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Chebyshev Polynomial Bounded For Analytic And Bi-Univalent Functions With Respect To Symmetric Conjugate Points*

Abbas Kareem Wanas[†], Abdulrahman Hameed Majeed[‡]

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Abstract

In the present investigation, we use the Chebyshev polynomial expansions to derive estimates on the initial coefficients for a new subclass of analytic and bi-univalent functions with respect to symmetric conjugate points. Also, we solve Fekete-Szegö problem for functions in this class.

1 Introduction

The importance of Chebyshev polynomial in numerical analysis is increased in both theoretical and practical points of view. There are four kinds of Chebyshev polynomials. Several researchers dealing with orthogonal polynomials of Chebyshev family, contain mainly results of Chebyshev polynomials of first kind $T_n(t)$, the second kind $U_n(t)$ and their numerous uses in different applications one can refer [5, 7, 11]. The Chebyshev polynomials of the first and second kinds are well known and they are defined by

$$T_n(t) = \cos n\theta \qquad \text{and} \qquad U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta} \quad (-1 < t < 1),$$

where n indicates the polynomial degree and $t = \cos n\theta$.

Let $\mathcal{A}$ stand for the family of functions f which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ that have the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Also, let S be the subclass of $\mathcal{A}$ consisting of the form (1) which are univalent in U.

For two functions f and g analytic in U, we say that the function f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$, if there exists a Schwarz function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$,$(z \in U)$.

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[†]Department of Mathematics, College of Science, University of Al-Qadisiyah, Iraq
[‡]Department of Mathematics, College of Science, University of Baghdad, Iraq

It is well known (see [6]) that every function $f \in S$ has an inverse f^{-1}, defined by $f^{-1}(f(z)) = z$, $(z \in U)$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f), r_0(f) \geq \frac{1}{4})$, where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \quad (2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ indicate the class of bi-univalent functions in U given by (1). Some examples of functions in the class Σ are

$$\frac{z}{1-z}, \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) \quad \text{and} \quad -\log(1-z)$$

with the corresponding inverse functions

$$\frac{w}{1+w}, \quad \frac{e^{2w}-1}{e^{2w}+1} \quad \text{and} \quad \frac{e^{w}-1}{e^{w}},$$

respectively. Other common examples of functions is not a member of Σ are

$$z - \frac{z^2}{2} \quad \text{and} \quad \frac{z}{1-z^2}.$$

Recently, many authors introduced various subclasses of the bi-univalent functions class Σ and investigated non sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1) (see [1, 2, 3, 4, 10, 12, 13]).

The Fekete-Szegö functional $\left|a_3 - \mu a_2^2\right|$ for $f \in S$ is well known for its rich history in the field of Geometric Function Theory. Its origin was in the disproof by Fekete and Szegö [9] conjecture of Littlewood and Paley that the coefficients of odd univalent functions are bounded by unity.

El-Ashwah and Thomas [8] introduced the class S_{sc}^{*} of functions called starlike with respect to symmetric conjugate points, they are the functions $f \in S$ satisfy the condition

$$Re\left\{\frac{zf'(z)}{f(z) - \overline{f(-\overline{z})}}\right\} > 0, \quad z \in U.$$

A function $f \in S$ is called convex with respect to symmetric conjugate points, if

$$Re\left\{\frac{(zf'(z))'}{\left(f(z) - \overline{f(-\overline{z})}\right)'}\right\} > 0, \quad z \in U.$$

The class of all convex functions with respect to symmetric conjugate points is denoted by C_{sc}.

DEFINITION 1. For $0 < \alpha \leq 1$ and $t \in (\frac{1}{2}, 1]$, a function $f \in \Sigma$ is said to be in the class $\mathcal{F}_{\Sigma}^{sc}(\alpha, t)$ if it satisfies the subordinations:

$$\frac{2zf'(z)}{f(z) - \overline{f(-\overline{z})}} + \frac{2\left(zf'(z)\right)'}{\left(f(z) - \overline{f(-\overline{z})}\right)'} - \frac{2\alpha z^2 f''(z) + 2zf'(z)}{\alpha z\left(f(z) - \overline{f(-\overline{z})}\right)' + (1-\alpha)\left(f(z) - \overline{f(-\overline{z})}\right)}$$
$$\prec H(z,t) = \frac{1}{1 - 2tz + z^2}$$

and

$$\frac{2wg'(w)}{g(w)-\overline{g(-\overline{w})}}+\frac{2\left(wg'(w)\right)'}{\left(g(w)-\overline{g(-\overline{w})}\right)'}-\frac{2\alpha w^2g''(w)+2wg'(w)}{\alpha w\left(g(w)-\overline{g(-\overline{w})}\right)'+(1-\alpha)\left(g(w)-\overline{g(-\overline{w})}\right)}$$
$$\prec H(w,t)=\frac{1}{1-2tw+w^2},$$

where the function $g=f^{-1}$ is given by (2).

