^*(\|\psi - \phi\|),$$

for every $t \in [0,T], \psi, \phi \in X$.

(H_2) Let $I_k : X \to X$ be functions such that there exist positive constants L_k satisfying

$$\|I_k(x) - I_k(y)\| \leq L_k \|x - y\|, \quad x, y \in X, \quad k = 1, 2, ..., m.$$

(H_3) There exist nonnegative constants ϵ_3, δ_k such that

$$\|f(t, \phi) - \bar{f}(t, \phi)\| \leq \epsilon_3, \qquad \|I_k(\phi) - \bar{I}_k(\phi)\| \leq \delta_k.$$

(H_4) There exist nonnegative constants $\epsilon_n, \delta_n, \delta_{kn}$ such that

$$\|f(t, \phi) - f_n(t, \phi)\| \leq \epsilon_n, \qquad \|x_0 - y_{n0}\| \leq \delta_n, \qquad \|I_k\phi(\tau_k) - I_{kn}\phi(\tau_k)\| \leq \delta_{kn}.$$

with $\epsilon_n \to 0, \delta_n \to 0, \delta_{kn} \to 0$ as $n \to \infty$.

3 Bound on Difference Between Approximate Solutions

THEOREM 3.1. If $x_1(t)$ and $x_2(t)$ are ε_i approximate solutions of equation (1) with (4) and (5) and the hypotheses (H_1) and (H_2) are satisfied, then the following inequality holds:

$$\|x_1 - x_2\|_{PC} \leq \Big[\frac{(\varepsilon_1 + \varepsilon_2)}{\Gamma(q+1)} T^q + \|x_0^1 - x_0^2\| \Big] C^{**},$$

where C^{**} is some constant.

PROOF. Let $x_i (i = 1, 2)$ be ε_i approximate solutions of equation (1) with (4) and (5). Then we have,

$$\|{}^cD_t^q x(t) - f(t, x(t))\| \leq \varepsilon_i.$$

Operating I^q on both the sides, we get

$$\begin{aligned}
I^q \varepsilon_i &\geq I^q \|{}^cD_t^q x(t) - f(t, x(t))\| \\
&\geq \Big\| x_i(t) - x_0^i - \frac{1}{\Gamma(q)} \sum_{0<\tau_k<t} \int_{\tau_{k-1}}^{\tau_k} (\tau_k - s)^{q-1} f(s, x_i(s)) ds \\
&- \frac{1}{\Gamma(q)} \int_{\tau_k}^{t} (t-s)^{q-1} f(s, x_i(s)) ds \\
&- \sum_{0<\tau_k<t} I_k x_i(\tau_k^-) \Big\|.
\end{aligned}$$

Using the inequalities $\|u_1 - v_1\| \leq \|u_1\| + \|v_1\|$ and $|\|u_1\| - \|v_1\|| \leq \|u_1 - v_1\|$, we get

$$\begin{aligned}
\frac{(\varepsilon_1 + \varepsilon_2)}{\Gamma(q+1)} t^q \geq & \Big\| x_1(t) - x_0^1 - \frac{1}{\Gamma(q)} \sum_{0<\tau_k<t} \int_{\tau_{k-1}}^{\tau_k} (\tau_k - s)^{q-1} f(s, x_1(s)) ds \\
& - \frac{1}{\Gamma(q)} \int_{\tau_k}^{t} (t-s)^{q-1} f(s, x_1(s)) ds - \sum_{0<\tau_k<t} I_k x_1(\tau_k^-) \Big\| \\
& + \Big\| x_2(t) - x_0^2 - \frac{1}{\Gamma(q)} \sum_{0<\tau_k<t} \int_{\tau_{k-1}}^{\tau_k} (\tau_k - s)^{q-1} f(s, x_2(s)) ds \\
& - \frac{1}{\Gamma(q)} \int_{\tau_k}^{t} (t-s)^{q-1} f(s, x_2(s)) ds - \sum_{0<\tau_k<t} I_k x_2(\tau_k^-) \Big\| \\
& \geq \|x_1(t) - x_2(t)\| - \|x_0^1 - x_0^2\| \\
& - \frac{1}{\Gamma(q)} \sum_{0<\tau_k<t} \int_{\tau_{k-1}}^{\tau_k} (\tau_k - s)^{q-1} \|f(s, x_1(s)) - f(s, x_2(s))\| ds \\
& - \frac{1}{\Gamma(q)} \int_{\tau_k}^{t} (t-s)^{q-1} \|f(s, x_1(s)) - f(s, x_2(s))\| ds \\
& - \sum_{0<\tau_k<t} \|I_k x_1(\tau_k^-) - I_k x_2(\tau_k^-)\|.
\end{aligned}$$

