Applied Mathematics E-Notes, 19(2019), 179-188 © ISSN 1607-2510
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Topological Sensitivity Analysis For The Transient Heat Problem And Applications*

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Received 7 February 2018

Abstract

In this paper we use the heat transfer equation as a parabolic model problem and we extend the topological sensitivity notion for the non-stationary regime. We derive a topological asymptotic formula valid for a large class of shape functions and arbitrary shaped geometric perturbations. The proposed approach is applied for solving a geometric inverse problem. The leading term of the obtained asymptotic expansion is used for building a one-iteration detection algorithm. The efficiency and accuracy of the proposed algorithm are illustrated by two numerical examples.

1 Introduction

Consider a non-homegeneous heated domain $\Omega \subset \mathbb{R}^2$ with a smooth boundary $\Sigma := \partial\Omega$. The temperature field ϕ inside Ω satisfies the system

$$\left\{ \begin{array}{rlll} \dfrac{\partial \phi}{\partial t} - div(c(x)\nabla\phi) & = Q & \text{in} & \Omega \times (0,T), \\ \phi & = \phi_d & \text{on} & \Sigma \times (0,T), \\ \phi(.,0) & = 0 & \text{in} & \Omega, \end{array} \right. \tag{1}$$

where the parameter c is a given smooth positive function describing the physical properties of the medium Ω, $Q \in L^2(0,T,L^2(\Omega))$ is a given function describing the heat generated source, $\phi_d \in L^2(0,T,H^{1/2}(\Sigma))$ is a given boundary temperature, and $T > 0$ is the computational time. It is well known that the problem (1) admits a unique solution ϕ that belongs to the time dependent functional space $H^1(0,T;H^1(\Omega))$, for more details one can consult [4].

The aim of this work is the detection of an unknown domain $\mathcal{D}^*$ strictly included in a non homogeneous material Ω such that the domain $\Omega\backslash\overline{\mathcal{D}^*}$ solves the optimal design problem

$$(\mathcal{P}_{inv}) \quad \min_{\mathcal{D}\subset\Omega} \mathcal{S}(\Omega\backslash\overline{\mathcal{D}}),$$

*Mathematics Subject Classifications: 49N45, 49Q12, 65K05.
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where $\mathcal{S}$ is a shape function of the form

$$\mathcal{S}(\Omega\backslash\overline{\mathcal{D}}) = \int_0^T \mathcal{F}_{\Omega\backslash\overline{\mathcal{D}}}(\phi_{\mathcal{D}}(.,t))dt,$$

with $\phi_{\mathcal{D}}$ the solution to the following heat transfer problem in $\Omega\backslash\overline{\mathcal{D}}$

$$\begin{cases} \dfrac{\partial\phi_{\mathcal{D}}}{\partial t} - \Delta\phi_{\mathcal{D}} = Q & \text{in} \quad \Omega\backslash\overline{\mathcal{D}}\times(0,T), \\ \nabla\phi_{\mathcal{D}}.n = \phi_n & \text{on} \quad \Sigma\times(0,T), \\ \phi_{\mathcal{D}} = 0 & \text{on} \quad \partial\mathcal{D}\times(0,T), \\ \phi_{\mathcal{D}}(.,0) = 0 & \text{in} \quad \Omega\backslash\overline{\mathcal{D}}. \end{cases}$$

To solve this optimization problem, we propose an accurate approach based on the topological sensitivity analysis method. The first step of our approach consists in finding the place where creating a small geometric perturbation will bring the best improvement of the shape function to be minimized.