In particular, we set $\mathcal{F}^{sc}_{\Sigma}(0,t)=\mathcal{F}^{sc}_{\Sigma}(t)$ for the class of functions $f\in\Sigma$ given by (1) and satisfying the following subordinations:

$$\frac{2\left(zf'(z)\right)'}{\left(f(z)-\overline{f(-\overline{z})}\right)'}\prec H(z,t)=\frac{1}{1-2tz+z^2}$$

and

$$\frac{2\left(wg'(w)\right)'}{\left(g(w)-\overline{g(-\overline{w})}\right)'}\prec H(w,t)=\frac{1}{1-2tw+w^2},$$

where the function $g=f^{-1}$ is given by (2).

We note that if $t=\cos\beta$, where $\beta\in(\frac{-\pi}{3},\frac{\pi}{3})$, then

$$H(z,t)=\frac{1}{1-2\cos\beta z+z^2}=1+\sum_{n=1}^{\infty}\frac{\sin(n+1)\beta}{\sin\beta}z^n,\ z\in U.$$

Therefore

$$H(z,t)=1+2\cos\beta z+\left(3\cos^2\beta-\sin^2\beta\right)z^2+\cdots,\ z\in U.$$

From [14], we can write

$$H(z,t)=1+U_1(t)z+U_2(t)z^2+\cdots\ \left(z\in U,t\in(-1,1)\right),$$

where

$$U_{n-1}=\frac{\sin(n\,arc\cos t)}{\sqrt{1-t^2}}\ \left(n\in\mathbb{N}=\{1,2,\cdots\}\right),$$

are the Chebyshev polynomials of the second kind. Also, it is known that

$$U_n(t)=2tU_{n-1}(t)-U_{n-2}(t)$$

and

$$U_1(t)=2t,\ U_2(t)=4t^2-1,\ U_3(t)=8t^3-4t,\cdots. \tag{3}$$

The generating function of the first kind of Chebyshev polynomial $T_n(t)$, $t\in[-1,1]$ is given by

$$\sum_{n=0}^{\infty}T_n(t)z^n=\frac{1-tz}{1-2tz+z^2},\ z\in U.$$

The Chebyshev polynomials of first kind $T_n(t)$ and of the second kind $U_n(t)$ are connected by

$$\frac{dT_n(t)}{dt} = nU_{n-1}(t),\ T_n(t) = U_n(t) - tU_{n-1}(t),\ 2T_n(t) = U_n(t) - U_{n-2}(t).$$

The main object of the present paper is to introduce a new subclass $\mathcal{F}_{\Sigma}^{sc}(\alpha, t)$ of analytic and bi-univalent functions with respect to other points and seek the initial coefficients for functions in $\mathcal{F}_{\Sigma}^{sc}(\alpha, t)$ by using Chebyshev polynomial expansions.

2 Coefficient Bounds for the Function Class $\mathcal{F}_{\Sigma}^{sc}(\alpha, t)$

THEOREM 1. For $0 < \alpha \le 1$ and $t \in (\frac{1}{2}, 1]$, let f given by (1) be in the class $\mathcal{F}_{\Sigma}^{sc}(\alpha, t)$. Then

$$|a_2| \le \frac{t\sqrt{2t}}{\sqrt{\left|(2-\alpha)^2 - 2\left(2\alpha^2 - 6\alpha + 5\right)t^2\right|}}$$

and

$$|a_3| \le \frac{t^2}{(2-\alpha)^2} + \frac{t}{3-2\alpha}.$$

PROOF. Let $f \in \mathcal{F}_{\Sigma}^{sc}(\alpha, t)$. Then there exists two analytic functions $u, v : U \longrightarrow U$ given by