By using hypotheses (H_1) and (H_2), we get

$$\begin{aligned}
\frac{(\varepsilon_1 + \varepsilon_2)}{\Gamma(q+1)} t^q \geq & \|x_1(t) - x_2(t)\| - \|x_0^1 - x_0^2\| \\
& - \frac{1}{\Gamma(q)} \sum_{0<\tau_k<t} \int_{\tau_{k-1}}^{\tau_k} (\tau_k - s)^{q-1} p^* \|x_1(s) - x_2(s)\| ds \\
& - \frac{1}{\Gamma(q)} \int_{\tau_k}^{t} (t-s)^{q-1} p^* \|x_1(s)) - x_2(s)\| ds \\
& - \sum_{0<\tau_k<t} L_k \|x_1(\tau_k^-) - x_2(\tau_k^-)\|.
\end{aligned}$$

Let $z(t) = \|x_1(t) - x_2(t)\|$. Then we have

$$\frac{(\varepsilon_1+\varepsilon_2)}{\Gamma(q+1)}t^q \geq z(t) - \|x_0^1 - x_0^2\| - \frac{1}{\Gamma(q)}\sum_{0<\tau_k<t}\int_{\tau_{k-1}}^{\tau_k}(\tau_k - s)^{q-1}p^* z(s)ds$$
$$- \frac{1}{\Gamma(q)}\int_{\tau_k}^{t}(t-s)^{q-1}p^* z(s)ds - \sum_{0<\tau_k<t} L_k z(\tau_k^-).$$

Therefore we get

$$z(t) \leq \frac{(\varepsilon_1+\varepsilon_2)}{\Gamma(q+1)}t^q + \|x_0^1 - x_0^2\| + \frac{1}{\Gamma(q)}\sum_{0<\tau_k<t}\int_{\tau_{k-1}}^{\tau_k}(\tau_k - s)^{q-1}p^* z(s)ds$$
$$+ \frac{1}{\Gamma(q)}\int_{\tau_k}^{t}(t-s)^{q-1}p^* z(s)ds + \sum_{0<\tau_k<t} L_k z(\tau_k^-).$$

Applying the impulsive fractional inequality given in Lemma 2.5, we get

Case I: For $0 < q \leq \frac{1}{2}$,

$$\begin{aligned} & z(t) \\ \leq\; & \Big[\frac{(\varepsilon_1+\varepsilon_2)}{\Gamma(q+1)}t^q \\ & +\|(x_0^1 - x_0^2)\|\Big]\prod_{0<\tau_k<t}\Bigg\{(1+L_k)e^{\left(\int\limits_{\tau_{k-1}}^{\tau_k}\frac{(t-\xi)^{q-1}}{\Gamma(q)}p^* d\xi\right)} + \frac{p^*}{\Gamma(q)}\left(\frac{e^{\mu_k\tau_k}}{\mu_k^{q^2}}\Gamma(q^2)\right)^{\frac{1}{\mu_k}} \\ & \times\left(\int\limits_{\tau_{k-1}}^{\tau_k} e^{v_k\left(\int\limits_{\tau_{k-1}}^{s}\frac{(t-\xi)^{q-1}}{\Gamma(q)}p^* d\xi - s\right)}ds\right)^{\frac{1}{v_k}}\Bigg\}e^{\left(\int\limits_{\tau_l}^{t}\frac{(t-\xi)^{q-1}}{\Gamma(q)}p^* d\xi\right)} \end{aligned} \tag{6}$$