More precisely, let $\mathcal{H}_{z,\varepsilon}\subset\Omega$ be a small geometric perturbation that is centered at $z\in\Omega$ and has the shape $\mathcal{H}_{z,\varepsilon} = z+\varepsilon\mathcal{H}$, where $\varepsilon>0$ and $\mathcal{H}\subset\mathbb{R}^2$ is a fixed bounded domain containing the origin. The topological sensitivity analysis method leads to an asymptotic expansion of the shape functional $\mathcal{S}$ on the form

$$\mathcal{S}(\Omega\backslash\overline{\mathcal{H}}_{z,\varepsilon}) = \mathcal{S}(\Omega) + \rho(\varepsilon)\delta\mathcal{S}(z) + o(\rho(\varepsilon)),\ \forall z\in\Omega,$$

where $\varepsilon\mapsto\rho(\varepsilon)$ is a scalar positive function going to zero with ε. This formula is called the topological asymptotic expansion and $\delta\mathcal{S}$ the topological sensitivity function or the topological gradient. Obviously, if we want to minimize $\mathcal{S}$, the best location to insert a small geometric perturbation $\mathcal{H}_{z,\varepsilon}$ in Ω is where $\delta\mathcal{S}$ is most negative. In fact, if $\delta\mathcal{S}(z)<0$, we have $\mathcal{S}(\Omega\backslash\overline{\mathcal{H}}_{z,\varepsilon}) < \mathcal{S}(\Omega)$ for small ε. Starting with this observation, the topological sensitivity analysis allows us to build fast and accurate numerical algorithms that can solve a large class of problems in applications.

The theoretical aspect of the topological sensitivity analysis method has been derived for various operators; one can consult [10] for the Laplace equation, [9] for the elasticity problem, [1, 6, 11] for the Stokes system, [12] for the Helmholtz equation and [5] for the acoustic problem.

However, the most significant contributions in this context have been focused on problems associated with stationary partial differential equations (PDE) [2, 3]. Until recently, there have been very few investigations dealing with the transient regime.

In this paper, we consider the heat transfer equation as a parabolic model problem and we extend the topological sensitivity notion for the non-stationary case.

The rest of this paper is organized as follows. In the next Section, we examine the influence of a small geometric perturbation on the heat transfer problem's solution. In Section 3, we extend the topological sensitivity analysis notion for the non-stationary regime and we derive a topological asymptotic expansion valid for a large class of shape functions. Section 4 is devoted to some numerical applications.

2 Estimate of the Perturbed Solution

Let $\mathcal{H}_{z,\varepsilon}$ be a small geometric perturbation created inside the initial domain. We denote by ϕ_ε the solution to the following heat transfer problem in the perturbed domain $\Omega\backslash\overline{\mathcal{H}}_{z,\varepsilon}$ with a Dirichlet boundary condition on the boundary $\partial\mathcal{H}_{z,\varepsilon}$

$$\begin{cases} \dfrac{\partial\phi_\varepsilon}{\partial t} - div(c(x)\nabla\phi_\varepsilon) &= Q \quad \text{in} \quad \Omega\backslash\overline{\mathcal{H}}_{z,\varepsilon}\times(0,T), \\ \phi_\varepsilon &= g_D \quad \text{on} \quad \Sigma\times(0,T), \\ \phi_\varepsilon &= 0 \quad \text{on} \quad \partial\mathcal{H}_{z,\varepsilon}\times(0,T), \\ \phi_\varepsilon(.,0) &= 0 \quad \text{in} \quad \Omega\backslash\overline{\mathcal{H}}_{z,\varepsilon}. \end{cases} \tag{2}$$

Note that for $\varepsilon = 0$, we have $\Omega\backslash\overline{\mathcal{H}_{z,\varepsilon}} = \Omega$ and ϕ_ε coincides with the temperature field ϕ in the non-perturbed domain Ω.