$$u(z) = u_1 z + u_2 z^2 + u_3 z^3 + \cdots \quad (z \in U) \tag{4}$$

and

$$v(w) = v_1 w + v_2 w^2 + v_3 w^3 + \cdots \quad (w \in U), \tag{5}$$

with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(w)| < 1$, $z, w \in U$ such that

$$\frac{2zf'(z)}{f(z) - \overline{f(-\overline{z})}} + \frac{2\left(zf'(z)\right)'}{\left(f(z) - \overline{f(-\overline{z})}\right)'} - \frac{2\alpha z^2 f''(z) + 2zf'(z)}{\alpha z\left(f(z) - \overline{f(-\overline{z})}\right)' + (1-\alpha)\left(f(z) - \overline{f(-\overline{z})}\right)}$$
$$= 1 + U_1(t)u(z) + U_2(t)u^2(z) + \cdots \tag{6}$$

and

$$\frac{2wg'(w)}{g(w) - \overline{g(-\overline{w})}} + \frac{2\left(wg'(w)\right)'}{\left(g(w) - \overline{g(-\overline{w})}\right)'} - \frac{2\alpha w^2 g''(w) + 2wg'(w)}{\alpha w\left(g(w) - \overline{g(-\overline{w})}\right)' + (1-\alpha)\left(g(w) - \overline{g(-\overline{w})}\right)}$$
$$= 1 + U_1(t)v(w) + U_2(t)v^2(w) + \cdots. \tag{7}$$

Combining (4)–(7), we obtain

$$\frac{2zf'(z)}{f(z) - \overline{f(-\overline{z})}} + \frac{2\left(zf'(z)\right)'}{\left(f(z) - \overline{f(-\overline{z})}\right)'} - \frac{2\alpha z^2 f''(z) + 2zf'(z)}{\alpha z\left(f(z) - \overline{f(-\overline{z})}\right)' + (1-\alpha)\left(f(z) - \overline{f(-\overline{z})}\right)}$$
$$= 1 + U_1(t)u_1 z + \left[U_1(t)u_2 + U_2(t)u_1^2\right]z^2 + \cdots \tag{8}$$

and

$$\frac{2wg'(w)}{g(w)-\overline{g(-\overline{w})}}+\frac{2\left(wg'(w)\right)'}{\left(g(w)-\overline{g(-\overline{w})}\right)'}-\frac{2\alpha w^2 g''(w)+2wg'(w)}{\alpha w\left(g(w)-\overline{g(-\overline{w})}\right)'+(1-\alpha)\left(g(w)-\overline{g(-\overline{w})}\right)}$$
$$=1+U_1(t)v_1w+\left[U_1(t)v_2+U_2(t)v_1^2\right]w^2+\cdots. \tag{9}$$

It is well-known that if $|u(z)|<1$ and $|v(w)|<1$, $z,w\in U$, then

$$|u_i|\leq 1 \quad \text{and} \quad |v_i|\leq 1 \text{ for all } i\in\mathbb{N}. \tag{10}$$

Comparing the corresponding coefficients in (8) and (9), after simplifying, we have

$$2(2-\alpha)a_2=U_1(t)u_1, \tag{11}$$

$$2(3-2\alpha)a_3=U_1(t)u_2+U_2(t)u_1^2, \tag{12}$$

$$-2(2-\alpha)a_2=U_1(t)v_1 \tag{13}$$

and

$$2(3-2\alpha)(2a_2^2-a_3)=U_1(t)v_2+U_2(t)v_1^2. \tag{14}$$

It follows from (11) and (13) that

$$u_1=-v_1 \tag{15}$$

and

$$8\left(2-\alpha\right)^2a_2^2=U_1^2(t)(u_1^2+v_1^2). \tag{16}$$

If we add (12) to (14), we find that

$$4(3-2\alpha)a_2^2=U_1(t)(u_2+v_2)+U_2(t)(u_1^2+v_1^2). \tag{17}$$

Substituting the value of $u_1^2+v_1^2$ from (16) in the right hand side of (17), we get

$$\left[4(3-2\alpha)-\frac{8U_2(t)}{U_1^2(t)}\left(2-\alpha\right)^2\right]a_2^2=U_1(t)(u_2+v_2). \tag{18}$$

Further computations using (3), (10) and (18), we obtain

$$|a_2|\leq\frac{t\sqrt{2t}}{\sqrt{\left|\left(2-\alpha\right)^2-2\left(2\alpha^2-6\alpha+5\right)t^2\right|}}.$$