where $\mu_k = q+1$, $v_k = 1 + \frac{1}{q}$ and $t_l = \max\{\tau_k : t \geq \tau_k, k = 1, 2, ..\}$. Let

$$I_1 = e^{\left(\int\limits_{\tau_{k-1}}^{\tau_k}\frac{(t-\xi)^{q-1}}{\Gamma(q)}p^* d\xi\right)},$$
$$I_2 = e^{v_k\left(\int\limits_{\tau_{k-1}}^{s}\frac{(t-\xi)^{q-1}}{\Gamma(q)}p^* d\xi\right) - v_k s},$$
$$I_3 = \int\limits_{\tau_{k-1}}^{\tau_k} I_2 ds.$$

Since $t - \tau_k \leq T$, $t - \tau_{k-1} \leq T$ and $t \leq T$, we obtain $I_1 \leq C_1$ and $I_3 \leq C_2$ where C_1 and C_2 are some arbitrary constants. Now substituting the above values in inequality

(6), we get the following inequality

$$
\begin{aligned}
\|x_1(t)-x_2(t)\| \le & \Big[\frac{(\varepsilon_1+\varepsilon_2)}{\Gamma(q+1)}t^q + \|(x_0^1-x_0^2)\|\Big] \prod_{0<\tau_k<t}\big\{(1+L_k)\big\} \\
& \{C_1 + \frac{p^*}{\Gamma(q)}\left(\frac{e^{\mu_k\tau_k}}{\mu_k^{q^2}}\Gamma(q^2)\right)^{\frac{1}{\mu_k}} C_2\} e^{\left(\int_{t_l}^{t}\frac{(t-\xi)^{q-1}}{\Gamma(q)}p^* d\xi\right)} \\
& \le \Big[\frac{(\varepsilon_1+\varepsilon_2)}{\Gamma(q+1)}t^q + \|(x_0^1-x_0^2)\|\Big] \prod_{0<\tau_k<t}\big\{(1+L_k)\big\} C_3 e^{\frac{-p^*(t-t_l)^q}{\Gamma(q+1)}} \\
& \le \Big[\frac{(\varepsilon_1+\varepsilon_2)}{\Gamma(q+1)}T^q + \|(x_0^1-x_0^2)\|\Big] \prod_{0<\tau_k<t}\big\{(1+L_k)\big\} C_3 e^{\frac{-p^*T^q}{\Gamma(q+1)}} \\
& \le \Big[\frac{(\varepsilon_1+\varepsilon_2)}{\Gamma(q+1)}T^q + \|(x_0^1-x_0^2)\|\Big] \prod_{0<\tau_k<t}\big\{(1+L_k)\big\} C^*
\end{aligned}
$$

where $C_3 = \{C_1 + \frac{p^*}{\Gamma(q)}\left(\frac{e^{\mu_k\tau_k}}{\mu_k^{q^2}}\Gamma(q^2)\right)^{\frac{1}{\mu_k}} C_2\}$, and $C^* = C_3 e^{\frac{-p^*T^q}{\Gamma(q+1)}}$ are some constants. This gives

$$
\|x_1-x_2\|_{PC} \le \Big[\frac{(\varepsilon_1+\varepsilon_2)}{\Gamma(q+1)}T^q + \|(x_0^1-x_0^2)\|\Big] C^{**} \tag{7}
$$

where $C^{**} = \prod_{0<\tau_k<t}\big\{(1+L_k)\big\}C^*$.

Case II: For $q > \frac{1}{2}$, we similarly obtain the same inequality (7).

REMAR. If we put $\varepsilon_1 = \varepsilon_2 = 0$ and $x_0^1 = x_0^2, t \in (0,T]$ in inequality (7) then the uniqueness of solutions of (1)–(3) is established.