Next, we discuss the influence of the small geometric perturbation on the temperature distribution. We will derive the asymptotic behavior of the variation $\phi_\varepsilon - \phi$ with respect to the perturbation size ε. We will prove that the leading term of the variation $\phi_\varepsilon - \phi$ is defined as

$$\Theta(x,t) = 2\pi\phi(z,t)\Gamma(x-z),\ \forall(x,t)\in\Omega\backslash\overline{\mathcal{H}}_{z,\varepsilon}\times(0,T), \tag{3}$$

where Γ is the fundamental solution of the Laplace operator in $\mathbb{R}^2$ (see [7])

$$\Gamma(y) = \frac{1}{2\pi}\log|y|, \forall y\in\mathbb{R}^2.$$

Consequently, we derive the following estimate.

PROPOSITION 1. Let $\mathcal{H}_{z,\varepsilon}$ be a small geometric perturbation created inside the domain Ω with a Dirichlet boundary condition on $\partial\mathcal{H}_{z,\varepsilon}$. Then, the perturbed solution ϕ_ε satisfies the estimate

$$\|\phi_\varepsilon - \phi - \frac{1}{\log(\varepsilon)}\Theta\|_{L^\infty(0,T;L^2(\Omega\backslash\overline{\mathcal{H}}_{z,\varepsilon}))} + \|\phi_\varepsilon - \phi - \frac{1}{\log(\varepsilon)}\Theta\|_{L^2(0,T;H^1(\Omega\backslash\overline{\mathcal{H}}_{z,\varepsilon}))} \leq \frac{c}{\sqrt{-\log\varepsilon}}.$$

PROOF. The established estimate can be derived using Green formula, elliptic inequality and Trace theorem. For more details and similar results, one can consult [8].

3 Topological Sensitivity Analysis

In this section, we derive an asymptotic expansion describing the variation of $\mathcal{S}$ with respect to the presence of a small geometric perturbation, valid for a large class of shape functions $\mathcal{S}$. More precisely, we will establish an asymptotic formula of the form

$$\mathcal{S}(\Omega\backslash\overline{\mathcal{H}}_{z,\varepsilon}) = \mathcal{S}(\Omega) - \frac{1}{\log\varepsilon}\delta\mathcal{S}(z) + o(\frac{-1}{\log\varepsilon}),$$

valid for all shape function $\mathcal{S}$ having the form

$$\mathcal{S}(\Omega\backslash\overline{\mathcal{H}}_{z,\varepsilon}) = \int_0^T \mathcal{F}_\varepsilon(\phi_\varepsilon(.,t))dt,$$

where ϕ_ε is the solution to (2) and $\mathcal{F}_\varepsilon$ is a cost function defined on $H^1(\Omega\backslash\overline{\mathcal{H}}_{z,\varepsilon})$ and satisfying the following assumption:

Assumption $(\mathcal{A})$. In the term $D\mathcal{F}_0(\phi(.,t))(\phi_\varepsilon - \phi)$, the solution ϕ_ε is extended by zero inside the domain $\mathcal{H}_{z,\varepsilon}$.

Using the assumption $(\mathcal{A})$, the variation of the shape function j reads

$$\mathcal{S}(\Omega\backslash\overline{\mathcal{H}_{z,\varepsilon}}) - \mathcal{S}(\Omega) = \int_0^T D\mathcal{F}_0(\phi)(\phi_\varepsilon - \phi)dt - \frac{1}{\log\varepsilon}\delta\mathcal{F}(z) + o(\frac{-1}{\log\varepsilon}).$$

Let ψ be the solution to the associated adjoint problem. It satisfies the following system

$$\begin{cases} -\frac{\partial\psi}{\partial t} - div(c(x)\nabla\psi) &= -D\mathcal{F}_0(\phi) \quad \text{in} \quad \Omega\times(0,T), \\ \psi &= 0 \quad \text{on} \quad \Sigma\times(0,T), \\ \psi(.,T) &= 0 \quad \text{in} \quad \Omega. \end{cases} \tag{4}$$