Next, if we subtract (14) from (12), we deduce that

$$4(3-2\alpha)(a_3-a_2^2)=U_1(t)(u_2-v_2)+U_2(t)(u_1^2-v_1^2). \tag{19}$$

In view of (15) and (16), we get from (19)

$$a_3=\frac{U_1^2(t)}{8\left(2-\alpha\right)^2}(u_1^2+v_1^2)+\frac{U_1(t)}{4(3-2\alpha)}(u_2-v_2).$$

Thus applying (3), we obtain

$$|a_3| \leq \frac{t^2}{(2-\alpha)^2} + \frac{t}{3-2\alpha}.$$

Putting $\alpha = 0$ in Theorem 1, we conclude the following result:

COROLLARY 1. For $t \in (\frac{1}{2}, 1]$, let f given by (1) in the class $\mathcal{F}^{sc}_{\Sigma}(t)$. Then

$$|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{|2-5t^2|}}$$

and

$$|a_3| \leq \frac{t(3t+4)}{12}.$$

3 Fekete-Szegö Problem for the Function Class $\mathcal{F}^{sc}_{\Sigma}(\alpha, t)$

THEOREM 2. For $0 < \alpha \leq 1$, $t \in (\frac{1}{2}, 1]$ and $\mu \in \mathbb{R}$, let f given by (1) be in the class $\mathcal{F}^{sc}_{\Sigma}(\alpha, t)$. Then

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{t}{3-2\alpha} & \text{for } |\mu - 1| \leq \frac{1}{2(3-2\alpha)} \left| \frac{(2-\alpha)^2}{t^2} - 2(2\alpha^2 - 6\alpha + 5) \right|, \\ \frac{2t^3|\mu-1|}{\left|2(3-2\alpha)t^2 - (2-\alpha)^2(4t^2-1)\right|} & \text{for } |\mu - 1| \geq \frac{1}{2(3-2\alpha)} \left| \frac{(2-\alpha)^2}{t^2} - 2(2\alpha^2 - 6\alpha + 5) \right|. \end{cases}$$

PROOF. It follows from (18) and (19) that

$$\begin{aligned} a_3 - \mu a_2^2 &= (1-\mu)\frac{U_1^3(t)(u_2+v_2)}{4(3-2\alpha)U_1^2(t) - 8U_2(t)(2-\alpha)^2} + \frac{U_1(t)}{4(3-2\alpha)}(u_2 - v_2) \\ &= U_1(t)\left[\left(\psi(\mu) + \frac{1}{4(3-2\alpha)}\right)u_2 + \left(\psi(\mu) - \frac{1}{4(3-2\alpha)}\right)v_2\right], \end{aligned}$$

where

$$\psi(\mu) = \frac{U_1^2(t)(1-\mu)}{4\left[(3-2\alpha)U_1^2(t) - 2U_2(t)(2-\alpha)^2\right]}.$$

According to (3), we find that

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{t}{3-2\alpha}, & 0 \leq |\psi(\mu)| \leq \frac{1}{4(3-2\alpha)}, \\ 4t\,|\psi(\mu)|, & |\psi(\mu)| \geq \frac{1}{4(3-2\alpha)}. \end{cases}$$

After some computations, we obtain

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{t}{3-2\alpha} & \text{for } |\mu-1| \leq \frac{1}{2(3-2\alpha)}\left|\frac{(2-\alpha)^2}{t^2}-2(2\alpha^2-6\alpha+5)\right|, \\ \frac{2t^3|\mu-1|}{\left|2(3-2\alpha)t^2-(2-\alpha)^2(4t^2-1)\right|} & \text{for } |\mu-1| \geq \frac{1}{2(3-2\alpha)}\left|\frac{(2-\alpha)^2}{t^2}-2(2\alpha^2-6\alpha+5)\right|. \end{cases}$$

Putting $\mu = 1$ in Theorem 2, we conclude the following result:

COROLLARY 2. For $0 < \alpha \leq 1$ and $t \in (\frac{1}{2}, 1]$, let f given by (1) be in the class $\mathcal{F}_{\Sigma}^{sc}(\alpha, t)$. Then

$$\left|a_3 - a_2^2\right| \leq \frac{t}{3-2\alpha}.$$

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