4 Nearness and Convergence of Solutions

Consider the FIIVP (1)–(3), along with the following FIIVP

$$
{}^cD_t^q y(t) = \bar{f}(t, y(t)), \quad t \in (0,T], \quad t \neq \tau_k, k = 1,2,..,m, \tag{8}
$$

$$
y(0) = y_0, \tag{9}
$$

$$
\Delta y(\tau_k) = \bar{I}_k y(\tau_k), \quad k = 1,2,...,m, \tag{10}
$$

where, $\bar{f} : [0,T] \times X \to X$ and $\bar{I}_k : X \to X$.

THEOREM 4.1. Let $x(t)$ and $y(t)$ be respective solutions of initial value problem (1)–(3) and (8)–(10) on $[0,T]$. Suppose that the functions f, $\bar{f}$, I_k and $\bar{I}_k$ in (1)–(3) and (8)–(10) satisfy hypotheses (H_1)–(H_3). Then the following inequality holds:

$$
\|x-y\|_{PC} \le \left\{\|x_0-y_0\| + \frac{2\epsilon_3}{q\Gamma(q)}T^q + \delta_k\right\}C^{**}.
$$

PROOF. Using the facts that $x(t)$ and $y(t)$ be respectively solutions of initial value problem (1)–(2) and (8)–(10) and hypotheses (H_1)–(H_3), we get

$$\begin{aligned}
&\|x(t)-y(t)\| \\
\leq\ & \|x_0-y_0\|+\frac{1}{\Gamma(q)}\sum_{0<\tau_k<t}\int_{\tau_{k-1}}^{\tau_k}(t-s)^{q-1}\|f(s,x(s))-\bar{f}(s,y(s))\|ds \\
&+\frac{1}{\Gamma(q)}\int_{\tau_k}^{t}(t-s)^{q-1}\|f(s,x(s))-\bar{f}(s,y(s))\|ds+\sum_{0<\tau_k<t}L_k\|x(\tau_k)-y(\tau_k)\|+\delta_k \\
\leq\ & [\|x_0-y_0\|+\frac{\epsilon_3}{q\Gamma(q)}\big((t-\tau_k)^q+\sum_{0<\tau_k<t}(\tau_k-\tau_{k-1})^q\big)+\delta_k] \\
&+\frac{1}{\Gamma(q)}\sum_{0<\tau_k<t}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{q-1}p^*\|x(s)-y(s)\|ds \\
&+\frac{1}{\Gamma(q)}\int_{\tau_k}^{t}(t-s)^{q-1}p^*\|x(s)-y(s)\|ds+\sum_{0<\tau_k<t}L_k\|x(\tau_k)-y(\tau_k)\|.
\end{aligned}$$

Let $z(t)=\|x(t)-y(t)\|$. Then

$$\begin{aligned}
z(t)=\|x(t)-y(t)\| &\leq [\|x_0-y_0\|+\frac{\epsilon_3}{q\Gamma(q)}\big((t-\tau_k)^q+\sum_{0<\tau_k<t}(\tau_k-\tau_{k-1})^q\big)+\delta_k] \\
&+\frac{1}{\Gamma(q)}\sum_{0<\tau_k<t}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{q-1}p^*z(s)ds+\frac{1}{\Gamma(q)}\int_{\tau_k}^{t}(t-s)^{q-1}p^*z(s)ds \\
&+\sum_{0<\tau_k<t}L_kz(\tau_k).
\end{aligned}\tag{11}$$

Now applying the Lemma 2.5, we get

$$\begin{aligned}
&\|x(t)-y(t)\| \\
\leq\ & [\|x_0-y_0\|+\frac{\epsilon_3}{q\Gamma(q)}\big((t-\tau_k)^q \\
&+\sum_{0<\tau_k<t}(\tau_k-\tau_{k-1})^q\big)+\delta_k]\prod_{0<t_k<t}\big\{(1+L_k)\big\}C_3e^{\left(\int\limits_{t_l}^{t}\frac{(t-\xi)^{q-1}}{\Gamma(q)}p^*d\xi\right)} \\
\leq\ & \left[\|x_0-y_0\|+\frac{\epsilon_3}{\Gamma(q+1)}\big((t-\tau_k)^q+\sum_{0<\tau_k<t}(\tau_k-\tau_{k-1})^q\big)+\delta_k\right] \\
&\times\prod_{0<t_k<t}\big\{(1+L_k)\big\}C_3e^{\frac{p^*T^q}{\Gamma(q+1)}}
\end{aligned}$$