With the help of the weak formulation of (4), the shape function variation can be rewritten as

$$\begin{aligned} \mathcal{S}(\Omega\backslash\overline{\mathcal{H}_{z,\varepsilon}}) - \mathcal{S}(\Omega) \quad = \quad & -\int_0^T\int_\Omega \frac{\partial(\phi_\varepsilon-\phi)}{\partial t}\psi dxdt - \int_0^T\int_\Omega c(x)\nabla(\phi_\varepsilon-\phi).\nabla\psi dxdt \\ & -\frac{1}{\log\varepsilon}\delta\mathcal{F}(z) + o(\frac{-1}{\log\varepsilon}). \end{aligned}$$

From the fact that $\phi_\varepsilon = 0$ in $\mathcal{H}_{z,\varepsilon}$, it follows

$$\begin{aligned} \mathcal{S}(\Omega\backslash\overline{\mathcal{H}_{z,\varepsilon}}) - \mathcal{S}(\Omega) = & \int_0^T\int_{\mathcal{H}_{z,\varepsilon}} \frac{\partial\phi}{\partial t}\psi dxdt + \int_0^T\int_{\mathcal{H}_{z,\varepsilon}} c(x)\nabla\phi.\nabla\psi dxdt \\ & -\int_0^T\int_{\Omega\backslash\overline{\mathcal{H}}_{z,\varepsilon}} \frac{\partial(\phi_\varepsilon-\phi)}{\partial t}\psi dxdt - \int_0^T\int_{\Omega\backslash\overline{\mathcal{H}}_{z,\varepsilon}} c(x)\nabla(\phi_\varepsilon-\phi).\nabla\psi dxdt \\ & -\frac{1}{\log\varepsilon}\delta\mathcal{F}(z) + o(\frac{-1}{\log\varepsilon}). \end{aligned}$$

Applying a Green formula, we obtain

$$\begin{aligned} \mathcal{S}(\Omega\backslash\overline{\mathcal{H}_{z,\varepsilon}}) - \mathcal{S}(\Omega) = & \int_0^T\int_{\mathcal{H}_{z,\varepsilon}} \frac{\partial\phi}{\partial t}\psi dxdt + \int_0^T\int_{\mathcal{H}_{z,\varepsilon}} c(x)\nabla\phi.\nabla\psi dxdt \\ & -\int_0^T\int_{\partial\mathcal{H}_{z,\varepsilon}} c(x)\nabla(\phi_\varepsilon-\phi).n\psi dsdt - \frac{1}{\log\varepsilon}\delta\mathcal{F}(z) + o(\frac{-1}{\log\varepsilon}). \end{aligned} \tag{5}$$

We are now ready to present the main results of this section.

THEOREM 1. Let $\mathcal{H}_{z,\varepsilon} = z + \varepsilon\mathcal{H}$ be a small geometric perturbation created inside the domain Ω. If $\mathcal{F}_\varepsilon$ satisfies the assumption $(\mathcal{A})$, then the shape function $\mathcal{S}$ admits the following expansion

$$\mathcal{S}(\Omega\backslash\overline{\mathcal{H}}_{z,\varepsilon}) = \mathcal{S}(\Omega) - \frac{1}{\log\varepsilon}\delta\mathcal{S}(z) + o\left(\frac{-1}{\log\varepsilon}\right),$$

with the topological sensitivity function $\delta\mathcal{S}$ defined by

$$\delta\mathcal{S}(x) = 2\pi c(x)\int_0^T \phi(x,t)\psi(x,t)dt + \delta\mathcal{F}(x), \forall x \in \Omega.$$

The term $\delta\mathcal{F}$ depends on the considered function $\mathcal{F}_\varepsilon$.

PROOF. The obtained asymptotic formula can be established with the help of Proposition 1 and some Cauchy-Schwartz inequalities. The term $\frac{-1}{\log\varepsilon}$ follows from the asymptotic behavior of the fundamental solution of the Laplace operator. We refer to [8] for more details and similar results.

In the next section, we will present two numerical examples and we will calculate the variations $\delta\mathcal{F}$ for two considered cost functions.