which gives

$$\begin{aligned}
\|x-y\|_{PC} &\leq [\|x_0-y_0\|+\frac{2\epsilon_3}{\Gamma(q+1)}T^q+\delta_k]\prod_{0<t_k<t}\big\{(1+L_k)\big\}C_3e^{\frac{p^*T^q}{\Gamma(q+1)}}. \\
&\leq \big[\|x_0-y_0\|+\frac{2\epsilon_3}{\Gamma(q+1)}T^q+\delta_k\big]C^{**}.
\end{aligned}$$

This completes the proof.

REMARK. If f is close to $\bar{f}$, x_0 is close to y_0 then the corresponding solutions of initial value problem (1)–(3) and (8)–(10) are close to each other.

Consider the initial value problem (1)–(3) with the initial value problem

$$^cD_t^q y_n(t) = f_n(t, y_n(t)), \quad t \in (0, T], \quad t \neq \tau_k, k = 1, 2, .., m, \tag{12}$$

$$y_n(0) = y_{n0}, \tag{13}$$

$$\Delta y_n(\tau_k) = I_{kn} y_n(\tau_k), \quad k = 1, 2, ..., m, \tag{14}$$

where $f_n : [0, T] \times X \to X$ and $I_{kn} : X \to X$

COROLLARY 4.2. Let $x(t)$ and $y_n(t), n = 1, 2, ...$ be respectively solutions of initial value problems (1)–(3) and (12)–(14) on $[0, T]$. Suppose that the functions f, f_n, I_k and I_{kn} in (1)-(3) and (12)–(14) satisfy the hypotheses (H_1), (H_2) and (H_4). Then $y_n(t) \to x(t)$ as $n \to \infty$ on $(0, T]$.

REMARK. The result obtained in this corollary provides sufficient conditions that ensures solutions of FIIVP problem (12)–(14) will converge to solutions of initial value problem (1)–(3).

5 Continuous Dependence of Solutions On Parameters

Consider the following FIIVP

$$^cD_t^q x(t) = f(t, x(t), \delta), \quad t \in (0, T], \quad t \neq \tau_k, k = 1, 2, .., m \tag{15}$$

$$x(0) = x_0, \tag{16}$$

$$\Delta x(\tau_k) = I_k x(\tau_k), \quad k = 1, 2, ..., m. \tag{17}$$

and

$$^cD_t^q y(t) = f(t, y(t), \delta'), \quad t \in (0, T], \quad t \neq \tau_k, k = 1, 2, .., m \tag{18}$$

$$y(0) = y_0, \tag{19}$$

$$\Delta y(\tau_k) = I_k y(\tau_k), \quad k = 1, 2, ..., m. \tag{20}$$

where $f : [0, T] \times X \times R \to X$, $I_k : X \to X$, and δ, δ' are real parameters.

COROLLARY 5.1. Let $x(t)$ and $y(t)$ be solutions of equations (15)–(17) and (18)–(20), respectively. Assume the hypothesis $[H_2]$ holds and

$$\left\| f(t, \psi, \delta) - f(t, \phi, \delta') \right\| \leq L^*(\|\psi - \phi\| + \|\delta - \delta'\|).$$

Then

$$\|x - y\|_{PC} \leq \left[\|x_0 - y_0\| + \frac{L^* \|\delta - \delta'\|}{\Gamma(q + 1)} 2T^q \right] C^{**}.$$

PROOF. It is an easy consequence of our main result so we have omitted the proof.

Acknowledgment: One of the author Ms. S. D. Kadam would like to acknowledge DST-INSPIRE, New Delhi for providing INSPIRE fellowship.

References

[1] S. Abbas, Existence of solutions to fractional order ordinary and delay differential equations and applications, Electron. J. Differential Equations, 2011, No. 9, 11 pp.

[2] T. Burton, An existence theorem for a fractional differential equation using progressive contractions, J. Fract. Calc. Appl., 8(2017), 168–172.

[3] L. Debnath, Recent applications of fractional calculus to science and engineering, Int. J. Math. Math. Sci., 2003, no. 54, 3413–3442.

[4] H. L. Tidke and M. B. Dhakne: Approximate solutions to nonlinear mixed type integrodifferential equation with nonlocal condition, Comm. Appl. Nonlinear Anal., 17(2010), 35–44.

[5] H. L. Tidke and M. B. Dhakne: Approximate solutions of abstract nonlinear integrodifferential equation of second order with nonlocal conditions, Far East J. Appl. Math., 41(2010), 121–135.

[6] T. Lian, Q. Dong and G. Li, Picard's iterative method for singular fractional differential equations, Int. J. Nonlinear Sci., 22(2016), 54–60.

[7] R. S. Jain, Nearness and convergence properties of mild solutions of impulsive nonlocal cauchy problem, International J. of Math. and its Appl., 4(2016), 35–40.

[8] R. S. Jain, B. Surendranath Reddy and S. D. Kadam, Approximate solutions of impulsive integro-differential equations, Arab. J. Math., 7(2018), 273–279.

[9] T. L. Holambe, M. M. Haque and G. P. Kamble, Approximations to the solution to Cauchy type weighted nonlocal fractional differential equation, Nonlinear Anal. Differ. Equ., 4(2016), 697–717.

[10] A. Yarkar and M. E. Koksal, Existece results for solutions of nonlinear fractional differential equations, Hindawai Publishing Corporation, Abstarct and Applied Analysis, 2012, Article ID267108, 12 pages.

[11] B. G. Pachpatte, Approximate solutions for integrodifferential equations of the neutral type, Comment. Math. Univ. Carolin., 513(2010), 489–501.

[12] D. B. Pachpatte, On some approximate soltion of nonlinear integrodifferential equations on time scale, Differ. Equ. Dyn. Syst., 20(2012), 441–451.

[13] I. Podlubny, Fractional Differential Equations, Acad. Press, London, 1999.

[14] M. Rahimy, Applications of fractional differential equations, Appl. Math. Sci., 4(2010), 2453–2461.

[15] K. S. Miller and B. Ross, An Introduction To Fractional Calculus And Differential Equations, John Wiley And Sons, INC.

[16] V. lashmikanthamm, D. D. Bainov and P. S. Simeonov, Theory of Impulsive Differential Equations, Series in Modern Applied Mathematics, 6. World Scientific Publishing Co., Inc., Teaneck, NJ, 1989.

[17] D. Bainov and P. Simeonov, Impulsive Differential Equations, Periodic Solutions and Applications, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1993.

[18] K. B. Oldham and J. Spanier, The Fractional Calculus, Theory and applications of differentiation and integration to arbitrary order. With an annotated chronological bibliography by Bertram Ross. Mathematics in Science and Engineering, Vol. 111. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1974.

[19] K. D. Kucche and S. T. Sutar, On Existence and Stability Results for Nonlinear Fractional Delay Differential Equations, Bol. Soc. Parana. Mat., 36(2018), 55–75.

[20] W. Liengtragulngam, P. Thiramanus, S. K. Ntouyas and J. Tariboon, Impulsive inequalities with nonlocal jumps and their applications to impulsive fractional integral conditions, J. Inequal. Appl. 2015, 2015:189.

[21] A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[22] K. D. Kucche and J. J. Trujillo, Theory of system of nonlinear Fractional differential equations, Progress in Fractional Differentiation and Applications, 3(2017), 1–12.

[23] K. D. Kucche, J. J. Nieto and V. Venktesh, Theory of nonlinear implicit fractional differential equations, Differ. Equ. Dyn. Syst., (2016), 1–17.

[24] J. Wang, M. Feckan and Y. Zhou, A survey on impulsive fractional differential equations, Fract. Calc. Appl. Anal., 19(2016), 806–831.