4 Numerical Experiments

This section is devoted to some numerical investigations. Based on the obtained theoretical result, we present in the next section a one-iteration detection algorithm for solving the considered geometric inverse problem. The proposed algorithm will be applied in sections 4.2 and 4.3 for solving two numerical examples.

4.1 Detection Procedure

To solve the inverse problem $(\mathcal{P}_{inv})$, we propose a fast and efficient detection procedure. The unknown domain $\mathcal{D}^*$ will be located at spots where the topological sensitivity function $\delta\mathcal{S}$ is most negative. The main steps of the proposed detection procedure are summarized by the following one-iteration algorithm.

The one-iteration detection algorithm

- Compute the temperature field ϕ and the adjoint state ψ in the initial domain Ω.
- Compute the topological sensitivity function $\delta\mathcal{S}(x), \forall x \in \Omega$.
- Determine $\zeta^* \in [0,1]$ such that $\mathcal{S}(\Omega\backslash\overline{\mathcal{H}_{\zeta^*}}) \leq \mathcal{S}(\Omega\backslash\overline{\mathcal{H}_\zeta})$, $\forall\zeta \in [0,1]$, where $\mathcal{H}_\zeta = \{x \in \Omega;\ \delta\mathcal{S}(x) < \zeta\delta_{min}\}$ with δ_{min} is the most negative value of the function $\delta\mathcal{S}$ in Ω.

Next, we will apply this algorithm for solving the geometric inverse problem $(\mathcal{P}_{inv})$ and we will present some numerical simulations. Two examples will be considered. The first one concerns the quadratic boundary function (7). The second one is associated with the H^1-semi norm function (9). In all the numerical simulations, the solutions ϕ and ψ are computed in the space domain $\Omega = [0, 3/2] \times [0, 1]$, with the data time $T = 1$.

4.2 First Numerical Example

This example concerns the detection of an unknown domain $\mathcal{D}^* \subset \Omega$ solution to

$$(\mathcal{P}_{inv}^1) \quad \min_{\mathcal{D}\subset\Omega} \int_0^T \int_\Sigma |\phi_D - \varphi_m|^2 ds dt, \tag{6}$$

where φ_m is a measured boundary datum.

It is easy to see that this example is associated with the cost function $\mathcal{F}_\varepsilon$ defined on the exterior boundary Σ by

$$\mathcal{F}_\varepsilon(\phi) = \int_\Sigma |\phi - \varphi_m|^2 ds, \, \forall \phi \in H^1(\Omega \backslash \overline{\mathcal{H}}_{z,\varepsilon}). \tag{7}$$

The function (7) satisfies the assumption $(\mathcal{A})$ with

$$D\mathcal{F}_0(\phi(.,t))h = 2\int_\Sigma [\phi(.,t) - \varphi_m(.,t)]h ds, \, \forall h \in H^1(\Omega) \text{ and } \delta\mathcal{F}(x) = 0, \; x \in \Omega.$$

From Theorem 1, one can deduce that the topological sensitivity function $\delta\mathcal{S}$ associated to this geometric inverse problem is given by

$$\delta\mathcal{S}(x) = 2\pi \int_0^T \phi(x,t)\psi(x,t)dt, \, \forall x \in \Omega.$$

Next, we will present some detection results obtained by the proposed numerical algorithm.

In Figure 1, we present the detection result for an elliptical-shaped domain $\mathcal{D}^*$. As one can observe, the unknown domain $\mathcal{D}^*$ is located at zone where the topological sensitivity function $\delta\mathcal{S}$ is most negative. The boundary $\partial\mathcal{D}^*$ of the unknown domain (black line) is approximated by a level set curve of the scalar function $\delta\mathcal{S}$: $\partial\mathcal{D}^* = \{x \in \Omega; \, \delta\mathcal{S}(x) = \zeta\delta_{min}\}$.

In Figures 2 and 3, we consider the case where the unknown domain $\mathcal{D}^*$ is composed of two connected sub-domains. The detection result of two elliptical shaped-domains is illustrated in Figure 2.

In Figure 3, we present the detection result for two circular-shaped domains located near the corner.

From Figures 1–3, one can observe that the proposed algorithm gives a good detection result for the unknown domains located close to the boundary Σ. But what happens if the unknown domain $\mathcal{D}^*$ is not close to the boundary? To discuss this case, we apply our numerical algorithm for detecting a circular-shaped domain located at

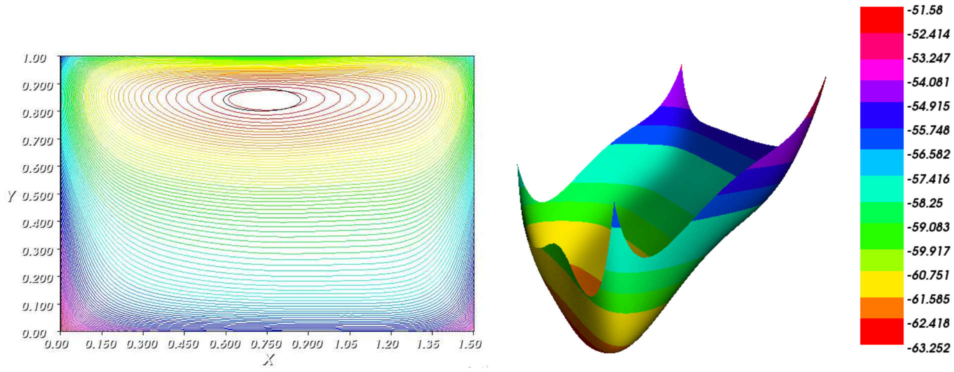

Figure 1: Detection result of an elliptical-shaped domain.

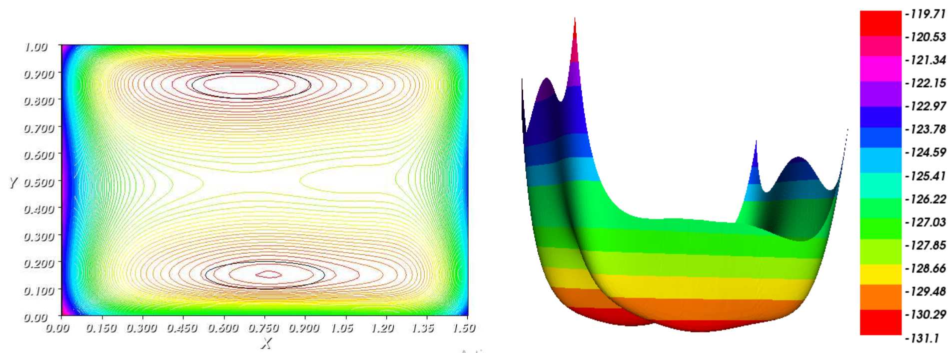

Figure 2: Detection of two elliptical-shaped domains.

the center of Ω. The obtained result is shown in Figure 4. As it is expected, the algorithm cannot detect the unknown domain in this case. This can be explained by the fact the topological sensitivity function $\delta\mathcal{S}$ has a poor information from the boundary measured data. Such information comes from the term $\delta\mathcal{F}$ which is negligible for this shape function example.

We consider in the next section a shape function example with significant variation term $\delta\mathcal{F}$.

4.3 Second Numerical Example

Here, we aim to detect the unknown domain $\mathcal{D}^*$ solution to the following geometric inverse problem

$$(\mathcal{P}^2_{inv}) \min_{\mathcal{D}\subset\Omega} \int_0^T \int_{\Omega\setminus\overline{\mathcal{D}}} |\nabla\phi_D - \nabla\varphi_w|^2 dxdt, \tag{8}$$

where φ_w is a given wanted state.

The associated cost function $\mathcal{F}_\varepsilon$ is defined by the H^1-semi-norm

$$\mathcal{F}_\varepsilon(\phi) = \int_{\Omega\setminus\overline{\mathcal{H}}_{z,\varepsilon}} |\nabla\phi - \nabla\varphi_w|^2 dx, \ \forall\phi\in H^1(\Omega\setminus\overline{\mathcal{H}}_{z,\varepsilon}). \tag{9}$$

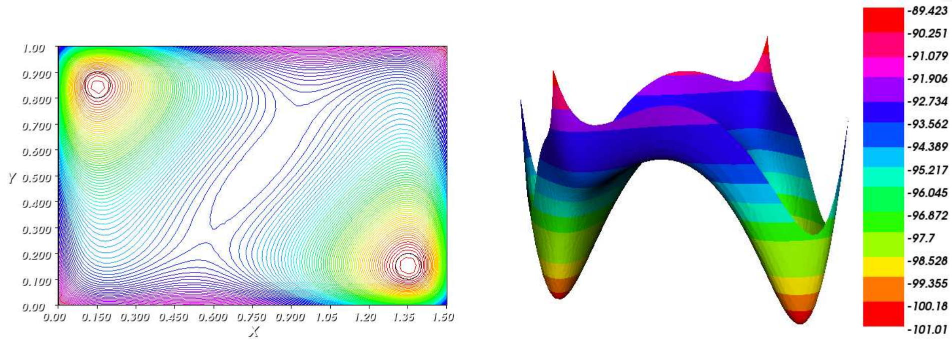

Figure 3: Detection of two circular-shaped domains.

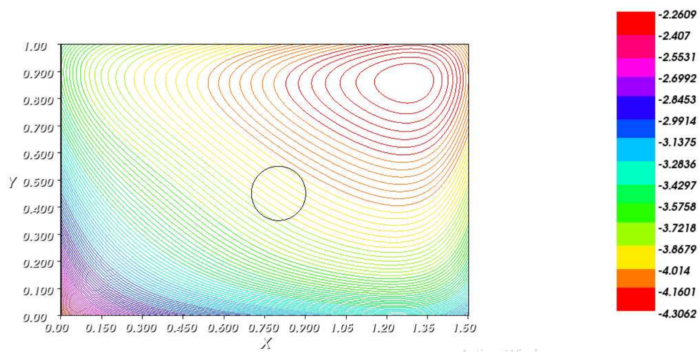

Figure 4: Detection result for a circular-shaped domain located away from the boundary.

The function (9) satisfies the assumption $(\mathcal{A})$ with

$$\begin{aligned} D\mathcal{F}_0(\phi(.,t))h &= 2\int_\Omega [\nabla\phi(.,t) - \nabla\varphi_w(.,t)]\nabla h dx, \forall h \in H^1(\Omega), \\ \delta\mathcal{F}(x) &= 2\pi c(x)\int_0^T |\phi(x,t)|^2 dt,\ x \in \Omega. \end{aligned}$$

From Theorem 1, it follows that the topological sensitivity function is given by

$$\delta\mathcal{S}(x) = 2\pi\int_0^T \left[\phi(x,t)\psi(x,t) + |\phi(x,t)|^2\right] dt, \forall x \in \Omega.$$

Next, we present some detection results for different size, shape and location domains unknown.

We start this paragraph by the detection of unknown circular and elliptical shaped domains $\mathcal{D}^*$. The target domain $\mathcal{D}^*$ and the iso-values of the topological sensitivity function $\delta\mathcal{S}$ are described in Figures 5 and 6. As one can observe, the unknown domain $\mathcal{D}^*$ (black line) is approximated by a level set curve of $\delta\mathcal{S}$. The last case is concerned

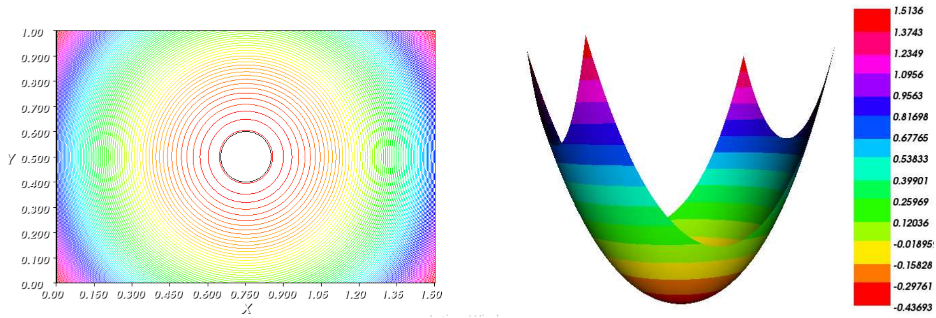

Figure 5: Detection of a circular-shaped domain.

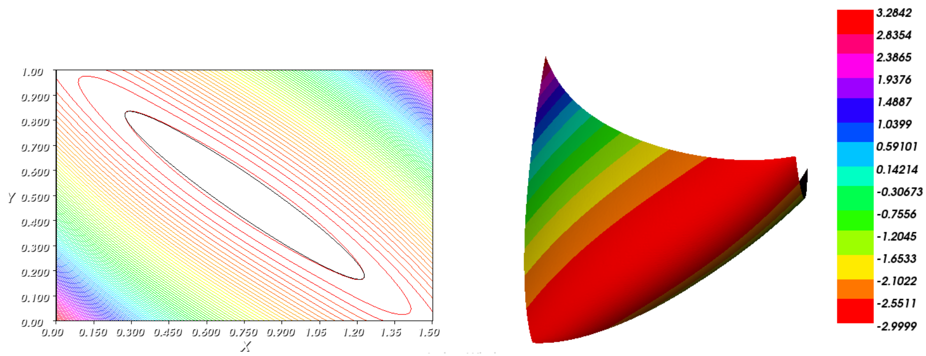

Figure 6: Detection of an elliptical-shaped domain.

with a singular geometric domain. We present in Figure 7, the detection result for a square-shaped domain.

5 Conclusion

A topological asymptotic expansion is derived for a parabolic type operator. The established asymptotic formula is valid for a large class of shape functions. The leading term of the shape function variation is exploited for building a fast and simple detection algorithm. The constructed numerical procedure is applied for solving two geometric inverse problems. The obtained numerical simulations show that the proposed detection algorithm can successfully detect different size and unknown shaped domains.

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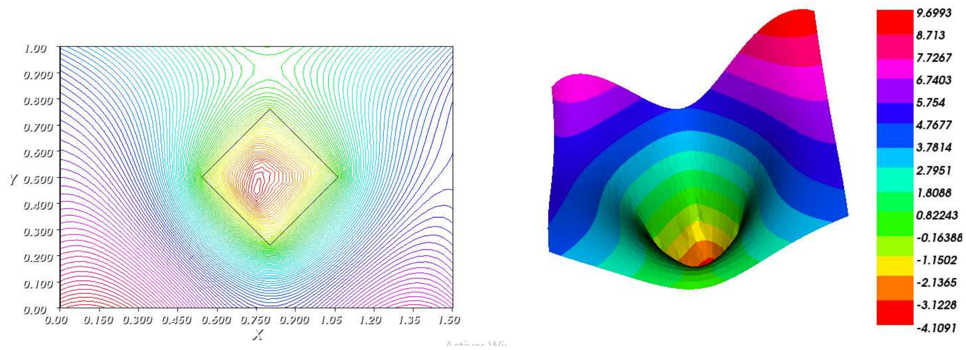

Figure 7: Detection of square-shaped domain.

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